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GENERALIZED STRONGLY NONLINEAR QUASI–VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we introduce and study a new class of variational inequalities, which are called the generalized strongly nonlinear quasi-variational inequalities. An algorithm for finding the approximate solution of generalized strongly nonlinear quasi-variational inequalities is also given. These variational inequalities include the previously known classes of variational inequalities as special cases.

1. Introduction

Variational inequality theory introduced by Stampacchia [12] has enjoyed vigorous growth for the last thirty years. Variational inequality theory describes a broad spectrum of interesting and important developments involving a link among various fields of mathematics, physics, economics, and engineering sciences [1,6].

In recent years, various extensions and generalizations of the variational inequalities have been proposed and analyzed. An important one is the quasi-variational inequality introduced and studied by Bensoussan and Lions [2]. For the recent applications, and numerical methods, see [4,5].

Key words and phrases. Variational inequalities, iterative algorithms.

In this paper, we obtain an existence theorem of solutions of a generalized strongly nonlinear quasi-variational inequality and construst a new iterative algorithm, which includes many known algorithms as special cases to slove variational inequalities and quasi-variational inequalities. Further, we prove the convergence of the iterative sequences generated by this algorithm. Our main results extend and improve the earlier and recent results of Noor[8,9,10], Siddiqi and Ansari[11].

2. Preliminaries

Let H be a Hilbert space. We denote by $< \cdot, \cdot >$ and $\| \cdot \|$ the inner product and norm on H, respectively. Let $K \subset H$ be a closed convex subset of H. Given mappings $m : H \to H$, $A : H \to H$, $g : H \to H$, $T : H \to 2^H$, and $V : H \to 2^H$, we consider the problem of finding $u \in H$, $y \in V(u)$, and $w \in T(u)$ such that $g(u) \in K(u)$ and

$$< v - g(u), w + Ay > \geq 0 \tag{2.1}$$

for all $v \in K(u)$, where $K(u) = m(u) + K$.

The problem (2.1) is known as the generalized strongly nonlinear quasi-variational inequality.

If $g \equiv I$, the identity operator, the problem (2.1) is equivalent to finding $u \in K(u)$, $y \in V(u)$, and $w \in T(u)$ such that

$$< v - u, w + Ay > \geq 0 \tag{2.2}$$

for all $v \in K(u)$. The problem (2.2) is called the multivalued strongly nonlinear quasi-variational inequality(see Noor[10]).

If $K(u) \equiv K$, the problem (2.2) is equivalent to finding $u \in K$, $y \in V(u)$, and $w \in T(u)$ such that

$$< v - u, w + Ay > \geq 0 \tag{2.3}$$

for all $v \in K$, which is called the multivalued strongly nonlinear variational inequality(see Noor[10]).

If $T : H \to H$ is a single valued operator and $V : H \to H$ is the identity operator, the problem (2.3) is equivalent to finding $u \in K$ such that

$$< v - u, T(u) + A(u) > \geq 0$$

for all $v \in K$, which is called the strongly nonlinear variational inequality(see Noor[10]).

LEMMA 2.1[6]. *If $K \subset H$ is a closed convex set and $z \in H$ is a given point, then $u \in K$ satisfies the inequality*

$$< u - z, v - u > \geq 0$$

for all $v \in K$ if and only if

$$u = P_K z. \tag{2.4}$$

LEMMA 2.2[6]. *The mapping P_K defined by (2.4) is nonexpansive, that is,*

$$\|P_K u - P_K v\| \leq \|u - v\|$$

for all $u, v \in H$.

LEMMA 2.3[7]. *If $K(u) = m(u) + K$ and $K \subset H$ is a closed convex set, then for any $u, v \in H$, we have*

$$P_{K(u)}(v) = m(u) + P_K(v - m(u)).$$

Let (X, d) be a metric space, 2^X be the family of all nonempty subsets of X. For any $A, B \in 2^X$, define

$$\delta(A, B) = \sup\{d(x, y) : x \in A, y \in B\}.$$

Let $P = \{d(x, y) : x, y \in X\}$, $\bar{P}$ denotes the closure of P. A mapping $F : X \to 2^X$ is said to be the φ-contraction mapping if

$$\delta(Fx, Fy) \leq \varphi(d(x, y))$$

for all $x, y \in X$, where $\varphi : \bar{P} \to [0, \infty)$ satisfies $\varphi(t) < t$ for $t \in \bar{P} - \{0\}$.

By the proof of Theorem 1 and 2 of Boyd and Wong [3], it is easy to see that the following theorem holds.

THEOREM 2.1. *Let (X, d) be a complete metrically convex metric space and $F : X \to 2^X$ be a φ-contractive mapping. Then F has a fixed point and for any $x_0 \in X$, $x_n \in F(x_{n-1})$, $n \geq 1$, $\{x_n\}$ converges to a fixed point of F in X.*

DEFINITION 2.1. Let D be a nonempty subset of H, $T : D \to 2^H$ and $\Phi, \Psi : [0, \infty) \to [0, \infty)$. We call

(1) T is Φ-Lipschitz continuous if

$$\delta(Tx, Ty) \leq \|x - y\| \Phi(\|x - y\|)$$

for all $x, y \in D$.

(2) T is Ψ-strongly monotone if

$$< u - v, x - y > \geq \|x - y\|^2 \Psi(\|x - y\|)$$

for all $x, y \in D$, $u \in T(x)$, and $v \in T(y)$.

DEFINITION 2.2. An operator $g : H \to H$ is said to be

(i) strongly monotone if there exists a constant $\delta > 0$ such that

$$(g(u) - g(v), u - v) \geq \alpha \|u - v\|^2 \quad \text{for all} \quad u, v \in H;$$

(ii) Lipschitz continuous if there exists a constant $\sigma > 0$ such that

$$\|g(u) - g(v)\| \leq \sigma \|u - v\| \quad \text{for all} \quad u, v \in H.$$

3. Main Results

THEOREM 3.1. *Let K be a nonempty closed convex subset of H. Then $u \in H$, $y \in V(u)$, and $w \in T(u)$ are a solution of problem (2.1) if and only if, for some given $\rho > 0$, the mapping $F : H \to 2^H$ defined by*

$$F(u) = \cup_{w \in T(u)} \cup_{y \in V(u)} [u - g(u) + m(u) + P_K(g(u) - \rho(w + Ay) - m(u))]$$

has a fixed point.

Proof. Let $u \in H$, $y \in V(u)$, and $w \in T(u)$ be a solution of problem (2.1). Then we have $g(u) \in K(u)$ and

$$< w + Ay, v - g(u) > \geq 0$$

for all $v \in K(u)$, and hence for any given $\rho > 0$,

$$< g(u) - (g(u) - \rho(w + Ay)), v - g(u) > \geq 0$$

for all $v \in K(u)$. By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} g(u) &= P_{K(u)}(g(u) - \rho(w + Ay)) \\ &= m(u) + P_K(g(u) - \rho(w + Ay) - m(u)). \end{aligned}$$

Hence we get

$$\begin{aligned} u &= u - g(u) + m(u) + P_K(g(u) - \rho(w + Ay) - m(u)) \\ &\in \cup_{w \in T(u)} \cup_{y \in V(u)} [u - g(u) + m(u) \\ &\qquad + P_K(g(u) - \rho(w + Ay) - m(u))] \\ &= F(u), \end{aligned}$$

i.e., u is a fixed point of F.

Now let u be a fixed point of F. By the definition of F, there exist $y \in V(u)$ and $w \in T(u)$ such that

$$u = u - g(u) + m(u) + P_K(g(u) - \rho(w + Ay) - m(u))$$

Therefore

$$\begin{aligned} g(u) &= m(u) + P_K(g(u) - \rho(w + Ay) - m(u)) \\ &= P_{K(u)}(g(u) - \rho(w + Ay)). \end{aligned}$$

Hence

$$g(u) \in K(u)$$

and by Lemma 2.1,

$$< g(u) - (g(u) - \rho(w + Ay)), v - g(u) > \geq 0$$

for all $v \in K(u)$. Note $\rho > 0$, and we have

$$< w + Ay, v - g(u) > \geq 0$$

for all $v \in K(u)$. i.e., $u \in H$, $y \in V(u)$, and $w \in T(u)$ are a solution of problem (2.1).

THEOREM 3.2. *Let K be a closed convex subset of H, $T: H \to 2^H$ be Φ-Lipschitz continuous and Ψ-strongly monotone, and $V: H \to 2^H$ be Γ-Lipschitz continuous, $g: H \to H$ be Lipschitz continuous and strongly monotone, and $A, m: H \to H$ be Lipschitz continuous. Suppose that there exists a constant $\rho > 0$ such that $\rho\xi\Gamma(t) < 1 - k$ and for all $t \in [0, \infty)$*

$$\frac{1}{\rho}\{1 - [1 - (k + \rho\xi\Gamma(t))]^2 + \rho^2\Phi^2(t)\} < 2\Psi(t) < \frac{1}{\rho} + \rho\Phi^2(t) \tag{3.1}$$

and

$$k = 2(\sqrt{1 - 2\delta + \sigma^2} + \mu) < 1,$$

where δ is a strong monotonicity constant of g and ξ, σ, μ are Lipschitz constants of A, g, and m, respectively. Then, (2.1) has a solution.

Proof. Define a mapping $F: H \to 2^H$ as

$$\begin{aligned} F(u) = \cup_{w \in T(u)} \cup_{y \in V(u)} [&u - g(u) + m(u) \\ &+ P_K(g(u) - \rho(w + Ay) - m(u))] \end{aligned}$$

for each $u \in H$. By Theorem 3.1, it suffices to prove that F has a fixed point in H. For any $u_1, u_2 \in H$, $w_1 \in T(u_1)$, $w_2 \in T(u_2)$, $y_1 \in V(u_1)$, and $y_2 \in V(u_2)$, by Lemma 2.2, we have

$$
\begin{aligned}
&\|(u_1 - g(u_1) + m(u_1) + P_K(g(u_1) - \rho(w_1 + Ay_1) - m(u_1))) \\
&\quad - (u_2 - g(u_2) + m(u_2) + P_K(g(u_2) - \rho(w_2 + Ay_2) - m(u_2)))\| \\
&\quad \leq \|u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2)\| \\
&\qquad + \|P_K(g(u_1) - \rho(w_1 + Ay_1) - m(u_1)) \\
&\qquad\quad - P_K(g(u_2) - \rho(w_2 + Ay_2) - m(u_2))\| \\
&\quad \leq 2\|u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2)\| \\
&\qquad + \|u_1 - u_2 - \rho(w_1 - w_2)\| + \rho\|Ay_1 - Ay_2\|. \qquad (3.2)
\end{aligned}
$$

Since T is Φ-Lipschitz continuous and Ψ-strongly monotone, it can be obtained that

$$
\begin{aligned}
&\|u_1 - u_2 - \rho(w_1 - w_2)\|^2 \\
&\quad = \|u_1 - u_2\|^2 - 2\rho < w_1 - w_2, u_1 - u_2 > +\rho^2\|w_1 - w_2\|^2 \\
&\quad \leq \|u_1 - u_2\|^2 - 2\rho\|u_1 - u_2\|^2\Psi(\|u_1 - u_2\|) + \rho^2\delta^2(T(u_1), T(u_2)) \\
&\quad \leq \|u_1 - u_2\|^2 - 2\rho\|u_1 - u_2\|^2\Psi(\|u_1 - u_2\|) \\
&\qquad + \rho^2\|u_1 - u_2\|^2\Phi^2(\|u_1 - u_2\|) \\
&\quad = [1 - 2\rho\Psi(\|u_1 - u_2\|) + \rho^2\Phi^2(\|u_1 - u_2\|)]\|u_1 - u_2\|^2. \qquad (3.3)
\end{aligned}
$$

By using the Lipschitz continuity of g and m, and the strong monotonicity of g, we easily see that

$$
\begin{aligned}
&\|u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2)\| \\
&\quad \leq \|u_1 - u_2 - (g(u_1) - g(u_2))\| + \|m(u_1) - m(u_2)\| \\
&\quad \leq (\sqrt{1 - 2\delta + \sigma^2} + \mu)\|u_1 - u_2\|. \qquad (3.4)
\end{aligned}
$$

Further, since A is Lipschitz continuous and V is Γ-Lipschitz continuous, we have

$$\begin{aligned} \|Ay_1 - Ay_2\| &\leq \xi\|y_1 - y_2\| \\ &\leq \xi\delta(V(u_1), V(u_2)) \\ &\leq \xi\|u_1 - u_2\|\Gamma(\|u_1 - u_2\|). \end{aligned} \tag{3.5}$$

From (3.2)-(3.5), it follows that

$$\begin{aligned} \delta(F(u_1), F(u_2)) \leq &\,[2(\sqrt{1 - 2\delta + \sigma^2} + \mu) \\ &+ (1 - 2\rho\Psi(\|u_1 - u_2\|) + \rho^2\Phi^2(\|u_1 - u_2\|))^{\frac{1}{2}} \\ &+ \rho\xi\Gamma(\|u_1 - u_2\|)]\|u_1 - u_2\| \\ \leq &\,\varphi(\|u_1 - u_2\|) \end{aligned}$$

for all $u_1, u_2 \in H$, where

$$\varphi(t) = t[k + (1 - 2\rho\Psi(t) + \rho^2\Phi^2(t))^{\frac{1}{2}} + \rho\xi\Gamma(t)]$$

and $k = 2(\sqrt{1 - 2\delta + \sigma^2} + \mu)$.

Clearly, each Hilbert space is a metrically convex metric space and by (3.1), $\varphi(t) < t$ for each $t \in [0, \infty)$. By Theorem 2.1, F has a fixed point u in H and hence (2.1) has a solution $u \in H$, $y \in V(u)$, and $w \in T(u)$.

THEOREM 3.3. *Let K be a closed convex subset of H, $T : H \to 2^H$ be Φ-Lipschitz continuous and Ψ-strongly monotone, and $V : H \to 2^H$ be Γ-Lipschitz continuous, $g : H \to H$ be Lipschitz continuous and strongly monotone,* *and $A, m, : H \to H$ be Lipschitz continuous. Suppose that there exists $\rho > 0$ and $h \in [0, 1)$ such that for all $t \in [0, \infty)$,*

$$0 < [1 - 2\rho\Psi(t) + \rho^2\Phi^2(t)]^{\frac{1}{2}} \leq h - k - \rho\xi\Gamma(t), \qquad (3.6)$$

$$\overline{\lim_{t \to 0+}} \Phi(t) \neq \infty, \quad \overline{\lim_{t \to 0+}} \Gamma(t) \neq \infty.$$

and

$$k = 2(\sqrt{1 - 2\delta + \sigma^2} + \mu) < h,$$

where δ is a strong monotonicity constant of g and ξ, σ, and μ are Lipschitz constants of A, g, and m, respectively. Then for any $u_0 \in H$, the iterative scheme defined by

$$\begin{aligned} u_{n+1} = (1 - \alpha_n)u_n + \alpha_n[u_n - g(u_n) + m(u_n) \\ + P_K(g(u_n) - \rho(w_n + Ay_n) - m(u_n))], \end{aligned} \qquad (3.7)$$

$$w_n \in T(u_n), \quad y_n \in V(u_n),$$

$$0 \leq \alpha_n \leq 1 \quad \text{for each} \quad n \leq 0, \quad \sum_{n=0}^{\infty} \alpha_n \quad \text{diverges},$$

satisfies that $\{u_n\}$ converges to u strongly in H, $\{w_n\}$ and $\{y_n\}$ converge to w and y strongly in H, respectively, and $u \in H$, $y \in V(u)$, and $w \in T(u)$ is a solution of the problem (2.1).

Proof. By the assumption (3.6), for each $t \in [0, \infty)$, we have

$$\frac{1}{\rho}\{1 - [1 - (k + \rho\xi\Gamma(t))]^2 + \rho^2\Phi^2(t)\} < 2\Psi(t) < \frac{1}{\rho} + \rho\Phi^2(t).$$

By Theorem 3.2, the problem (2.1) has a solution $u \in H$, $y \in V(u)$, $w \in T(u)$, and

$$u = u - g(u) + m(u) + P_K(g(u) - \rho(w + Ay) - m(u)).$$

Hence, by Lemma 2.2, we have

$$\begin{aligned}
\|u_{n+1} - u\| \\
&\le (1-\alpha_n)\|u_n - u\| \\
&\quad + \alpha_n\{\|u_n - u - (g(u_n) - g(u)) + m(u_n) - m(u)\| \\
&\quad + \|P_K(g(u_n) - \rho(w_n + Ay_n) - m(u_n)) \\
&\quad - P_K(g(u) - \rho(w + Ay) - m(u))\|\} \\
&\le (1-\alpha_n)\|u_n - u\| \\
&\quad + \alpha_n\{2\|u_n - u - (g(u_n) - g(u)) + m(u_n) - m(u)\| \\
&\quad + \|u_n - u - \rho(w_n - w)\| + \rho\|Ay_n - Ay\|\}.
\end{aligned} \tag{3.8}$$

Since T is Φ-Lipschitz continuous and Ψ-strongly monotone, it can be obtained that

$$\begin{aligned}
&\|u_n - u - \rho(w_n - w)\|^2 \\
&\quad \le (1 - 2\rho\Psi(\|u_n - u\|) + \rho^2\Phi^2(\|u_n - u\|))\|u_n - u\|^2.
\end{aligned} \tag{3.9}$$

By using the Lipschitz continuity of g and m, and the strongly monotonicity of g, we easily see that

$$\begin{aligned}
&\|u_n - u - (g(u_n) - g(u)) + m(u_n) - m(u)\| \\
&\quad \le (\sqrt{1 - 2\delta + \sigma^2} + \mu)\|u_n - u\|.
\end{aligned} \tag{3.10}$$

Further, since A is Lipschitz continuous and V is Γ-Lipschitz continuous, we have

$$\|Ay_n - Ay\| \leq \xi\Gamma(\|u_n - u\|)\|u_n - u\|. \tag{3.11}$$

It follows from (3.8)-(3.11) that

$$\begin{aligned}
&\|u_{n+1} - u\| \\
&\quad \leq (1-\alpha_n)\|u_n - u\| + \alpha_n\{2(\sqrt{1-2\delta+\sigma^2} + \mu) \\
&\qquad + [1 - 2\rho\Psi(\|u_n - u\|) + \rho^2\Phi^2(\|u_n - u\|)]^{\frac{1}{2}} \\
&\qquad + \rho\xi\Gamma(\|u_n - u\|)\}\|u_n - u\| \\
&\quad \leq (1-\alpha_n)\|u_n - u\| + \alpha_n h\|u_n - u\| \\
&\quad = (1-(1-h)\alpha_n)\|u_n - u\| \\
&\quad \leq \Pi_{j=0}^{n}(1-(1-h)\alpha_j)\|u_0 - u\|.
\end{aligned}$$

Since $\sum_{j=0}^{\infty}\alpha_j$ diverges and $1 - h > 0$,

$$\Pi_{j=0}^{\infty}(1-(1-h)\alpha_j) = 0,$$

and hence $\{u_n\}$ converges u strongly. Since $w_n \in T(u_n)$, $w \in T(u)$, and T is Φ-Lipschitz continuous, we have

$$\begin{aligned}
\|w_n - w\| &\leq \delta(T(u_n), T(u)) \\
&\leq \Phi(\|u_n - u\|)\|u_n - u\|
\end{aligned}$$

and hence $\{w_n\}$ converges to w strongly. Similarly, we can prove $\{v_n\}$ converges to v strongly. This completes the proof.

REMARK. For a suitable choice of the operators T, V, A, g, and m, we obtain several known results[8,9,11] as special cases of Theorem 3.3.

References

1. C. Baiocchi and A. Capelo, *Variational and quasi-variational inequalities*, John Wiley and Sons, New York (1984).
2. A. Bensoussan and J. L. Lions, *Applications des inequations variationelles en control et en stochastiques*, Dunod, Paris (1978).
3. D. W. Boyd and J. S. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458-464.
4. S. S. Chang and N. J. Huang, *Generalized strongly nonlinear quasi-complementarity problems in Hilbert spaces*, J. Math. Anal. Appl. **158** (1991), 194-202.
5. J. Crank, *Free and moving boundary problems*, Clarendon Press, Oxford (1984).
6. D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York (1980).
7. M. A. Noor, *An iterative scheme for a class of quasi-variational inequalities*, J. Math. Anal. Appl. **110** (1985), 463-468.
8. M. A. Noor, *On the nonlinear complementarity problem*, J. Math. Anal. Appl. **123** (1987), 455-460.
9. M. A. Noor, *Iterative methods for a class of complementarity problems*, J. Math. Anal. Appl. **133** (1988), 366-382.
10. M. A. Noor, *Multivalued strongly nonlinear quasi-variational inequalities*, Chinese J. Math. **23** (1995), 275-286.
11. A. H. Siddiqi and Q. H. Ansari, *An iterative method for generalized variational inequalities*, Math. Japonica **34(3)** (1989), 475-481.
12. G. Stampacchia, *Formes bilinearies coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris **258** (1964), 4413-4416.

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