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Lower Estimates of Dimensions for Quasi Periodic Orbits

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1. Introduction

Let X be a Banach space with its norm denoted by $\|\cdot\|$ and consider a X-valued quasi-periodic function:

$$f(t)=g(w_1t, w_2t, \cdots, w_nt, t)$$

where g has period T in each of its arguments separately and the frequencies are not rationally related. In our previous papers [5], [7],[6] assuming that g is Hölder continuous with exponent $\delta_1 \in (0,1]$ and, using Diophantine (simultaneous) approximation, we have shown that the fractal dimension of its orbit is majorized by the value $(n+1)/\delta_1$.

The Hausdorff and fractal dimensions of orbits or attractors in nonlinear dynamical systems have been investigated by several authors to specify chaotic or strange properties or to estimate complexity of systems (cf. [3],[4], [8], [10]). While there have been various arguments on chaotic behavior, we can note that quasi-periodic states occupy some important positions as gateways in routes to chaos. In the present paper, using the simultaneous Diophantine approximation for the frequency parameters $w_1, w_2, \ldots$ with "badly approximable" property, we can estimate the lower bounds of the fractal dimensions of quasi-periodic orbits. Having shown that the lower bound of its fractal dimension is given by the value n/δ_2 where δ_2 is an exponent in an inverse Hölder's inequality, we can propose that the two parameters δ, n are essential for chaotic behavior or complexity of system.

As remarkable examples of the quasi-periodic functions, which satisfy our conditions, we investigate a Weierstrass-type (abr. W-type) function given by

$$h(t)=\sum_{k=1}^{\infty}(\lambda^k)^{-\delta}e^{i2\pi\lambda^k t}\varphi_k$$

for some constants $\lambda>1,\ 0<\delta<1$ and an orthonormal system $\{\varphi_k\}$ in a Hilbert space. The real or complex valued Weierstrass functions were studied and its fractal dimensions of its graph were calculated in the 2-dimensional space (cf. [3]). Here, in the setting of an infinite dimensional space, we obtain ranges for the dimension of the orbit Σ of $h(t)$ according to the algebraic properties of the parameter λ as follows:

(i) If $\lambda = (p)^{1/n}$ for a prime number p and $n \geq 2$, $\mathcal{D}_F(\Sigma) = n/\delta$.

(ii) If $\lambda = (q/p)^{1/n}$ for positive integers $p, q : q > p$ and $n \geq 1$,

$$n \leq \mathcal{D}_F(\Sigma) \leq \frac{n}{\delta}(1 + \frac{\log p}{\log q - \log p}).$$

(iii) If λ is a transcendental real number, $\mathcal{D}_F(\Sigma) = \infty$.

Since orthonormal systems are given by eigenfunctions of a differential operator in various P.D.E. examples, we investigate an abstract differential equation on a Hilbert space with its perturbation term given by a W-type function. Under a condition for harmonization between the frequency parameters and the eigenvalues of the differential operator, we estimate the dimension of its quasi-periodic attractor. Furthermore, in view of the case (iii) above, we can conclude that an arbitrarily small change of the parameter λ in the W-type function converts any finite dimensional quasi-periodic attractor to the one which is chaotic ($\mathcal{D}_F(\Sigma) = \infty$).

The plan of this paper is as follows: We show in section 2 that the dimension of the orbit for the quasi-periodic function g is greater than n/δ_2 when the irrational numbers $w_1, w_2, ...$ satisfy badly approximable conditions. In section 3 we apply this estimate to the W-type functions which take the values in a separable Hilbert space. In section 4 we investigate an abstract differential equation with its perturbation term given by a W-type function and give a condition for harmonization.

2. Fractal dimensions of quasi-periodic orbits

The purpose of this section is to estimate the upper and the lower bound of the fractal dimension for the orbit of a quasi-periodic function.

Let $N_\varepsilon(A)$, $\varepsilon > 0$, denote the minimum number of balls of X radius ε which is necessary to cover a subset A of X. The fractal dimension of A, which is also called the box dimension of A (cf. [3] or [10]), is the number

$$\mathcal{D}_F(A) = \lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(A)}{\log 1/\varepsilon}. \tag{2.1}$$

To estimate the lower bound of the dimension we need the following alternative expression given by

$$\mathcal{D}'_F(A) = \lim_{\varepsilon \to 0} \frac{\log L_\varepsilon(A)}{\log 1/\varepsilon} \tag{2.2}$$

where $L_\varepsilon(A)$ is the maximum number of mutually disjoint balls of X with radius ε and centers in A. If X is finite dimensional, it is known that $\mathcal{D}_F(A) = \mathcal{D}'_F(A)$ (see chapter 3 of [3]). In the infinite dimensional case we can show the following lemma.

Lemma 1. *Let A be a subset of X and assume that there exist constants $K, \vartheta > 0$:*

$$L_\varepsilon(A) \geq K\varepsilon^{-\vartheta} \quad [\textit{resp.} \quad L_\varepsilon(A) \leq K\varepsilon^{-\vartheta}]. \tag{2.3}$$

Then we have

$$\mathcal{D}_F(A) \geq \vartheta \quad [\textit{resp.} \quad \mathcal{D}_F(A) \leq \vartheta].$$

Proof. In view of the definitions (2.1) and (2.2), consider mutually disjoint balls with radius δ: $B_\delta(x_i), i = 1, ..., L_\delta(A) : x_i \in A$ and open balls with radius $\delta/2$: $U_j, j = 1, ..., N_{\delta/2}(A)$, which cover A. Then we can choose an open ball U_{j_i} for each center $x_i \in B_\delta(x_i)$, which satisfies

$$x_i \in U_{j_i} \subset B_\delta(x_i), \qquad U_{j_i} \cap U_{j_{i'}} = \varphi \ \text{ if } \ i \neq i'.$$

It follows that

$$L_\delta(A) \leq N_{\delta/2}(A).$$

Thus we can estimate

$$\begin{aligned} \mathcal{D}_F(A) &= \lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(A)}{-\log \varepsilon} \\ &\geq \lim_{\varepsilon \to 0} \frac{\log L_{2\varepsilon}(A)}{-\log \varepsilon} \\ &\geq \lim_{\varepsilon \to 0} \frac{\log K(2\varepsilon)^{-\vartheta}}{-\log \varepsilon} = \vartheta. \end{aligned}$$

Next, consider again the disjoint balls $B_\delta(x_i), i = 1, ..., L_\delta(A) : x_i \in A$. Then, since

$$d(x, \bigcup_{i=1}^{L_\delta} B_\delta(x_i)) < \delta$$

for every $x \in A$, we have

$$A \subset \bigcup_{i=1}^{L_\delta} B_{2\delta}(x_i).$$

It follows that

$$N_{2\delta}(A) \leq L_\delta(A).$$

Thus, applying the similar estimate as above, we can obtain the converse inequality. □

Remark 1. It is obvious that Lemma 1 also holds by substituting $L_\varepsilon(A), D_F(A)$ by $N_\varepsilon(A), D'_F(A)$, respectively. Furthermore, if there exist constants $K_1, K_2 > 0$ such that

$$K_1\varepsilon^{-\vartheta} \leq L_\varepsilon(A) \leq K_2\varepsilon^{-\vartheta},$$

then, following the argument of the proof, we can easily show that

$$\mathcal{D}_F(A) = \mathcal{D}'_F(A).$$

Consider a function $g(\cdot, \cdots, \cdot) : R^{n+1} \to X$, which satisfies the following conditions.

(G1) The function g has period T in each of its arguments separately;

$$\begin{aligned} g(t_1+T, t_2, \cdots, t_{n+1}) &= g(t_1, t_2+T, t_3, \cdots) \\ = \cdots &= g(t_1, t_2, \cdots, t_{n+1}+T) = g(t_1, \cdots, t_{n+1}). \end{aligned}$$

(G2) g is Hölder continuous; there exist constants $c_1 > 0$, $0 < \delta_i \leq 1$, $i = 1, \cdots, n+1$ and a small constant $\varepsilon_0 > 0$ such that

$$\|g(t_1, t_2, \cdots, t_{n+1}) - g(t'_1, t'_2, \cdots, t'_{n+1})\| \leq c_1 \sum_{i=1}^{n+1} |t_i - t'_i|^{\delta_i} \tag{2.4}$$

for $|t_i - t'_i| < \varepsilon_0$, $i = 1, \cdots, n+1$.

Consider an n-tuple of irrational numbers $w_1, w_2, \cdots, w_n$, which are rationally independent. Then the simultaneous approximation for these irrational numbers gives the sequences $l_i, r_{k,i} \in \mathbf{N}, i = 1, 2, \cdots$, and $k = 1, \cdots, n$, which satisfy

$$|l_i w_k - r_{k,i}| < \frac{1}{l_i^{1/n}}, \quad k = 1, \cdots, n. \tag{2.5}$$

(See [9].) We need the assumptions on the growth rate of the denominators l_i.

(D1) There exist positive constants $K_1, K_2 : K_2 > K_1 > 1$ such that

$$K_1 l_{j-1} < l_j < K_2 l_{j-1} \quad \text{for } j = 1, 2, \cdots . \tag{2.6}$$

Here we introduce the definition of the almost periodicity and our previous result.

A function $f : R \to X$ is called almost periodic if for each $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that for every $a \in R$ there exists an element $\alpha \in [a, a + l_\varepsilon]$ with the property

$$\|f(t+\alpha) - f(t)\| \leq \varepsilon \quad \text{for all } t \in \mathcal{R}. \tag{2.7}$$

Here the point α is called an ε-almost period and l_ε is called an inclusion length for ε-almost period.

Lemma 2. ([5]) *Let $f : R \to X$ be an almost periodic function, which satisfies a Hölder condition: there exists a constant $\delta : 0 < \delta \leq 1$ such that*

$$\sup_{t,s \in \mathcal{R}, t \neq s} \frac{|f(t) - f(s)|}{|t-s|^\delta} := c_0 < \infty. \tag{2.8}$$

If the inclusion length for ε-almost period of the function $f(t)$ satisfies the following estimate

$$l_\varepsilon \leq K\varepsilon^{-\vartheta} \tag{2.9}$$

for some $K > 0$ and $\vartheta > 0$, then the fractal dimension of its orbit $\Sigma := \cup_{t\in\mathcal{R}} f(t)$ satisfies

$$\mathcal{D}_F(\Sigma) \leq \vartheta + \frac{1}{\delta}. \tag{2.10}$$

Define a quasi-periodic function $f : R \to X$ by

$$f(t) = g(w_1 t, \cdots, w_n t, t)$$

and denote $\Sigma = \cup_{t\in R} f(t)$. Let $\gamma_1 = \min\{\delta_1, \cdots, \delta_{n+1}\}$, γ_2 be the secondary minimum, and put $\gamma_3 = \max\{\delta_1, \cdots, \delta_{n+1}\}$. Then, we can estimate the upper bound of the dimension by slightly modifying the results in [6].

Lemma 3. *Assume* **(G1)**, **(G2)** *and* **(D1)**, *then we have*

$$\mathcal{D}_F(\Sigma) \leq \frac{1}{\gamma_1} + \frac{n}{\gamma_2}. \tag{2.11}$$

In the present paper we consider the estimate of the dimension from below. Assume that the function g satisfies the following condition.
(G3) There exist constants $c_2 > 0$ and $\mu_i : 0 < \mu_i \leq 1$, $i = 1, \cdots, n+1$, such that

$$\|g(t_1, t_2, \cdots, t_{n+1}) - g(t'_1, t'_2, \cdots, t'_{n+1})\| \geq c_2 \sum_{i=1}^{n+1} |t_i - t'_i|^{\mu_i} \tag{2.12}$$

for $|t_i - t'_i| < \frac{T}{2}$, $i = 1, \cdots, n+1$.
Here we assume that the n-tuple of irrational numbers $\{w_1, w_2, \cdots, w_n\}$ are badly approximable (cf. [9]):
(D2) There exists a constant $k(n) > 0$, which depends on only the n-tuple and satisfies the following inequality

$$\max_{1\leq k\leq n} |lw_k - r| > k(n)\left(\frac{1}{l}\right)^{\frac{1}{n}} \tag{2.13}$$

for every positive integers l, r.

In case $n = 1$, that is, when an irrational real number α is badly approximable, the partial quotients in the continued fraction expansion are bounded (see [9]). In the simultaneous approximation case, we can show that the condition **(D2)** yields **(D1)**. (See [6] or [7] for the proof.)

Lemma 4. *The condition* **(D2)** *yields the condition* **(D1)**.

Remark 2. If $\{w_1, w_2, \cdots, w_n\}$ are any numbers in a real algebraic number field of degree $n+1$ such that $\{1, w_1, w_2, \cdots, w_n\}$ are linearly independent over the rationals, then $\{w_1, w_2, \cdots, w_n\}$ are badly approximable (see Theorem III, p.79 in [1]).

Let $\nu_1 = \max\{\mu_1, \cdots, \mu_{n+1}\}$, and $\nu_2 = \max\{\{\mu_1, \cdots, \mu_{n+1}\} - \{\nu_1\}\}$ and $\nu_3 = \min\{\mu_1, \cdots, \mu_{n+1}\}$. Then we can show the lower estimate.

Lemma 5. *Assume* **(G1)**, **(G3)** *and* **(D2)**. *Then the fractal dimension of the orbit* Σ *of* $f(t)$ *satisfies*

$$\mathcal{D}_F(\Sigma) \geq \max\{\frac{n}{\nu_1} + \frac{1}{\nu_3}, \frac{1}{\nu_1} + \frac{n}{\nu_2}\}. \tag{2.14}$$

Now we obtain the upper and lower estimate of the dimension by Lemma 3, 4 and 5.

Theorem 1. *Assume* **(G1)-(G3)** *and* **(D2)**. *Then*

$$\max\{\frac{n}{\nu_1} + \frac{1}{\nu_3}, \frac{1}{\nu_1} + \frac{n}{\nu_2}\} \leq \mathcal{D}_F(\Sigma) \leq \frac{1}{\gamma_1} + \frac{n}{\gamma_2}. \tag{2.15}$$

Consequently, if $\delta := \delta_1 = \cdots = \delta_n = \mu_1 = \cdots = \mu_n$,

$$\mathcal{D}_F(\Sigma) = \frac{n+1}{\delta}.$$

We give the proof of Lemma 5 by using the definition (2.2).

Proof of Lemma 5. Let $\varepsilon > 0$ be a small constant and put

$$\mu_M = \max\{\mu_1, \cdots, \mu_n\}, \quad \xi = (\frac{2\varepsilon}{c_2})^{1/\mu_{n+1}}.$$

Then, take large natural numbers $L_\varepsilon, S_\varepsilon \in \mathbf{N}$ given by

$$L_\varepsilon = [c(n)(2\varepsilon)^{-\frac{n}{\mu_M}}], \quad c(n) = (k(n)c_2)^{\frac{n}{\mu_M}} T^n$$
$$S_\varepsilon = [\frac{T}{2\xi}] \tag{2.16}$$

where $[\cdot]$ indicates the integer part of a real number. In the subset $\{f(t) : t \in [0, TL_\varepsilon]\}$ of Σ, take the points, which are considered as the centers of the mutually disjoint balls with radius ε, as follows.

$$\Sigma_0 = \{f(Tm + \xi l) : 0 \le m \le L_\varepsilon, 0 \le l \le S_\varepsilon,\ m, l \in \mathbf{N} \cup \{0\}\}.$$

Put $\tau = Tm + \xi l, \tau' = Tm' + \xi l', l, l', m, m' \in \mathbf{N} \cup \{0\} : 0 \le m' \le m \le L_\varepsilon,\ 0 \le l' \le l \le S_\varepsilon$. First we cinsider the case where $l = l'$ and $m > m'$. Note that $0 \le l\xi \le T/2$. Then, using **(G1)** and **(G3)**, we can find the natural numbers $p_k, k = 1, ..., n$, which satisfy

$$\begin{aligned}\|f(\tau) - f(\tau')\| &= \|g(w_1\tau, w_2\tau, \cdots, w_n\tau, \tau) - g(w_1\tau', w_2\tau', \cdots, w_n\tau', \tau')\| \\ &\ge \sum_{k=1}^{n} c_2 T^{\mu_M} |w_k m - w_k m' - p_k|^{\mu_M}\end{aligned}$$

where

$$|w_k(m - m') - p_k| < \frac{1}{2}, \quad k = 1, ..., n,$$

hold. It follows from **(D2)** that

$$\begin{aligned}\|f(\tau) - f(\tau')\| &\ge \sum_{k=1}^{n} c_2 T^{\mu_M} |w_k(m - m') - p_k|^{\mu_M} \\ &\ge k(n)c(T)(\frac{1}{m - m'})^{\mu_M/n} \\ &\ge k(n)c(T)L_\varepsilon^{-\mu_M/n} > 2\varepsilon\end{aligned}$$

where, for the minimum number μ_{min} of $\{\mu_1, \cdots, \mu_n\}$,

$$c(T) := \begin{cases} c_2 T^{\mu_{min}} & \text{if } T \ge 1 \\ c_2 T^{\mu_M} & \text{if } 0 \le T < 1. \end{cases}$$

Next, if $l \ne l'$, we have

$$\begin{aligned}\|f(\tau) - f(\tau')\| &\ge c_2 |\xi(l - l')|^{\mu_{n+1}} \\ &\ge c_2 \xi^{\mu_{n+1}} > 2\varepsilon.\end{aligned}$$

Thus the ε-balls, which have the centers in Σ_0, are mutually disjoint. Since the lower bound of $L_\varepsilon(\Sigma)$: the maximam numbers of the disjoint balls is given by

$$\begin{aligned}L_\varepsilon(\Sigma) &\ge L_\varepsilon \times S_\varepsilon \\ &> K\varepsilon^{-\frac{n}{\mu_M} - \frac{1}{\mu_{n+1}}},\end{aligned}$$

it follows from the definition (2.2) and Lemma 1 that

$$\mathcal{D}_F(\Sigma) \geq \lim_{\varepsilon \downarrow 0} \frac{\log L_\varepsilon}{-\log \varepsilon} = \frac{n}{\mu_M} + \frac{1}{\mu_{n+1}}.$$

Using the change of variation $s = t/w_k$ for each $k = 1, \cdots, n$, and applying the argument above to the function

$$f(s) = g(w_1's, \cdots, w_{k-1}'s, s, w_{k+1}'s, \cdots, w_n's, w_k's), \quad w_i' = \frac{w_i}{w_k}, \quad w_k' = \frac{1}{w_k},$$

we obtain

$$\mathcal{D}_F(\Sigma) \geq \frac{n}{\mu_M^{(k)}} + \frac{1}{\mu_k}$$

for each $k = 1, \cdots, n+1$ where $\mu_M^{(k)} = \max\{\mu_1, \cdots, \mu_k, \mu_{k+1}, \cdots, \mu_{n+1}\}$. Since

$$\max_k \{\frac{n}{\mu_M^{(k)}} + \frac{1}{\mu_k}\} = \max\{\frac{n}{\nu_1} + \frac{1}{\nu_3}, \frac{1}{\nu_1} + \frac{n}{\nu_2}\},$$

we obtain the conclusion. □

3. Weierstrass type functions

In this section we investigate Weierstrass type (abr. W-type) functions and estimate the dimensions of the orbits. Let H be a separable Hilbert space with its norm also denoted by $\|\cdot\|$ and $\{\varphi_i\}$ be a complete orthonormal system in H. First we consider a H-valued W-type function $h : R \to H$ defined by

$$h(t) = \sum_{k=1}^{\infty} (\lambda^k)^{-\delta} e^{i2\pi\lambda^k t} \varphi_k \tag{3.1}$$

for some constants $\lambda > 1$, $0 < \delta < 1$.

Lemma 6. *The function $h(t)$ satisfies*

$$\|h(t) - h(t')\| \leq d_1 |t - t'|^\delta, \tag{3.2}$$

$$\|h(t) - h(t')\| \geq d_2 |t - t'|^\delta \tag{3.3}$$

for $t, t' \in R : |t - t'| < (2\lambda)^{-1}$ and $d_1 = d_1(\lambda, \delta), d_2 = d_2(\lambda, \delta)$.

Proof. Since $|t - t'| < (2\lambda)^{-1}$, there exists an integer N such that

$$\frac{\lambda^{-(N+1)}}{2} \leq |t - t'| \leq \frac{\lambda^{-N}}{2}. \tag{3.4}$$

Using the above inequality and

$$2\pi\lambda^N|t-t'| \le \pi, \quad |e^{i\vartheta}-1| \le |\vartheta|, \text{ for } |\vartheta| \le \pi,$$

we obtain

$$\begin{aligned} \|h(t)-h(t')\|^2 &= \sum_{k=1}^{\infty}(\lambda^{2k})^{-\delta}|e^{i2\pi\lambda^k(t-t')}-1|^2 \\ &\le \sum_{k=1}^{N}(\lambda^{2k})^{-\delta}(2\pi\lambda^k)^2|t-t'|^2 + \sum_{k=N+1}^{\infty}4(\lambda^{2k})^{-\delta} \\ &\le \frac{4\pi^2\lambda^{2N(1-\delta)}}{1-\lambda^{2(\delta-1)}}|t-t'|^2 + \frac{4\lambda^{-2(N+1)\delta}}{1-\lambda^{-2\delta}}. \end{aligned}$$

It follows from (3.4) that

$$\begin{aligned} \|h(t)-h(t')\|^2 &\le [\frac{\pi^2 2^{2\delta}}{1-\lambda^{2(\delta-1)}} + \frac{4\cdot 2^{2\delta}}{1-\lambda^{-2\delta}}]|t-t'|^{2\delta} \\ &\le d_1^2|t-t'|^{2\delta}. \end{aligned}$$

Next, assume that $t, t' \in R$ satisfy(3.4), then, applying an elementary inequality

$$|e^{i\vartheta}-1| \ge 2|\sin\frac{\vartheta}{2}| \ge \frac{2}{\pi}|\vartheta|, \quad -\pi \le \vartheta \le \pi,$$

we obtain

$$\begin{aligned} \|h(t)-h(t')\|^2 &\ge \sum_{k=1}^{N}(\lambda^{2k})^{-\delta}|e^{i2\pi\lambda^k(t-t')}-1|^2 \\ &\ge \lambda^{-2N\delta}|e^{i2\pi\lambda^N(t-t')}-1|^2 \\ &\ge \lambda^{-2N\delta}(\frac{2}{\pi}2\pi\lambda^N(t-t'))^2 \\ &\ge 4\cdot 2^{2\delta}\lambda^{2(\delta-1)}|t-t'|^{2\delta}. \quad \square \end{aligned}$$

Remark 3. Since we can take $d_2 = 2\cdot 2^{\delta}\lambda^{\delta-1}$, it is obvious that $d_1 > d_2$.

Let $\{\delta_j\}$ be a periodic sequence of real numbers such that

$$0 < \delta_j \le 1, \quad \delta_{j+n} = \delta_j, \ j = 1, 2, \cdots,$$

and use the similar notations of its minimum and maximum numbers as those in the previous section;

$$\begin{aligned} &\gamma_1 = \min\{\delta_1, \cdots, \delta_n\}, \ \gamma_2 = \min\{\{\delta_1, \cdots, \delta_n\} - \{\gamma_1\}\}, \\ &\nu_1 = \max\{\delta_1, \cdots, \delta_n\}, \ \nu_2 = \max\{\{\delta_1, \cdots, \delta_n\} - \{\nu_1\}\}. \end{aligned}$$

Now we consider the following Weierstrass type function:

$$u(t)=\sum_{k=1}^{\infty}(\lambda^k)^{-\delta_k}e^{i2\pi\lambda^k t}\varphi_k.$$

Theorem 2. *Let p be a positive and square free integer, that is, p cannot be devided by the square of a prime and put $\lambda=p^{\frac{1}{n}}$, $n\geq 2$. Then the fractal dimension of the orbit $\Sigma=\bigcup_{t\in R}h(t)$, given by the W-type function $h(t)$ of (3.1), satisfies*

$$\max\{\frac{n-1}{\nu_1}+\frac{1}{\gamma_1},\frac{n-1}{\nu_2}+\frac{1}{\nu_1}\}\leq \mathcal{D}_F(\Sigma)\leq \frac{1}{\gamma_1}+\frac{n-1}{\gamma_2}. \tag{3.5}$$

Obviously, if $\delta:=\delta_1=\cdots=\delta_n$,

$$\mathcal{D}_F(\Sigma)=\frac{n}{\delta}.$$

Proof. The function $h:R\to X$ is given by

$$u(t)=\sum_{k=1}^{\infty}(p^{\frac{k}{n}})^{-\delta_k}\exp[i2\pi p^{\frac{k}{n}}t]\varphi_k.$$

Using functions $g_j:R\to H, j=1,...,n$, defined by

$$g_j(t)=\sum_{m=0}^{\infty}p^{-m\delta_j}e^{i2\pi p^{m+1}t}\varphi_{nm+j},\qquad j=1,...,n$$

and considering a residue class (mod n), we can describe the function $h(pt)$ as follows.

$$u(pt)=\sum_{k=1}^{\infty}(p^{\frac{k}{n}})^{-\delta_k}\exp[i2\pi p^{\frac{k}{n}+1}t]\varphi_k=\sum_{j=1}^{n}p^{-\frac{j\delta_j}{n}}g_j(p^{\frac{j}{n}}t).$$

Since $g_j(t+\frac{1}{p})=g_j(t)$ and it follows from lemma 6 that each $g_j(t)$ satisfies Hölder's conditions corresponding to (3.2) and (3.3), we can apply the argument in section 2 by the following correspondence.

$$T=p^{-1},\quad w_1=p^{1/n},\quad w_2=p^{2/n},\dots\ w_{n-1}=p^{(n-1)/n},\quad f(t)=h(pt).$$

It follows from Theorem 1 that

$$\begin{aligned}\max\{\frac{n-1}{\nu_1}+\frac{1}{\gamma_1},\frac{1}{\nu_1}+\frac{n-1}{\nu_2}\}\ &\leq\ \mathcal{D}_F(\bigcup_{t\in R}u(pt))=\mathcal{D}_F(\bigcup_{t\in R}u(t))\\ &\leq\ \frac{1}{\gamma_1}+\frac{n-1}{\gamma_2},\end{aligned}$$

since $w_1, w_2, ..., w_{n-1}$ are badly approximable (cf. [1]). □

For some other cases of the parameter λ we can show the following theorem by applying Lemma 2.

Theorem 3. *Let $\lambda > 1$. Then we have the following estimation for the fractal dimension of the orbit Σ given by the W-type function $u(t)$.*

(i) *If $\lambda = (q/p)^{1/n}$, $n \in \mathbf{N}$, $q, p : q > p$ are positive integers, then*

$$n \leq \mathcal{D}_F(\Sigma) \leq \frac{n}{\gamma_1}(1 + \frac{\log p}{\log q - \log p}). \tag{3.6}$$

Consequently, if $\lambda \in \mathbf{N}$, we have

$$1 \leq \mathcal{D}_F(\Sigma) \leq \frac{1}{\gamma_1}.$$

(ii) *If λ is a transcendental real number, then*

$$\mathcal{D}_F(\Sigma) = \infty. \tag{3.7}$$

Proof. [(i)] First we prove the case $n = 1$. Let P_N denote an orthogonal projection from H to the N-dimensional subspace spanned by $\{\varphi_1, ..., \varphi_N\}$. Then, since P_N is nonexpansive,

$$\|P_N u - P_N v\| \leq \|u - v\|, \quad u, v \in H,$$

and the projections of every covering of Σ also cover the subset $P_N\Sigma$, it follows from the definition of the fractal dimension that

$$\mathcal{D}_F(P_N\Sigma) \leq \mathcal{D}_F(\Sigma). \tag{3.8}$$

Since

$$P_N u(t) = \sum_{k=1}^{N} (\frac{q}{p})^{-k\delta_k} \exp[i2\pi(\frac{q}{p})^k t]\varphi_k$$

and each function in the summation is smooth (consequently, $\delta = 1$) and has a period of a rational value, $P_N u(t)$ has a periodic orbit in the N-dimensional subspace. It follows that

$$1 = \mathcal{D}_F(P_N\Sigma) \leq \mathcal{D}_F(\Sigma).$$

To show the second inequality in (3.6) we calculate the inclusion length of the almost periodic function $u(t)$. For a given small constant $\varepsilon > 0$, there exists a large number N:

$$\|u(t) - P_N u(t)\| < \frac{\varepsilon}{2}, \quad \forall t \in R.$$

Note that $P_N u(t)$ has period $\tau := p^N/q$, then we can estimate the inclusion length $l_\varepsilon \simeq p^N/q$ by using the following inequality

$$\begin{aligned} \|u(t+\tau) - u(t)\| &= \|u(t+\tau) - P_N u(t+\tau) + P_N u(t) - u(t)\| \\ &\leq \varepsilon. \end{aligned}$$

Since we can take the large number N, which satisfies

$$\begin{aligned} \|u(t) - P_N u(t)\|^2 &= \sum_{k=N+1}^{\infty} (\frac{q}{p})^{-2k\delta_k} \leq \sum_{k=N+1}^{\infty} (\frac{q}{p})^{-2k\gamma_1} \\ &< \frac{(q/p)^{-2(N+1)\gamma_1}}{1-(q/p)^{-2\gamma_1}} < \frac{\varepsilon^2}{4}, \end{aligned}$$

we have

$$\varepsilon > \frac{2(q/p)^{-(N+1)\gamma_1}}{\sqrt{1-(q/p)^{-2\gamma_1}}} > c_1(p,q,\delta)(\frac{q}{p})^{-N\gamma_1}.$$

It follows that

$$N > \frac{\log \varepsilon^{-1} c_1}{\gamma_1(\log q - \log p)}.$$

Thus it is sufficient to choose a large number

$$N_1 = [\frac{\log \varepsilon^{-1} c_1}{\gamma_1(\log q - \log p)}] + 1,$$

then we have

$$l_\varepsilon < p^{N_1} < c_2(p,q,\delta)\varepsilon^{\frac{-\log p}{\gamma_1(\log q - \log p)}}.$$

Applying Lemma 2 with Lemma 6, we obtain the second inequality of (3.6).

In case $\lambda = (q/p)^{1/n}, \quad n \geq 2$, we have

$$u(t) = \sum_{k=1}^{\infty} (\frac{q}{p})^{-k\delta_k/n} \exp[i2\pi(\frac{q}{p})^{k/n} t]\varphi_k.$$

Using the functions $y_j(t), j = 1, ..., n$ defined by

$$y_j(t) = \sum_{m=0}^{\infty} (\frac{q}{p})^{-m\delta_j} \exp[i2\pi(\frac{q}{p})^m t]\varphi_{mn+j},$$

we can describe

$$u(t) = \sum_{j=1}^{n} h_j(t) = \sum_{j=1}^{n} (\frac{q}{p})^{-j\delta_j/n} y_j((\frac{q}{p})^{j/n} t).$$

Since the terms of $P_{nN} h_j(t)$ for each j are periodic with the periods $\{(p/q)^{j/n}, (p/q)^{1+(j/n)}, \dots, (p/q)^{N-1+(j/n)}\}$, we note that, for each j, $P_{nN} h_j(t)$ is periodic with its period $p^{N-1+(j/n)}/q^{j/n}$ and smooth ($\delta = 1$). Thus, applying Theorem 1, we have

$$n \le \mathcal{D}_F(\bigcup_{t \in R} P_{nN} u(t)) \le \mathcal{D}_F(\Sigma).$$

Next we show the second inequality. Let $Q_j, j = 1, \dots, n$ be a projection on the subspace spanned by $\{\varphi_{nm+j} : m = 0, 1, 2, \dots\}$. Then, considering a change of variation $\tau = (q/p)^{j/n} t$, we have

$$Q_j u((\frac{q}{p})\tau) = \sum_{m=0}^{\infty} (\frac{q}{p})^{-(m+\frac{j}{n})\delta_j} \exp[i2\pi(\frac{q}{p})^{m+1}\tau] \varphi_{mn+j}.$$

It follows from (i) that we can estimate

$$\mathcal{D}_F(\bigcup_t Q_j u(t)) = \mathcal{D}_F(\bigcup_\tau Q_j u((\frac{q}{p})\tau)) \le \frac{1}{\gamma_1}(1 + \frac{\log p}{\log q - \log p}).$$

On the other hand, by using ε-covering balls of $Q_j \Sigma$ on each subspace $Q_j H$ with its number denoted by $N_j(\varepsilon)$, we can construct $\sqrt{n}\varepsilon$-balls of Σ with its number $\Pi_{j=1}^{n} N_j(\varepsilon)$. It follows from the definition of fractal dimensions that

$$\mathcal{D}_F(\Sigma) = \mathcal{D}_F(\sum_{j=1}^{n} Q_j \Sigma) \le \sum_{j=1}^{n} \mathcal{D}_F(Q_j \Sigma),$$

which yields the second inequality.

[(ii)] If λ is a transcendental real number, λ^k, $k = 1, 2, \dots$ are also transcendental and $\{\lambda, \lambda^2, \dots, \lambda^N, \dots\}$ are linearly independent over the rationals. Since each term of $P_N u(t)$ is periodic with its period $\lambda^{-j}, j = 1, \dots, N$, we have

$$N = \mathcal{D}_F(\bigcup_t P_N u(t)) \le \mathcal{D}_F(\Sigma)$$

for arbitrarily large N. □

4. Example of quasi-periodic attractor

We consider a linear abstract equation on a separable Hilbert space H:

$$\frac{du}{dt} + Au = f^*(t), \quad t > 0,$$
$$u(0) = u_0. \tag{4.1}$$

We assume that A is a selfadjoint positive definite operator with dense domain $D(A)$ in H, and that A^{-1} exists and is compact. Then it is well known that there exist eigenvalues λ_j and corresponding eigenfunctions φ_j of the operator A satisfying the following conditions:

$$\begin{aligned}
&0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots, \quad \lim_{j\to\infty} \lambda_j = \infty,\\
&A\varphi_j = \lambda_j \varphi_j, \;\; j = 1, 2, \cdots,\\
&\{\varphi_j(\cdot)\} \;\; \text{forms a complete orthonormal system in } H.
\end{aligned}$$

Here we assume that the perturbation $f^*(t)$ takes values in $D(A)^*$. Thus we consider (4.1) in the distribution sense. (In [2] we can find the various examples in the control theory where the perturbations or the control functions are given in the distribution sense.) Denote the inner product in H by $(\cdot,\cdot)$ and the dual pair between $D(A)$ and $D(A)^*$ by $<\cdot,\cdot>$. Define a W-type function $f : R \to H$ by

$$f(t) = \sum_{k=1}^{\infty} (\mu^{-\delta_k})^k e^{i2\pi\mu^k t} \varphi_{j_k}$$

where $\mu > 1$, $\{\delta_k\}$ is the n-periodic sequence: $0 < \delta_k \leq 1$ and the subsequence $\{j_k\}$ will be determined later. We consider a $D(A^*)$-valued functions f^* given by

$$f^*(t) \simeq \sum_{k=1}^{\infty} (\mu^{-\delta_k})^k \lambda_{j_k} e^{i2\pi\mu^k t} \varphi_{j_k},$$

which means that, for $u = \sum_{j=1}^{\infty} u_j \varphi_j \in D(A)$,

$$< f^*, u >= \sum_{k=1}^{\infty} (\mu^{-\delta_k})^k \lambda_{j_k} e^{i2\pi\mu^k t} u_{j_k}. \tag{4.2}$$

Taking the dual pairs with φ_{j_k} in (4.1) and applying elementary calculations, we can show that the solution $u(t)$ converges to the following W-type function $u_\infty(t)$ in H as $t \to \infty$

$$u_\infty(t) = \sum_{k=1}^{\infty} (\mu^{-\delta_k})^k \frac{\lambda_{j_k}}{\lambda_{j_k} + i2\pi\mu^k} e^{i2\pi\mu^k t} \varphi_{j_k}.$$

In fact, for the ordinary differential equations

$$\begin{aligned}
&\dot{u}_{j_k}(t) = -\lambda_{j_k} u_{j_k}(t) + \mu^{-\delta_k k} \lambda_{j_k} e^{i2\pi\mu^k t},\\
&u_{j_k}(0) = u_{j_k,0}, \qquad\qquad k = 1, 2, \ldots
\end{aligned}$$

where $u(t) = \sum_k u_k(t)\varphi_k$, we have

$$u_{j_k}(t) = e^{-\lambda_{j_k} t} u_{j_k,0} + \frac{\mu^{-\delta k} \lambda_{j_k}}{\lambda_{j_k} + i2\pi\mu^k} \{e^{i2\pi\mu^k t} - e^{-\lambda_{j_k} t}\}.$$

It follows that

$$\|u(t)-u_\infty(t)\|^2 \le \sum_{k=1}^{\infty} |u_{j_k,0} - \frac{\mu^{-\delta_k k}\lambda_{j_k}}{\lambda_{j_k}+i2\pi\mu^k}|^2 e^{-2\lambda_{j_k}t} + \sum_{j\notin\{j_k\}} |u_{j,0}|^2 e^{-2\lambda_j t} \to 0$$

as $t\to\infty$.

To harmonize the frequency parameter μ^k with the eigenvalue λ_{j_k}, considering a suitable parameter $\mu = p^{1/n}$ for a prime integer p and a natural number n, we choose a subsequence j_k, which satisfies

$$\mu^k \le C\lambda_{j_k} \tag{4.3}$$

for some constant $C>0$. Then, applying the proof of Lemma 6 with the following estimate

$$\frac{1}{\sqrt{1+(2\pi C)^2}} \le |\frac{\lambda_{j_k}}{\lambda_{j_k}+i2\pi\mu^k}| \le 1, \tag{4.4}$$

we can show that the W-type function $g_l(t)$, defined by

$$g_l(t) = \sum_{m=0}^{\infty} p^{-\delta_l m} \frac{\lambda_{j_{nm+l}}}{\lambda_{j_{nm+l}} + i2\pi p^{m+\frac{l}{n}}} \exp[i2\pi p^{m+1}t]\varphi_{j_{nm+l}},$$

satisfies Hölder's conditions corresponding to (3.2) and (3.3). Then we can put

$$u_\infty(pt) = \sum_{l=1}^{n} p^{-\frac{l\delta_l}{n}} g_l(p^{\frac{l}{n}}t).$$

Thus, applying the proof of Theorem 2, we obtain the following theorem.

Theorem 4. *Under the perturbation $f^*(t)$ of the W-type function given by (4.2) with the parameter $\mu = p^{1/n}$ for a prime integer p and the subsequence λ_{j_k}, which satisfies (4.3), system (4.1) admits a quasi-periodic global attractor $\Sigma = \bigcup_{t\in R} u_\infty(t)$ which satisfies*

$$\max\{\frac{n-1}{\nu_1}+\frac{1}{\gamma_1}, \frac{n-1}{\nu_2}+\frac{1}{\nu_1}\} \le \mathcal{D}_F(\Sigma) \le \frac{1}{\gamma_1}+\frac{n-1}{\gamma_2}.$$

Obviously, if $\delta := \delta_1 = \cdots = \delta_n$,

$$\mathcal{D}_F(\Sigma) = \frac{n}{\delta}.$$

Remark 4. As the condition for harmonization it is sufficient to assume that

$$\limsup_{k\to\infty} \frac{\mu^k}{\lambda_{j_k}} \le C < \infty,$$

since we can also obtain (4.4).

Remark 5. Applying Theorem 3, we can classify the dimensions of the quasi-periodic attractors by using the algebraic properties of the parameter λ. In view of Theorem 3-(iii), we can conclude that an arbitrarily small change of the parameter λ in the W-type function converts any finite dimensional quasi-periodic attractor to the one which is chaotic ($\mathcal{D}_F(\Sigma) = \infty$).

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