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ON WEAK CONVERGENCE TO FIXED POINTS OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we prove the following weak convergence theorem: Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let T be a nonexpansive mapping from C into itself with a fixed point. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, or $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$. Then $\{x_n\}$ converges weakly to a fixed point of T. This is a generalization of the results of Tan and Xu, and Takahashi and Kim.

1. INTRODUCTION

Let E be a real Banach space and let C be a nonempty closed convex subset of E. Then a mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For a mapping T from C into itself, we denote by $F(T)$ the set of fixed points of T. Now, we consider the following iteration scheme: $x_1 \in C$ and

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n \quad \text{for all } n \geq 1, \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Such an iteration scheme was introduced by Ishikawa [3]; see also Mann [4]. Recently Tan and Xu [8] proved the following interesting result (Corollary 1): Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable and let T be a nonexpansive mapping from C into itself with a fixed point. Then for any initial data x_1 in C, the iterates $\{x_n\}$ defined by (1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) < \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, converge weakly to a fixed point of T. On the other hand, Takahashi and Kim [7] proved the following (Corollary 2): Let C, E and T be as above and suppose $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$, or $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some a, b with

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$0 < a \le b < 1$. Then for any initial data x_1 in C, the iterates $\{x_n\}$ defined by (1) converge weakly to a fixed point of T. Note that Tan and Xu's result is applicable to the case of $\alpha_n = 1 - 1/n$ and $\beta_n = 1/n$ for all $n \ge 1$, while Takahashi and Kim's result is applicable to the case of $\alpha_n = \beta_n = 1/2$ for all $n \ge 1$.

In this paper, motivated by these two results, we prove the following weak convergence theorem: Let C, E and T be as above and suppose $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, or $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$. Then for any initial data x_1 in C, the iterates $\{x_n\}$ defined by (1) converge weakly to a fixed point of T. Compare this with Tan and Xu's result [8] and Takahashi and Kim's result [7].

2. Preliminaries

Let E be a Banach space. For each ε with $0 \le \varepsilon \le 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}.$$

Note that δ is nondecreasing and

$$\|\lambda x + (1-\lambda)y\| \le \max\{\|x\|, \|y\|\}\left[1 - 2\lambda(1-\lambda)\cdot\delta\left(\frac{\|x-y\|}{\max\{\|x\|,\|y\|\}}\right)\right]$$

for every $x, y \in E\setminus\{0\}$ and $\lambda \in [0,1]$; see [2]. E is called uniformly convex if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. The norm of E is called Fréchet differentiable if for each $x \in E$ with $\|x\| = 1$, $\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists and is attained uniformly in $y \in E$ with $\|y\| = 1$; see [2]. E is said to satisfy Opial's condition [5] if for any sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to $z \in E$, $\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \ne z$. All Hilbert spaces and $\ell^p (1 < p < \infty)$ satisfy Opial's condition, while L^p with $1 < p < \infty$ and $p \ne 2$ do not. The following lemma was proved by Reich [6]; see also [7].

Lemma 1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is Fréchet differentiable and let $\{T_1, T_2, T_3, \cdots\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and $S_n = T_nT_{n-1}\cdots T_1$ for all $n \ge 1$. Then the set $\left(\bigcap_{n=1}^{\infty} \overline{co}\{S_m x : m \ge n\}\right) \cap \left(\bigcap_{n=1}^{\infty} F(T_n)\right)$ consists of at most one point, where $\overline{co}\{S_m x : m \ge n\}$ is the closure of the convex hull of $\{S_m x : m \ge n\}$.*

3. Weak convergence theorem

In this section, we prove the following theorem which generalizes the results of Tan and Xu [8] and Takahashi and Kim [7].

Theorem. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let T be a nonexpansive mapping from C into itself with a fixed point. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, or $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Before proving it, we need some definitions and lemmas. We denote by $\mathbb{N}$ the set of positive integers. Let I be an infinite subset of $\mathbb{N}$. If $\{\lambda_n\}$ is a sequence of nonnegative numbers, then we denote by $\{\lambda_i : i \in I\}$ the subsequence of $\{\lambda_n\}$.

Lemma 2. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of nonnegative numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n\mu_n < \infty$. Then for $\varepsilon > 0$, there exists an infinite subset I of $\mathbb{N}$ such that $\sum\{\lambda_j : j \in \mathbb{N} \setminus I\} \leq \varepsilon$ and the subsequence $\{\mu_i : i \in I\}$ of $\{\mu_n\}$ converges to 0.*

Proof. For each $\varepsilon > 0$, first take $p_0 \in \mathbb{N}$ with $\sum_{n=p_0+1}^{\infty} \lambda_n\mu_n \leq \varepsilon/2$. From $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} \lambda_n\mu_n < \infty$, we have $\liminf_{n\to\infty} \mu_n = 0$. So, there exists $p_1 \in \mathbb{N}$ such that $p_1 > p_0$, $\mu_{p_1} < 1$ and

$$\sum\{\lambda_j\mu_j : j > p_1\} \leq \frac{\varepsilon}{2 \cdot 2^2}.$$

Similarly we can take $p_2, p_3, \cdots \in \mathbb{N}$ such that $p_k > p_{k-1}$, $\mu_{p_k} < 1/k$ and

$$\sum\{\lambda_j\mu_j : j > p_k\} \leq \frac{\varepsilon}{(k+1)\cdot 2^{k+1}}$$

for all $k = 2, 3, \cdots$. Define

$$I = \{1, 2, \cdots, p_0\} \bigcup \left(\bigcup_{k=1}^{\infty} \left\{ n : p_{k-1} < n \leq p_k, \mu_n < \frac{1}{k} \right\} \right).$$

Then, $\{\mu_i : i \in I\}$ is a subsequence of $\{\mu_n\}$ such that $\mu_i \to 0$. We also have

$$\sum\{\lambda_j : j \in \mathbb{N} \setminus I\} = \sum_{k=1}^{\infty} \sum \left\{ \lambda_n : p_{k-1} < n \leq p_k, \mu_n \geq \frac{1}{k} \right\}.$$

Putting $S_k = \{n : p_{k-1} < n \leq p_k, \mu_n \geq 1/k\}$, we have

$$\begin{aligned} \frac{1}{k}\sum\{\lambda_n : n \in S_k\} &\leq \sum\{\lambda_n\mu_n : n \in S_k\} \leq \sum\{\lambda_j\mu_j : j > p_{k-1}\} \\ &\leq \frac{\varepsilon}{k \cdot 2^k} \end{aligned}$$

and hence

$$\sum\{\lambda_j : j \in \mathbb{N} \setminus I\} \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

This completes the proof. □

Lemma 3. *Let $\{\lambda_n\}$ and $\{\mu_n\}$ be sequences of nonnegative numbers such that $\lambda_{n+1} \leq \lambda_n + \mu_n$ for all $n \in \mathbb{N}$. Suppose there exists a subsequence $\{\mu_i : i \in I\}$ of $\{\mu_n\}$ such that $\mu_i \to 0$, $\lambda_i \to \alpha$ and $\sum\{\mu_j : j \in \mathbb{N} \setminus I\} < \infty$. Then $\lambda_n \to \alpha$.*

Proof. Fix $\varepsilon > 0$ and take $n_0 \in I$ such that $|\lambda_i - \alpha| \leq \varepsilon$ and $\mu_i \leq \varepsilon$ for all $i \geq n_0$ and $\sum\{\mu_j : j > n_0, j \in \mathbb{N} \setminus I\} \leq \varepsilon$. For $n \in \mathbb{N} \setminus I$ with $n > n_0$, putting $k = \max\{i \in I : i < n\}$ and $\ell = \min\{i \in I : i > n\}$, we have

$$\lambda_n \leq \lambda_{n-1} + \mu_{n-1} \leq \cdots \leq \lambda_k + \sum_{j=k}^{n-1} \mu_j \leq \lambda_k + \mu_k + \varepsilon \leq \alpha + 3\varepsilon$$

and

$$\lambda_n \geq \lambda_{n+1} - \mu_n \geq \cdots \geq \lambda_\ell - \sum_{j=n}^{\ell-1} \mu_j \geq \lambda_\ell - \varepsilon \geq \alpha - 2\varepsilon > \alpha - 3\varepsilon.$$

So, we obtain the desired result. □

Lemma 4. *Let C be a closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping from C into itself with a fixed point. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$, where $\alpha_n, \beta_n \in [0,1]$. Then the following hold:*

(i) *If $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$;*

(ii) *if $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. We may assume that there exists $b \in (0,1)$ such that $\beta_n \leq b$ for all $n \in \mathbb{N}$. Fix $w \in F(T)$ and put $y_n = \beta_n T x_n + (1-\beta_n)x_n$ for all $n \in \mathbb{N}$. Then by the definition of $\{x_n\}$, we have

$$\begin{aligned}
\|x_{n+1} - w\| &= \|\alpha_n T y_n + (1-\alpha_n)x_n - w\| \\
&\leq \alpha_n\|Ty_n - w\| + (1-\alpha_n)\|x_n - w\| \\
&\leq \alpha_n\|y_n - w\| + (1-\alpha_n)\|x_n - w\| \\
&= \alpha_n\|\beta_n T x_n + (1-\beta_n)x_n - w\| + (1-\alpha_n)\|x_n - w\| \\
&\leq \alpha_n(\beta_n\|Tx_n - w\| + (1-\beta_n)\|x_n - w\|) + (1-\alpha_n)\|x_n - w\| \\
&\leq \|x_n - w\|
\end{aligned}$$

and hence the limit of $\{\|x_n - w\|\}$ exists. Put $c = \lim_{n\to\infty} \|x_n - w\|$. If $c = 0$, then (i) and (ii) hold. So, we assume that $c > 0$. We first prove (i). From $\|Ty_n - w\| \leq \|x_n - w\|$

for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\|x_{n+1} - w\| &= \|\alpha_n(Ty_n - w) + (1-\alpha_n)(x_n - w)\| \\ &\le \|x_n - w\| \left[1 - 2\alpha_n(1-\alpha_n)\cdot\delta\left(\frac{\|Ty_n - x_n\|}{\|x_n - w\|}\right)\right].\end{aligned}$$

Since

$$\begin{aligned}&\|x_n - w\| - \|x_{n+1} - w\| \\ &\quad \ge 2\|x_n - w\|\cdot\alpha_n(1-\alpha_n)\cdot\delta\left(\frac{\|Ty_n - x_n\|}{\|x_n - w\|}\right) \\ &\quad \ge 2c\cdot\alpha_n(1-\alpha_n)\cdot\delta\left(\frac{\|Ty_n - x_n\|}{\|x_n - w\|}\right)\end{aligned}$$

for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)\cdot\delta\left(\frac{\|Ty_n - x_n\|}{\|x_n - w\|}\right) < \infty.$$

By Lemma 2, there exists an infinite subset I_1 of $\mathbb{N}$ such that

$$\sum\{\alpha_j(1-\alpha_j) : j \in \mathbb{N}\setminus I_1\} < \infty \tag{2}$$

and $\left\{\delta\left(\frac{\|Ty_i - x_i\|}{\|x_i - w\|}\right) : i \in I_1\right\}$ converges to 0. Since $c = \lim_{n\to\infty}\|x_n - w\| > 0$, we obtain $\{\|Ty_i - x_i\| : i \in I_1\}$ converges to 0. From

$$\begin{aligned}\|Tx_i - x_i\| &\le \|Tx_i - Ty_i\| + \|Ty_i - x_i\| \\ &\le \|x_i - y_i\| + \|Ty_i - x_i\| \\ &= \beta_i\|Tx_i - x_i\| + \|Ty_i - x_i\| \\ &\le b\|Tx_i - x_i\| + \|Ty_i - x_i\|,\end{aligned}$$

we obtain

$$\limsup_{i\to\infty}\|Tx_i - x_i\| \le \limsup_{i\to\infty}\frac{1}{(1-b)}\|Ty_i - x_i\| = 0.$$

Hence we have

$$\lim_{i\to\infty}\|Tx_i - x_i\| = 0. \tag{3}$$

Since

$$
\begin{aligned}
&\|Tx_{n+1} - x_{n+1}\| \\
&\quad \le \|Tx_{n+1} - T(\alpha_n Tx_n + (1-\alpha_n)x_n)\| + \|T(\alpha_n Tx_n + (1-\alpha_n)x_n) - Tx_n\| \\
&\qquad + \|Tx_n - (\alpha_n Tx_n + (1-\alpha_n)x_n)\| + \|\alpha_n Tx_n + (1-\alpha_n)x_n - x_{n+1}\| \\
&\quad \le 2\|\alpha_n Tx_n + (1-\alpha_n)x_n - x_{n+1}\| + \|\alpha_n Tx_n + (1-\alpha_n)x_n - x_n\| \\
&\qquad + (1-\alpha_n)\|Tx_n - x_n\| \\
&\quad = 2\alpha_n\|Tx_n - Ty_n\| + \|Tx_n - x_n\| \\
&\quad \le 2\alpha_n\|x_n - y_n\| + \|Tx_n - x_n\| \\
&\quad = (1 + 2\alpha_n\beta_n)\|Tx_n - x_n\|
\end{aligned}
$$

and

$$
\begin{aligned}
&\|Tx_{n+1} - x_{n+1}\| \\
&\quad \le \|Tx_{n+1} - T(\alpha_n Ty_n + (1-\alpha_n)y_n)\| + \|T(\alpha_n Ty_n + (1-\alpha_n)y_n) - Ty_n\| \\
&\qquad + \|Ty_n - (\alpha_n Ty_n + (1-\alpha_n)y_n)\| + \|\alpha_n Ty_n + (1-\alpha_n)y_n - x_{n+1}\| \\
&\quad \le 2\|\alpha_n Ty_n + (1-\alpha_n)y_n - x_{n+1}\| + \|\alpha_n Ty_n + (1-\alpha_n)y_n - y_n\| \\
&\qquad + (1-\alpha_n)\|Ty_n - y_n\| \\
&\quad = 2(1-\alpha_n)\|x_n - y_n\| + \|Ty_n - y_n\| \\
&\quad \le 2(1-\alpha_n)\|x_n - y_n\| + \|Ty_n - Tx_n\| + \|Tx_n - y_n\| \\
&\quad \le 2(1-\alpha_n)\|x_n - y_n\| + \|y_n - x_n\| + \|Tx_n - y_n\| \\
&\quad = (1 + 2(1-\alpha_n)\beta_n)\|Tx_n - x_n\|
\end{aligned}
$$

for all $n \in \mathbb{N}$, we obtain

$$
\|Tx_{n+1} - x_{n+1}\| \le (1 + 4\alpha_n(1-\alpha_n)\beta_n)\|Tx_n - x_n\|. \tag{4}
$$

Since $\{\|Tx_n - x_n\|\}$ is bounded, from Lemma 3, (2), (3) and (4), we obtain $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We next prove (ii). From $\|Tx_n - w\| \le \|x_n - w\|$ for all $n \in \mathbb{N}$, we obtain

$$
\begin{aligned}
\|x_{n+1} - w\| &\le \alpha_n\|y_n - w\| + (1-\alpha_n)\|x_n - w\| \\
&= \alpha_n\|\beta_n(Tx_n - w) + (1-\beta_n)(x_n - w)\| + (1-\alpha_n)\|x_n - w\| \\
&\le \alpha_n\|x_n - w\| \left[1 - 2\beta_n(1-\beta_n)\cdot\delta\left(\frac{\|Tx_n - x_n\|}{\|x_n - w\|}\right)\right] \\
&\quad + (1-\alpha_n)\|x_n - w\|.
\end{aligned}
$$

From

$$\begin{aligned}
&\|x_n - w\| - \|x_{n+1} - w\| \\
&\quad \geq 2\|x_n - w\| \cdot \alpha_n\beta_n(1-\beta_n) \cdot \delta\left(\frac{\|Tx_n - x_n\|}{\|x_n - w\|}\right) \\
&\quad \geq 2c \cdot \alpha_n\beta_n(1-b) \cdot \delta\left(\frac{\|Tx_n - x_n\|}{\|x_n - w\|}\right)
\end{aligned}$$

for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \alpha_n\beta_n \cdot \delta\left(\frac{\|Tx_n - x_n\|}{\|x_n - w\|}\right) < \infty.$$

By Lemma 2, there exists an infinite subset I_2 of $\mathbb{N}$ such that

$$(5) \qquad \sum\{\alpha_j\beta_j : j \in \mathbb{N} \setminus I_2\} < \infty$$

and $\left\{\delta\left(\frac{\|Tx_i - x_i\|}{\|x_i - w\|}\right) : i \in I_2\right\}$ converges to 0. Since $c = \lim_{n\to\infty} \|x_n - w\| > 0$, we obtain

$$(6) \qquad \lim_{i\to\infty} \|Tx_i - x_i\| = 0.$$

Since $\{\|Tx_n - x_n\|\}$ is bounded, from Lemma 3, (4), (5) and (6), we obtain $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. □

Proof of Theorem. Note that by Lemma 4 and Browder [1], a weak subsequential limit of the sequence $\{x_n\}$ is a fixed point of T. Since E is reflexive and $\{x_n\}$ is bounded, to complete the proof, we prove that $\{x_n\}$ has at most one weak subsequential limit. In the case that E satisfies Opial's condition, we assume that z_1 and z_2 are two distinct weak sequential limit of the subsequence $\{x_i : i \in I\}$ and $\{x_j : j \in J\}$ of $\{x_n\}$ respectively. We obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{i\to\infty} \|x_i - z_1\| < \lim_{i\to\infty} \|x_i - z_2\| = \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{j\to\infty} \|x_j - z_2\| < \lim_{j\to\infty} \|x_j - z_1\| = \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This is a contradiction. In the case that the norm of E is Fréchet differentiable, for each $n \in \mathbb{N}$, we define a nonexpansive mapping T_n from C into itself by

$$T_n(x) = \alpha_n T[\beta_n Tx + (1-\beta_n)x] + (1-\alpha_n)x.$$

Then $\{x_n\}$ can be written as $x_{n+1} = T_nT_{n-1}\cdots T_1x_1$ and $F(T) \subset F(T_n)$ for all $n \in \mathbb{N}$. Let z be a subsequential limit of $\{x_n\}$ and put $S_n = T_nT_{n-1}\cdots T_1$ for all $n \in \mathbb{N}$. Then $z \in \left(\bigcap_{n=1}^{\infty} \overline{co}\{S_mx : m \geq n\}\right) \bigcap \left(\bigcap_{n=1}^{\infty} F(T_n)\right)$. So, by Lemma 1, $\{x_n\}$ has at most one weak subsequential limit. This completes the proof. □

As direct consequences of Theorem, we obtain the following corollaries.

Corollary 1 (Tan and Xu [8]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let T be a nonexpansive mapping from C into itself with a fixed point. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$, where $\alpha_n, \beta_n \in [0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) < \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, Then $\{x_n\}$ converges weakly to a fixed point of T.*

Corollary 2 (Takahashi and Kim [7]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let T be a nonexpansive mapping from C into itself with a fixed point. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n) x_n] + (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$, where $\alpha_n, \beta_n \in [0,1]$ such that $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$ or $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some a, b with $0 < a \le b < 1$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. It is obvious that $\limsup_{n\to\infty} \beta_n \le b < 1$. In the case of $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$, we obtain $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) \ge \sum_{n=1}^{\infty} a(1-b) = \infty$. In the case of $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ we obtain $\sum_{n=1}^{\infty} \alpha_n\beta_n \ge \sum_{n=1}^{\infty} a^2 = \infty$. This completes the proof. □

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