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NORM INEQUALITIES IN NONLINEAR TRANSFORMS

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Abstract

We shall introduce the recent paper [17]. In the paper, we show the existence of natural norm inequalities in some general nonlinear transforms of reproducing kernel Hilbert spaces and as its applications we derive typical concrete norm inequalities in the nonlinear transforms.

1. General principles

Let E be an arbitrary nonvoid abstract set and let $H_K(E)$ be a Hilbert (possibly finite-dimensional) space admitting a reproducing kernel $K(p,q)$ on E. Then, the Hilbert space $H_K(E)$ is composed of complex-valued functions $f(p)$ on E such that

$$K(\cdot,q) \in H_K(E) \quad \text{for any fixed } q \in E \tag{1.1}$$

and, for any member f of $H_K(E)$ and for any fixed point q of E,

$$(f(\cdot), K(\cdot,q))_{H_K} = f(q). \tag{1.2}$$

In general, a reproducing kernel $K(p,q)$ on E satisfying (1.1) and (1.2) is uniquely determined by the Hilbert space $H_K(E)$ and is a positive matrix in the sense that for any points $\{p_j\}_j$ of E and for any complex numbers $\{C_j\}_j$

$$\sum_{j,j'} C_j \overline{C_{j'}} K(p_{j'}, p_j) \geq 0. \tag{1.3}$$

Conversely, a positive matrix $K(p,q)$ on E satisfying (1.3) determines uniquely a functional Hilbert space (reproducing kernel Hilbert space $\equiv$ RKHS) $H_K(E)$ satisfying (1.1) and (1.2).

We shall consider the RKHS $H_K(E)$ as an input function space of the following nonlinear transform

$$\varphi: \quad f \in H_K(E) \longrightarrow \sum_{n=0}^{\infty} d_n(p) f(p)^n, \tag{1.4}$$

where $\{d_n(p)\}$ are any functions on E.

In this nonlinear transform φ, we shall show that the images $\varphi(f)$, $f \in H_K(E)$, belong to a Hilbert space $\mathbf{H}$ which is naturally determined by the nonlinear transform φ and there exits a natural norm inequality between the two norms $\|\varphi(f)\|_{\mathbf{H}}$ and $\|f\|_{H_K}$.

In order to show these facts we need the three basic ideas of Aronszajn [1] for reproducing kernels; that is, sums, products and restrictions of reproducing kernels.

For two positive matrices $K_1(p,q)$ and $K_2(p,q)$ on E, the sum $K_3(p,q) = K_1(p,q)+K_2(p,q)$ is, of course, a positive matrix on E. The RKHS H_{K_3} admitting the reproducing kernel $K_3(p,q)$ on E is composed of all functions

$$f = f_1 + f_2 \quad (f_j \in H_{K_j}) \tag{1.5}$$

and the norm in H_{K_3} is given by

$$\|f\|^2_{H_{K_3}} = \min\{\|f_1\|^2_{H_{K_1}} + \|f_2\|^2_{H_{K_2}}\}, \tag{1.6}$$

where the minimum is taken over all the expressions (1.5) for f.

The product $K_4(p_1,p_2;q_1,q_2) = K_1(p_1,q_1)K_2(p_2,q_2)$ on $(E\times E)\times(E\times E)$ is, of course, a positive matrix on $E \times E$. The RKHS H_{K_4} admitting the reproducing kernel $K_4(p_1,p_2;q_1,q_2)$ on $E \times E$ is composed of all functions

$$f(p_1,p_2) = \sum_{n=1}^{\infty} f_{1,n}(p_1)f_{2,n}(p_2) \quad (f_{j,n} \in H_{K_j}) \tag{1.7}$$

having finite norms

$$\|f\|^2_{H_{K_4}} = \sum_{n=1}^{\infty} \|f_{1,n}\|^2_{H_{K_1}} \|f_{2,n}\|^2_{H_{K_2}} < \infty. \tag{1.8}$$

The restriction $K_5(p,q) = K_4(p,p;q,q)$ to the diagonal set E of $E \times E$ is a positive matrix and the RKHS H_{K_5} admitting the reproducing kernel $K_5(p,q)$ on E is composed of all functions $f(p) \equiv f(p,p)$ in (1.7) satisfying (1.8). The norm in H_{K_5} is given by

$$\|f\|^2_{H_{K_5}} = \min \sum_{n=1}^{\infty} \|f_{1,n}\|^2_{H_{K_1}} \|f_{2,n}\|^2_{H_{K_2}}, \tag{1.9}$$

where the minimum is taken over all the expressions (1.7) satisfying (1.8) for $f(p) = f(p,p)$ on E.

For a RKHS H_K on E and for any function $s(p)$ on E,

$$K_s(p,q) = s(p)\overline{s(q)}K(p,q) \quad \text{on} \quad E \times E$$

is the reproducing kernel for the Hilbert space H_{K_s} comprising of all functions $f_s(p)$ on E which are expressible in the form

$$f_s(p) = f(p)s(p) \quad \text{on} \quad E \quad \text{for} \quad f \in H_K$$

and we have the inequality

$$\|f_s\|_{H_{K_s}} \leq \|f\|_{H_K},$$

as we see from (1.9).

For n-times sum and n-times product, the circumstances are similar. Hence, we have, in particular, for any $f_j \in H_{K_j}$ $(j = 1, 2, ..., N)$

$$\text{(1.10)} \qquad \|\sum_{j=1}^{N} f_j\|^2_{H_{(\sum_{j=1}^{N} K_j)}} \leq \sum_{j=1}^{N} \|f_j\|^2_{H_{K_j}}$$

and

$$\text{(1.11)} \qquad \|f^n\|^2_{H_{K^n}} \leq \|f\|^{2n}_{H_K}.$$

Hence, we obtain, in general

Theorem. *If*

$$\sum_{n=0}^{\infty} |h_n(p)|^2 K(p,p)^n < \infty \quad \textit{on} \quad E$$

and if, for $f \in H_K$

$$\sum_{n=0}^{\infty} (\|f\|_{H_K})^{2n} < \infty,$$

then, for the nonlinear transform $\varphi(f)$ *in (1.4),*

$$\varphi(f) = \sum_{n=0}^{\infty} d_n(p) f(p)^n$$

converges absolutely on E*, and*

$$\text{(1.12)} \qquad \varphi(f) \in H_{\mathbf{K_d}}$$

and

$$\text{(1.13)} \qquad \|\varphi(f)\|^2_{H_{\mathbf{K_d}}} \leq \sum_{n=0}^{\infty} (\|f\|_{H_K})^{2n},$$

where $H_{\mathbf{K_d}}$ is the RKHS admitting the reproducing kernel

$$\mathbf{K_d}(p,q) = \sum_{n=0}^{\infty} d_n(p)\overline{d_n(q)}K(p,q)^n \quad on \quad E,$$

which converges absolutely on $E \times E$.

In Theorem, the concrete realization of the norm in the RKHS $H_{\mathbf{K_d}}$ is, in general, involved. See, Hejhal [4] and Saitoh [6] for a profound result for $\hat{K}(z,\bar{u})^2$ in the case of the classical Szegö reproducing kernel $\hat{K}(z,\bar{u})$ and for a prototype result of Theorem, respectively.

At this moment, recall the fact that if

$$\mathbf{K_d}(p,q) \ll \tilde{K}(p,q) \quad \text{on} \quad E; \tag{1.14}$$

that is,

$$\tilde{K}(p,q) - \mathbf{K_d}(p,q)$$

is a positive matrix on E, then we have

$$H_{\mathbf{K_d}} \subset H_{\tilde{K}}$$

(as the classes of functions) and, as we see from (1.5) and (1.6)

$$\|f\|_{H_{\tilde{K}}} \le \|f\|_{H_{\mathbf{K_d}}} \quad \text{for all} \quad f \in H_{\mathbf{K_d}}.$$

Hence, for some suitable reproducing kernel $\tilde{K}(p,q)$ satisfying (1.14) whose norm can be determined, in a reasonable way, we can obtain the inequality

$$\|\varphi(f)\|^2_{H_{\tilde{K}}} \le \sum_{n=0}^{\infty}(\|f\|_{H_K})^{2n},$$

in Theorem.

When all the coefficients $\{d_n\}$ are constants, as a typical reproducing kernel $\tilde{K}(p,q)$ satisfying (1.14) and a large reproducing kernel for $K(p,q)$, we can consider the exponential of $K(p,q)$

$$\exp K(p,q) = 1 + K(p,q) + \frac{K(p,q)^2}{2!} + \cdots$$

which is a positive matrix on E. Note here that the constants $(n!)^{-1}$ are not essential in our arguments. Then, we have, in particular

Corollary 1.1. *If $\{d_n(p)\}$ are all constants, we have, in Theorem*

$$\|\varphi(f)\|^2_{H_{\exp K}} \leq \sum_{n=0}^{\infty} |d_n|^2 n! (\|f\|_{H_K})^{2n},$$

if the right hand side converges.

Corollary 1.2. *Let*

$$N(z) = \sum_{n=0}^{\infty} a_n z^n$$

be analytic around $z = 0$ which converges on the disc $\{|z| < R\}$. We define the analytic function $N^+(z)$ on $\{|z| < R\}$ by

$$N^+(z) = \sum_{n=0}^{\infty} |a_n| z^n.$$

We assume that for a reproducing kernel $K(p,q)$ on E,

$$K(p,p) < R.$$

Then, $N^+(K(p,q))$ converges absolutely on $E \times E$ and is a positive matrix on E. For the RKHS $H_{N^+(K)}$ admitting the reproducing kernel $N^+(K(p,q))$ and for a function f in H_K satisfying

$$N^+(\|f\|^2_{H_K}) < \infty,$$

we have the norm inequality

$$\|N(f)\|^2_{H_{N^+(K)}} \leq N^+(\|f\|^2_{H_K}).$$

In the theory of nonlinear partial differential equations, we meet nonlinear transforms, for example, for $u(x,t)$

$$u \longrightarrow u_t + 6uu_x + u_{xxx}$$

and

$$u \longrightarrow u_{tt} - u_{xx} + m^2 \sin u \quad (m > 0; \text{constant}).$$

For such nonlinear transforms we shall show that similar results are valid as in our Theorem.

In order simply to state the result, we shall assume that E is an open interval on $\mathbf{R}$. Then, for the smoothness of a RKHS $H_K(E)$, note that if

$$\frac{\partial^{(j+j')}K(x,y)}{\partial x^j \partial y^{j'}} \quad (j,j' \leq n)$$

are continuously differentiable on $E \times E$, then for any member f of $H_K(E)$, $f^{(j)}(j \leq n)$ are also continuously differentiable on E (Kreĭn [5]), and we have

$$f^{(n)} \in H_{K^{n,n}}$$

and

$$\|f^{(n)}\|_{K^{n,n}} \leq \|f\|_{H_K},$$

for the RKHS $H_{K^{n,n}}$ admitting the reproducing kernel

$$K^{n,n}(x,y) = \frac{\partial^{2n}K(x,y)}{\partial x^n \partial y^n} \quad \text{on} \quad E$$

(Saitoh [14]). Hence, for example, in the nonlinear transform

$$\psi : f \in H_K(E) \longrightarrow h_1(x)f''(x) + h_2(x)f'(x)^2 + h_3(x)|f(x)|^2 \tag{1.15}$$

for any functions $\{h_j(x)\}$ on E, the images $\psi(f)$ belong to the RKHS $H_{\psi^+(K)}$ admitting the reproducing kernel

$$\psi^+(K(x,y)) = h_1(x)\overline{h_1(y)}K^{2,2}(x,y) \tag{1.16}$$

$$+h_2(x)\overline{h_2(y)}K^{1,1}(x,y)^2 + h_3(x)\overline{h_3(y)}K(x,y)\overline{K(x,y)}$$

and we obtain the inequality

$$\|\psi(f)\|^2_{H_{\psi^+(K)}} \leq \|f\|^2_{H_K}(1 + 2\|f\|^2_{H_K}). \tag{1.17}$$

In some general linear transform of Hilbert spaces we could get essentially isometrical identities between the input and the output function spaces (see [9] and [13]), but in our nonlinear transforms we get norm inequalities, essentially and to determine the cases making the equalities hold in the inequalities is, in general, involved, in even the case of a finite dimensional RKHS H_K and we need case by case arguments to determine the cases. See, for example, [11], [10] and [6]. However, for many cases (not always), for the reproducing kernels $f(p) = K(p,q)(q \in E)$ equalities hold in our inequalities. See [10], for example.

Subsequently we shall derive typical norm inequalities in nonlinear transforms by applying our Theorem to typical reproducing kernels. For other examples, see the original paper[17] or the articles in References.

2. The most simple Sobolev Hilbert space

Note that

$$G(x,y) = \frac{1}{2}e^{-|x-y|} \tag{2.1}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\xi^2+1}e^{i\xi(x-y)}d\xi$$

is the reproducing kernel for the Sobolev Hilbert space S on $\mathbf{R}$ comprising of all real-valued and absolutely continuous functions $f(x)$ on $\mathbf{R}$ with finite norms

$$\left\{\int_{-\infty}^{\infty}(f'(x)^2+f(x)^2)dx\right\}^{\frac{1}{2}} < \infty, \tag{2.2}$$

(Saitoh [15]). Then, we have

$$K(x,y) = \sum_{n=1}^{\infty}G(x,y)^n \tag{2.3}$$

$$\ll \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{n=1}^{\infty}\frac{2n}{2^n}\right)\frac{1}{\xi^2+1}e^{i\xi(x-y)}d\xi$$

$$= D\cdot G(x,y); D = 2\sum_{n=1}^{\infty}\frac{n}{2^n} > 0.$$

Hence, for the nonlinear transform of $f \in S$

$$\varphi(f) = \sum_{n=1}^{\infty}d_n f(x)^n \quad (d_n : \text{constants}),$$

we have the inequality

$$\frac{1}{D}\int_{-\infty}^{\infty}\left\{\left(\sum_{n=1}^{\infty}d_n f(x)^n\right)'^{\,2} + \left(\sum_{n=1}^{\infty}d_n f(x)^n\right)^2\right\}dx$$

$$\leq \sum_{n=1}^{\infty}|d_n|^2\left\{\int_{-\infty}^{\infty}(f'(x)^2+f(x)^2)dx\right\}^n,$$

if the right hand side converges.

3. The Bergman-Selberg kernels

For $q > \frac{1}{2}$,

$$K_q(z, \overline{u}) = \frac{\Gamma(2q)}{(z+\overline{u})^{2q}}$$

is the Bergman-Selberg reproducing kernel on the half plane $R^+ = \{\Re z > 0\}$ comprising of all analytic functions $f(z)$ on R^+ with finite norms

$$\|f\|^2_{H_{K_q}} = \frac{1}{\pi\Gamma(2q-1)} \iint_{R^+} |f(z)|^2 [2\Re z]^{2q-2}\, dxdy. \tag{3.1}$$

For $q = \frac{1}{2}$, $K_{1/2}(z, \overline{u})$ is the Szegö reproducing kernel on R^+ comprising of all analytic functions $f(z)$ on R^+ with finite norms

$$\|f\|^2_{H_{K_{1/2}}} = \frac{1}{2\pi} \sup_{x>0} \int_{-\infty}^{\infty} |f(x+iy)|^2\, dy. \tag{3.2}$$

Then, a member $f(z)$ of $H_{K_{1/2}}$ has nontangential boundary values on the imaginary axis belonging to L_2 and we have

$$\|f\|^2_{H_{K_{1/2}}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(iy)|^2\, dy$$

(cf. [3]). Note that

$$\frac{\partial^2 K_1(z,\overline{u})}{\partial z \partial \overline{u}} = \frac{6}{(z+\overline{u})^4}, \quad \frac{\partial^4 K_1(z,\overline{u})}{\partial z^2 \partial \overline{u}^2} = \frac{120}{(z+\overline{u})^6},$$

$$\frac{\partial^2 K_{1/2}(z,\overline{u})}{\partial z \partial \overline{u}} = \frac{2}{(z+\overline{u})^3}, \quad \text{and} \quad \frac{\partial^4 K_{1/2}(z,\overline{u})}{\partial z^2 \partial \overline{u}^2} = \frac{24}{(z+\overline{u})^5}.$$

Hence, for the nonlinear transforms of $f \in H_{K_1}$

$$d_1 f' + d_2 f^2,$$

$$d_1 f'' + d_2 f' f + d_3 f^3,$$
$$d_1 f f'' + d_2 (f')^2 + d_3 f^4,$$
$$d_1 f' f'' + d_2 f (f')^2 + d_3 f^5,$$

and

$$d_1 (f'')^2 + d_2 (f')^3 + d_3 (f')^2 f^2 + d_4 f^6,$$

we have the specially simple norm inequalities. We have, for example,

$$\frac{6}{7}\|d_1 f' + d_2 f^2\|^2_{H_{K_2}} \le \|f\|^2_{H_{K_1}}\left(\frac{1}{6}|d_1|^2 + |d_2|^2\|f\|^2_{H_{K_1}}\right),$$

and

$$\frac{120}{127}\|d_1 f'' + d_2 f' f + d_3 f^3\|^2_{H_{K_3}}$$

$$\le \|f\|^2_{H_{K_1}}\left(\frac{1}{120}|d_1|^2 + \frac{1}{6}|d_2|^2\|f\|^2_{H_{K_1}} + |d_3|^2\|f\|^4_{H_{K_1}}\right).$$

For the nonlinear transforms of $f \in H_{K_{1/2}}$

$$d_1 f' + d_2 f^3, d_1 f f' + d_2 f^4, d_1 (f')^2 + d_2 f' f^3 + d_3 f^6, d_1 f'' + d_2 f' f^2 + d_3 f^5,$$

and

$$d_1 f f'' + d_2 (f')^2 + d_3 f^6,$$

we have the corresponding and specially simple norm inequalities.

4. A transform with nonconstant coefficients

As we see directly by using the Taylor expansion,

$$K(z, \overline{u}) = \frac{1}{\overline{u}z}\log\frac{1}{1-\overline{u}z}$$

is the reproducing kernel for the Hilbert space H_K comprising of all analytic functions $f(z)$ on U with finite norms

$$\left\{\frac{1}{\pi}\iint_U |f'(z)|^2 dxdy + \frac{1}{2\pi}\int_{\partial U} |f(z)|^2 |dz|\right\}^{\frac{1}{2}} < \infty,$$

([3]). Hence, from the identity

$$\frac{1}{1-\overline{u}z} = e^{z\overline{u}K(z,\overline{u})},$$

in the transform of H_K functions

$$f \longrightarrow e^{zf},$$

we have the inequality

$$\frac{1}{2\pi}\int_{\partial U} |e^{zf(z)}|^2 |dz|$$

$$\leq \exp\left[\frac{1}{\pi}\iint_U |f'(z)|^2 dxdy + \frac{1}{2\pi}\int_{\partial U} |f(z)|^2 |dz|\right].$$

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