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Covering dimension and nonlinear equations

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For a set $S$ in a Banach space, we denote by $\dim(S)$ its covering dimension ([1], p.42). Recall that, when $S$ is a convex set, the covering dimension of $S$ coincides with the algebraic dimension of $S$, this latter being understood as $\infty$ if it is not finite ([1], p.57). Also, $\overline{S}$ and $\text{conv}(S)$ will denote the closure and the convex hull of $S$, respectively.

In [3], we proved what follows.

THEOREM A ([3], Theorem 1). - Let $X,Y$ be two Banach spaces, $\Phi : X \to Y$ a continuous, linear, surjective operator, and $\Psi : X \to Y$ a continuous operator with relatively compact range.

Then, one has

$$\dim(\{x \in X : \Phi(x) = \Psi(x)\}) \geq \dim(\Phi^{-1}(0)).$$

In the present paper, we improve Theorem A by establishing the following result.

THEOREM 1. - Let $X,Y$ be two Banach spaces, $\Phi : X \to Y$ a continuous, linear, surjective operator, and $\Psi : X \to Y$ a completely continuous operator with bounded range.

Then, one has

$$\dim(\{x \in X : \Phi(x) = \Psi(x)\}) \geq \dim(\Phi^{-1}(0)).$$

PROOF. First, assume that $\Phi$ is not injective. For each $x \in X$, $y \in Y$, $r > 0$, we denote by $B_X(x,r)$ (resp. $B_Y(y,r)$) the closed ball in $X$ (resp. $Y$) of radius $r$ centered at $x$ (resp. $y$). By the open mapping theorem, there is $\delta > 0$ such that

$$B_Y(0, \delta) \subseteq \Phi(B_X(0,1)).$$

Since $\Psi(X)$ is bounded, there is $\rho > 0$ such that

$$\overline{\Psi(X)} \subseteq B_Y(0, \rho).$$

Consequently, one has

$$\overline{\Psi(X)} \subseteq \Phi\left(B_X\left(0, \frac{\rho}{\delta}\right)\right).$$

Now, fix any bounded open convex set $A$ in $X$ such that

$$B_X\left(0, \frac{\rho}{\delta}\right) \subseteq A.$$
Put 
\[ K = \overline{\Psi(A)}. \]

Since \( \Psi \) is completely continuous, \( K \) is compact. Fix any positive integer \( n \) such that \( n \leq \dim(\Phi^{-1}(0)) \). Also, fix \( z \in K \). Thus, \( \Phi^{-1}(z) \cap A \) is a convex set of dimension at least \( n \). Choose \( n + 1 \) affinely independent points \( u_{z,1}, \ldots, u_{z,n+1} \) in \( \Phi^{-1}(z) \cap A \). By the open mapping theorem again, the operator \( \Phi \) is open, and so, successively, the multifunctions \( y \to \Phi^{-1}(y), \ y \to \Phi^{-1}(y) \cap A, \) and \( y \to \Phi^{-1}(y) \cap \overline{A} \) are lower semicontinuous. Then, applying the classical Michael theorem ([2], p.98) to the restriction to \( K \) of the latter multifunction, we get \( n + 1 \) continuous functions \( f_{z,1}, \ldots, f_{z,n+1} \), from \( K \) into \( \overline{A} \), such that, for all \( y \in K, \ i = 1, \ldots, n+1 \), one has
\[ \Phi(f_{z,i}(y)) = y \]
and
\[ f_{z,i}(z) = u_{z,i}. \]

Now, for each \( i = 1, \ldots, n + 1 \), fix a neighbourhood \( U_{z,i} \) of \( u_{z,i} \) in \( A \) in such a way that, for any choice of \( w_i \) in \( U_{z,i} \), the points \( w_1, \ldots, w_{n+1} \) are affinely independent. Now, put
\[ V_z = \bigcap_{i=1}^{n+1} f_{z,i}^{-1}(U_{z,i}). \]

Thus, \( V_z \) is a neighbourhood of \( z \) in \( K \). Since \( K \) is compact, there are finitely many \( z_1, \ldots, z_p \in K \) such that \( K = \bigcup_{j=1}^{p} V_{z_j} \). For each \( y \in K \), put
\[ F(y) = \mathrm{conv}(\{f_{z_j,i}(y) : j = 1, \ldots, p, \ i = 1, \ldots, n+1\}). \]

Observe that, for some \( j \), one has \( y \in V_{z_j} \), and so \( f_{z_j,i}(y) \in U_{z,j,i} \) for all \( i = 1, \ldots, n + 1 \). Hence, \( F(y) \) is a compact convex subset of \( \Phi^{-1}(y) \cap \overline{A} \), with \( \dim(F(y)) \geq n \). Observe also that the multifunction is \( F \) is continuous ([2], p.86 and p.89) and that the set \( F(K) \) is compact ([2], p.90). Put
\[ C = \overline{\mathrm{conv}(F(K))}. \]

Furthermore, note that, by continuity, one has \( \Psi(\overline{A}) \subseteq K \). Finally, consider the multifunction \( G : C \to 2^C \) defined by putting
\[ G(x) = F(\Psi(x)) \]
for all \( x \in C \). Hence, \( G \) is a continuous multifunction, from the compact convex set \( C \) into itself, whose values are compact convex sets of dimension at least \( n \). Consequently, by the result of [4], one has
\[ \dim(\{x \in C : x \in G(x)\}) \geq n. \]
But, since
\[ \{ x \in C : x \in F(\Psi(x)) \} \subseteq \{ x \in C : x \in \Phi^{-1}(\Psi(x)) \} \]
the conclusion follows ([1], p.220). Finally, if \( \Phi \) is injective, the conclusion
means simply that the set \( \{ x \in X : \Phi(x) = \Psi(x) \} \) is non-empty, and this is
got readily proceeding as before. \( \triangle \) In [3], we indicated some
examples of application of Theorem A. We now point out an application of
Theorem 1 which cannot be obtained from Theorem A. For a Banach space
\( E \), we denote by \( \mathcal{L}(E) \) the space of all continuous linear operators from \( E \)
into \( E \), with the usual norm. Also, \( I \) will denote a (non-degenerate) compact
real interval. THEOREM 2. - Let \( E \) be an infinite-dimensional Banach
space, \( A : I \rightarrow \mathcal{L}(E) \) a continuous function and \( f : I \times E \rightarrow E \) a uniformly
continuous function with relatively compact range.

Then, one has
\[ \dim(\{ u \in C^1(I, E) : u'(t) = A(t)(u(t)) + f(t, u(t)) \forall t \in I \}) = \infty. \]

PROOF. Take \( X = C^1(I, E) \), \( Y = C^0(I, E) \) and \( \Phi(u) = u'(\cdot) - A(\cdot)(u(\cdot)) \)
for all \( u \in X \). So, by a classical result, \( \Phi \) is a continuous linear operator from
\( X \) onto \( Y \) such that \( \dim(\Phi^{-1}(0)) = \infty \). Next, put \( \Psi(u) = f(\cdot, u(\cdot)) \)
for all \( u \in X \). So, \( \Psi \) is an operator from \( X \) into \( Y \) with bounded range. From our
assumptions, thanks to the Ascoli-Arzelà theorem, it also follows that \( \Psi \) is
completely continuous. Then, the conclusion follows directly from Theorem
1. \( \triangle \)

Analogously, one gets from Theorem 1 the following

THEOREM 3. - Let \( A : I \rightarrow \mathcal{L}(\mathbb{R}^n) \) be a continuous function and \( f : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) a continuous and bounded function.

Then, one has
\[ \dim(\{ u \in C^1(I, \mathbb{R}^n) : u'(t) = A(t)(u(t)) + f(t, u(t)) \forall t \in I \}) \geq n. \]

THEOREM 4. - Let \( a_1, \ldots, a_k \) be \( k \) continuous real functions on \( I \). Fur-
ther, let \( f : I \times \mathbb{R}^k \rightarrow \mathbb{R} \) be a continuous and bounded function.

Then, one has
\[ \dim\left( \left\{ u \in C^k(I) : u^{(k)}(t) + \sum_{i=1}^{k} a_i(t)u^{(k-i)}(t) = f(t, u(t), u'(t), \ldots, u^{(k-1)}(t)) \forall t \in I \right\} \right) \geq k. \]

References


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