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<th>Title</th>
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</tr>
</thead>
<tbody>
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Kyoto University
Newton polyhedrons and a formal Gevrey space of double indices
for linear partial differential operators

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Abstract

In this paper, we shall study a necessary condition to become that
the below operator $L_m(t, x; \partial_t, \partial_x)$ is bijective on $G^{\{s_t, s_x\}}$, where it is said
that a formal power series $U(t, x) = \sum_{\ell \beta} U_{\ell \beta} t^{\ell \beta} / \ell! \beta!$ belongs to $G^{\{s_t, s_x\}}$
when $U(t, x) = \sum_{\ell \beta} U_{\ell \beta} t^{\ell \beta} / (\ell! \beta! s_t (\beta!) s_x)$ is convergent near the origin for
$s_t, s_x \geq 1$.

1 Introduction

Let us give the operator $L_m$ that we shall study in this paper.

Let $t = (t_1, \cdots, t_q) \in C^q$, $x = (x_1, \cdots, x_p) \in C^p$ and
t · $\partial_t = (t_1 \partial t_1, t_2 \partial t_2, \cdots, t_q \partial t_q)$. Set

(1.1) \[(t \cdot \partial_t)^j = (t_1 \partial t_1)^{j_1} (t_2 \partial t_2)^{j_2} \cdots (t_q \partial t_q)^{j_q}\]

for $j = (j_1, \cdots, j_q) \in N^q$ and

(1.2) \[P_m(t \cdot \partial_t) = \sum_{|j| \leq m} P_j(t \cdot \partial_t)^j,\]

where $P_\sigma \in C$, and $P_m$ is said to be of Fuchs type of order $m$ in $[M]$. Then we
consider the following operator:

(1.3) \[L_m = P_m(t \cdot \partial_t) + A(t, x; \partial_t; \partial_x) + B(t, x; \partial_t; \partial_x)\]

where

(1.4) \[A = \sum_{\substack{|\alpha| = 0 \\text{finite} \\text{finite} \\text{finite} \\text{finite}}}^{|\sigma'| = |\sigma| \leq m \\text{finite} \\text{finite} \\text{finite} \\text{finite}} \]

\[a_{\sigma, \sigma'}^\alpha(t, x)t^{\sigma'} \partial_t^\alpha \partial_x^\sigma,\]

and

(1.5) \[B = \sum_{\substack{|\alpha'| + |\alpha| > |\sigma| + |\alpha| \\text{finite} \\text{finite} \\text{finite} \\text{finite}}} \]

\[b_{\sigma, \sigma'}^{\alpha, \alpha'}(t, x)t^{\sigma'} x^{\alpha'} \partial_t^\sigma \partial_x^{\alpha},\]

where the coefficients $a_{\sigma, \sigma'}(t, x)$ and $b_{\sigma, \sigma'}^{\alpha, \alpha'}(t, x)$ are holomorphic functions in a
neighbourhood of the origin for any $(\sigma, \sigma', \alpha, \alpha') \in Z^q \times N^q \times Z^p \times N^p$.
Miyake and Hashimoto studied the unique solvability in $G^{\{s_t,1\}}$ for such type operator. They characterized the Gevrey index $s_t$ by Newton polygons in [M] and [MH].

Our motivation comes from the following facts. Put

$$L = (t\partial_t + 1) - 3t^3 x \partial_t^2 \partial_x - (t\partial_t + 1)x^2 \partial_x.$$  

This operator is not bijective in $G^{\{s_t,1\}}$ for any $s_t$, but is bijective in $G^{s_t,s_x}$ for $s_t \geq 3$ and $s_x \geq 2$.

So it is our purpose that we shall consider $G^{s_t,s_x}$ to obtain the unique solvability for this operator. We shall define a Newton polyhedrons to characterize double Gevrey indices.

In Section 2, we give our results after defining a function space and Newton polyhedron and listing some notations. In Section 3, we prove our theorems.

2 Statement of results

2.1 Notations.

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the set of non negative integers, integers, real numbers and complex numbers, respectively.

$\mathbb{C}[[t, x]]$ denotes the set of formal power series in $t \in \mathbb{C}^q$ and $x \in \mathbb{C}^q$ with coefficients in $\mathbb{C}$.

For multi indices $\sigma = (\sigma_1, \cdots, \sigma_q) \in \mathbb{Z}^q$ and $\alpha = (\alpha_1, \cdots, \alpha_p) \in \mathbb{Z}^p$, an integro-differential $\partial_t^\sigma \partial_x^\alpha U(t, x)$ of $U(t, x) = \sum_{l, \beta} U_{l\beta} t \theta^l / l! \beta! \in \mathbb{C}[[t, x]]$ is defined as follows:

$$(2.1) \quad \partial_t^\sigma \partial_x^\alpha U(t, x) := \sum_{l \in \mathbb{N}^q, l - \sigma \in \mathbb{N}^q} U_{l\beta} \frac{t^{-\sigma} x^{-\alpha}}{(l - \sigma)! (\beta - \alpha)!}.$$  

2.2 Formal Gevrey class $G^{\{s_t,s_x\}}(T, X; m)$.

For $U(t, x) \in \mathbb{C}[[t, x]]$, we set $U(t, x) = \sum_{l, \beta} U_{l\beta} t \theta^l / l! \beta!$, where $U_{l\beta} \in \mathbb{C}$, $l \in \mathbb{N}^q$ and $\beta \in \mathbb{N}^p$ and $\mathbb{R}_+$ denotes the set of positive real numbers.

Let $s_t, s_x \geq 1, T > 0, X > 0, \tau = (\tau_1, \cdots, \tau_q) \in \mathbb{R}_+^q, \rho = (\rho_1, \cdots, \rho_p) \in \mathbb{R}_+^p$ and $m \in \mathbb{N}$. Then we define a space $G^{\{s_t,s_x\}}(T, X; m) \subset \mathbb{C}[[t, x]]$ as follows.

$$(2.2) \quad G^{\{s_t,s_x\}}(T, X; m) := \left\{ U(t, x) \in \mathbb{C}[[t, x]]; \|U\|_{T/\tau, X/\rho, m} < \infty \right\},$$  

where $\|U\|_{T/\tau, X/\rho, m}$ is defined by

$$(2.3) \quad \|U\|_{T/\tau, X/\rho, m} := \max_{l \in \mathbb{N}^q, l - \sigma \in \mathbb{N}^q} \left( \sum_{\beta \in \mathbb{N}^p} \frac{1}{(l - \sigma)! (\beta - \alpha)!} \right)^{1/(l - \sigma)! (\beta - \alpha)!}.$$  

In Section 2, we give our results after defining a function space and Newton polyhedron and listing some notations. In Section 3, we prove our theorems.
where

\[
\|U\|^{(s_1, s_2)}_{T, x; r, m} := \sup_{l, \beta} |U_{l\beta}| \frac{|l|!^{m} T^{|l|} x^{|eta|}}{((s_t + m)|l| + s_x |eta|)! T^l \rho^\beta}
\]

and \( n! := \Gamma(n + 1) \).

Hence there exist positive constants \( R_t, R_x \) and \( C \) such that

\[
|U_{l\beta}| \leq C \frac{|l|!^{s_1} |\beta|!^{s_2}}{R_t^{|l|} R_x^{|eta|}}
\]

for any \( l \in \mathbb{N}^q \) and \( \beta \in \mathbb{N}^p \).

Here we define a formal Gevrey space as follows:

**Definition 2.1**

\[
G^{(s_1, s_2)} := \cup_{T, x > 0} G^{(s_1, s_2)}(T, X; m)
\]

**2.3 Newton polyhedron.**

Here we define Newton polyhedron for a linear partial integro-differential operator and state some remarks \( L_m \).

Let

\[
P = \sum_{finito} a_{\sigma, \sigma', \alpha, \alpha'}^\alpha(t, x) t^{\sigma'} x^{\alpha'} \partial_t^\sigma \partial_x^\alpha
\]

be a linear partial integro-differential operator of finite order with holomorphic coefficients in a neighbourhood of the origin.

In the space \( \mathbb{R}^3 \), we define the following lower half line for \((\sigma, \sigma', \alpha, \alpha') \in \mathbb{Z}^q \times \mathbb{N}^q \times \mathbb{Z}^p \times \mathbb{N}^p:\)

\[
Q(\sigma, \sigma', \alpha, \alpha') := \{(X, Y, Z) \in \mathbb{R}^3; X = |\sigma'| - |\sigma|, Y = |\alpha'| - |\alpha|, Z \leq |\sigma| + |\alpha|\}.
\]

**Definition 2.2** Newton polyhedron \( N(P) \) of the operator \( P \) is defined by

\[
N(P) := Ch\{Q(\sigma, \sigma', \alpha, \alpha'); (\sigma, \sigma', \alpha, \alpha') \text{ with } a_{\sigma, \sigma'}^\alpha(t, x) \neq 0\},
\]

where \( Ch\{\cdot\} \) denotes the convex hull of sets in \{\cdot\}.

Let \( N(L_m) \) be Newton polyhedron of \( L_m \).

**Remark 2.3** By the form of \( L_m \), the lower half line \{(0, 0, Z); Z \leq m\} becomes a side of \( N(L_m) \) and the point \((0, 0, m)\) becomes a vertex of \( N(L_m) \).
Next let 
(2.9) \[ \mathcal{A} = \{ (\mathcal{X}, \mathcal{Y}, Z); a\mathcal{X} + b\mathcal{Y} - Z + m \geq 0 \}. \]

Then we define the following set of pairing indices:
(2.10) \[ \mathcal{S} = \{ (s_t, s_x); s_t = a + 1, s_x = b + 1, N(L_m) \subseteq \mathcal{A} \}. \]

**Remark 2.4** Since the boundary of \( \mathcal{S} \) is a hyper plane which goes through the point \((0,0,m)\), there exists \((s_t, s_x)\) such that \(N(L_m) \subseteq \mathcal{A}\) by Remark 2.3.

**Remark 2.5** For any \((s_t, s_x)\) belonging to \(\mathcal{S}\), we obtain
(2.11) \[ s_t(|\sigma| - |\sigma'|) + s_x(|\alpha| - |\alpha'|) + |\sigma'| + |\alpha'| - m \leq 0 \]
for \((\sigma, \sigma', \alpha, \alpha')\) with \(Q(\sigma, \sigma', \alpha, \alpha') \in N(L_m)\).

### 2.4 Main results.

We assume the following additional condition.
(A.1) If \(m(|\sigma'| - |\sigma|) + |\sigma'| < m\) for \((\sigma, \sigma', \alpha, \alpha')\) with \(b_{\sigma\sigma}'(t, x) \neq 0\), then
(2.12) \[ (m + s_t)(|\sigma| - |\sigma'|) + s_x(|\alpha| - |\alpha'|) + |\alpha'| \leq 0, \]
for \((s_t, s_x)\) belonging to \(\mathcal{S}\).
(S.C) For any \(\epsilon > 0\), there exist \(\tau \in \mathbb{R}_+^q\) and \(\rho \in \mathbb{R}_+^p\) such that
(2.13) \[ \sum_{|\alpha| = 0}^{\text{finite}} |a_{\sigma, \sigma}'^{\alpha}| \tau^{\sigma - \sigma'} \rho^{\alpha} < \epsilon, \]
where \(a_{\sigma, \sigma}'^{\alpha} = a_{\sigma, \sigma}'^{\alpha}(0,0)\).

It is said that the condition (S.C) is Spectral condition in \([M]\).

Then we obtain the following results.

**Theorem 2.6** Assume that \(L_m\) satisfies the condition (A.1) and (S.C) and further assume that there exists a positive constant \(C\) such that
(2.14) \[ |P_m(l)| \geq C(|l| + 1)^m \quad \text{for all} \quad l \in \mathbb{N}^q. \]

Then the mapping
(2.15) \[ L_m : G^{(s_t, s_x)} \rightarrow G^{(s_t, s_x)} \]
is bijective for any \((s_t, s_x)\) belonging to \(\mathcal{S}\).
Next set

\[\delta = \max \left\{ \left\{ \frac{|\alpha'| + \max\{|\sigma'| - m, m(|\sigma'| - m)|}{|\alpha'| + |\sigma'| - |\alpha| - |\sigma|}; b_{\sigma\sigma'}(t, x) \neq 0 \right\}, 1 \right\}.\]

Then for \(s_x \geq \delta\), there exist indices \(s_t\) with \(s_t \geq 1\) such that if \(m(|\sigma'|-|\sigma|) + |\sigma'| - m \geq 0\), then

\[s_t(|\sigma|-|\sigma'|) + s_x(|\alpha|-|\alpha'|) + |\alpha'| - m \leq 0,\]

and if \(m(|\sigma'|-|\sigma|) + |\sigma'| - m < 0\), then

\[(s_t + m)(|\sigma| - |\sigma'|) + s_x(|\alpha| - |\alpha'|) + |\alpha'| \leq 0.\]

For example if \(s_x = s_t \geq \delta\), then the above formulas are satisfied. So we obtain the following Corollary.

**Corollary 2.7** Assume that \(L_m\) satisfies the condition \((S.C)\) and further assume the inequality (2.14). Then for \(s_x \geq \delta\), there exist indices \(s_t\) with \(s_t \geq 1\) such that the mapping (2.15) is bijective.

### 3 Proof of Theorem.

In this section first we estimate a operator of form \(t^{\sigma'}x^{\alpha'} \partial_t^{\sigma} \partial_x^{\alpha} P_m^{-1}\) on \(G_{\rho\tau}^{\{s_t,s_x\}}(T, X; m)\), next by using the estimate we show \(L_m P_m^{-1}\) is bijective on same space, at last we give a proof of Theorem 2.6.

#### 3.1 The estimate of operator \(t^{\sigma'}x^{\alpha'} \partial_t^{\sigma} \partial_x^{\alpha} P_m^{-1}\).

Here we study a estimate of the operator \(t^{\sigma'}x^{\alpha'} \partial_t^{\sigma} \partial_x^{\alpha} P_m^{-1}\) of the mapping

\[t^{\sigma'}x^{\alpha'} \partial_t^{\sigma} \partial_x^{\alpha} P_m^{-1} : \mathcal{S}_{\rho\tau}(T, X; m) \rightarrow \mathcal{S}_{\rho\tau}(T, X; m),\]

where

\[P_m^{-1} : \sum U_{l\beta} \frac{t^l x^\beta}{l!\beta!} \mapsto \sum P_m(l)^{-1} U_{l\beta} \frac{t^l x^\beta}{l!\beta!}.\]

**Lemma 3.1** Assume that the conditions in Theorem 2.6 are satisfied. Then for any \((s_t, s_x)\) belonging to \(S\), there exist a positive constant \(C\) such that the operator norm of the mapping (3.1) is estimated as follows:

\[||t^{\sigma'}x^{\alpha'} \partial_t^{\sigma} \partial_x^{\alpha} P_m^{-1}|| \leq CT^{|\alpha'| - |\alpha|} X^{|\sigma'| - |\sigma|} \rho^{|\sigma| - |\sigma'|},\]

where the constant \(C\) depends only on \(m, s_t, s_x, \sigma, \sigma', \alpha\) and \(\alpha'\), and \(||\cdot||\) denotes the operator norm on \(G_{\rho\tau}^{\{s_t,s_x\}}(T, X; m)\).
Proof. Let \( t^\sigma x^\alpha \partial_t^\sigma \partial_x^\alpha P_m^{-1} U(t, x) = \sum_{\beta l} V_{l\beta} t^l x^\beta / l! \beta! \). Then we obtain

\[
V_{l\beta} = \frac{l!}{(l - \sigma')!} \frac{\beta!}{(\beta - \alpha')!} P_m(l + \sigma - \sigma')^{-1} U_{l\beta\sigma - \alpha - \alpha'}
\]

where \( l + \sigma - \sigma' \in \mathbb{N}^q \) and \( \beta + \alpha - \alpha' \in \mathbb{N}^p \).

Therefore we have

\[
|V_{l\beta}| \leq C_0 \frac{|l|!}{(s_t + m)|l| + s_x|\beta|} \frac{|l|! |\beta|!}{\eta^l \xi^\beta}.
\]

By Remark 2.5 and the condition (A.1), we obtain the estimation (3.3). Q.E.D.

3.2 The estimate of operator \( L_m P_m^{-1} \).

For \( x \in \mathbb{C}^p \), we set \( |x| = x_1 + \cdots + x_p \) and \( ||x|| = |x_1| + \cdots + |x_p| \). For a domain \( \Omega \subset \mathbb{C}^p \), \( O(\Omega) \) denotes the set of holomorphic functions in \( \Omega \), \( O(\overline{\Omega}) := O(\Omega) \cap C(\overline{\Omega}) \). Similar notations will be used frequently for functions defined in a domain \( C_{t1x}^{q+p} \).

Let \( a(t, x) = \sum a_{l\beta} t^l x^\beta / l! \beta! \in O(||t|| \leq \kappa T, ||x|| \leq \kappa X) \). By Cauchy's integral formula on a polycircle \( \prod_{j=1}^q \{ |\iota_j| = \eta_j \kappa T \} \times \prod_{i=1}^p \{ |x_i| = \xi_i \kappa X \} \) (\( \eta_j > 0, \eta_1 + \cdots + \eta_q = 1 \)) and \( (\xi_i > 0, \xi_1 + \cdots + \xi_p = 1) \), we have

\[
|a_{l\beta}| \leq C \frac{l! \beta!}{(\kappa T)^{|l|} (\kappa X)^{|\beta|}} \frac{|l|! |\beta|!}{\eta^l \xi^\beta}.
\]

Since \( \eta^l \) and \( \xi^\beta \) take its maximum on the above mentioned domain at a point \( \eta = (l_1/|l|, \cdots, l_q/|l|) \) and \( \xi = (\beta_1/|\beta|, \cdots, \beta_p/|\beta|) \), we have

\[
|a_{l\beta}| \leq ||a||_{\kappa T, \kappa X} \frac{l! \beta!}{(\kappa T)^{|l|} (\kappa X)^{|\beta|}}
\]

Hence by Stirling's formula, we have

\[
|a_{l\beta}| \leq C(q, p) ||a||_{\kappa T, \kappa X} \frac{(|l| + [q/2])! (|\beta| + [p/2])!}{(\kappa T)^{|l|} (\kappa X)^{|\beta|}}.
\]
for some positive constant $C(q, p)$ depending only on the dimension $q$ of $t$ and
the dimension $p$ of $x$. Here $[q/2]$ (resp. $[p/2]$) denotes the integral part of $q/2$
(resp. $p/2$).
Then we have the following lemma.

**Lemma 3.2** Let $U(t, x) \in G_{\tau, \rho}^{s, s}(T, X; m)$ and
\[ a(t, x) \in \mathcal{O}\{||\tau t|| \leq \kappa T\} \times \{||\rho x|| \leq \kappa X\} \quad (\kappa > 0). \]
Then $a(t, x)U(t, x) \in G_{\tau, \rho}^{s, s}(T, X; m)$ for any $\kappa > 1$ and it holds
\[
||aU||_{T/\tau, X/\rho; m}^{s, s} \leq C(q, p) \frac{[q/2]!}{(1 - 1/\kappa)^{[q/2]+1}} \frac{[p/2]!}{(1 - 1/\kappa)^{[p/2]+1}} \times ||a||_{\kappa T, \kappa X} ||U||_{T/\tau, X/\rho; m}^{s, s}.
\]

**Proof.** We may be $\rho = (1, \cdots, 1) \in \mathbb{R}_+^p$ and $\tau = (1, \cdots, 1) \in \mathbb{R}_+^q$.
Set $a(t, x)U(t, x) = \sum V_{l\beta} x^\beta / l!\beta!$, where
\[
V_{l\beta} = \sum_{0 \leq n \leq l} a_{n\gamma l\beta\gamma} U_{n-\gamma} \frac{l!}{n!(l-n)!} \frac{\beta!}{\gamma!(\beta-\gamma)!}.
\]
Then we have
\[
|V_{l\beta}| \leq C(q, p) ||a||_{\kappa T, \kappa X} ||U||_{T/\tau, X/\rho; m}^{s, s} \sum_{0 \leq n \leq l} \frac{(|n| + [q/2])!(|\gamma| + [p/2])!}{(\kappa T)^{|n|}(\kappa X)^{|\gamma|}} \times \frac{l!}{n!(l-n)!} \frac{\beta!}{\gamma!(\beta-\gamma)!}.
\]
Hence we have
\[
||aU||_{T/\tau, X/\rho; m}^{s, s} \leq C(q, p) ||a||_{\kappa T, \kappa X} ||U||_{T/\tau, X/\rho; m}^{s, s} \frac{[q/2]!}{(1 - 1/\kappa)^{[q/2]+1}} \frac{[p/2]!}{(1 - 1/\kappa)^{[p/2]+1}}.
\]
Q.E.D
By Lemma 3.1 and Lemma 3.2, we obtain the following essential proposition.

**Proposition 3.3** Assume that the conditions of Theorem 2.6 are satisfied.
Then for any $(s_t, s_x)$ belonging to $S$, there exist a positive constant $R_0$, $\tau \in \mathbb{R}_+^q$ and $\rho \in \mathbb{R}_+^p$ such that the mapping
\[
L_m P_m^{-1} : G_{\tau, \rho}^{s_t, s_x}(R, R; m) \rightarrow G_{\tau, \rho}^{s_t, s_x}(R, R; m)
\]
is bijective for any $R$ with $0 < R < R_0$. 
Proof. Since $P_m P_m^{-1} = I$ on $G_{r+p}^{s_l, s_n}(R, R; m)$, it is sufficiently that we show that $Q = (A + B)P_m^{-1}$ is a contraction mapping. By Lemma 3.1, Lemma 3.2 and the condition (A.1), we obtain a estimation $||A P_m^{-1}|| = O(\epsilon) + O(R)$, and by Lemma 3.1, Lemma 3.2 and the condition of the operator $B (|\sigma'| + |\alpha'| > |\sigma| + |\alpha|)$, we obtain a estimation $||B P_m^{-1}|| = O(R)$. Hence the operator $Q$ is a construction mapping on $G_{r+p}^{s_l, s_n}(R, R; m)$ for sufficiently small $\epsilon$ and $R$. Q.E.D.

Proof of Theorem 2.6.
Let $P_m^{-1} U(t, x) = \sum P_m(l)^{-1} U_{l\beta} t^{\beta}/l!\beta!$ for $U(t, x) = \sum U_{l\beta} t^{\beta}/l!\beta!$. By Proposition 3.3, $L_m P_m^{-1}$ is bijective on $G_{r+p}^{s_l, s_n}(R, R; m)$, and since $P_m^{-1} P_m = P_m P_m^{-1} = I$ (identity) holds on $G_{r+p}^{s_l, s_n}(R, R; m)$, $L_m$ is bijective on $G_{r+p}^{s_l, s_n}(R, R; m)$. This completes the proof.

References


