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On Set-Valued Minimax Problems

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Abstract. The paper presents a study of minimax problems for vector-valued maps and set-valued maps. The aim consists of three parts: we give

(i) some formulations of minimax problems and minimax theorems for vector-valued maps and set-valued maps. For this kind research, we observe

(ii) some existence results on generalized saddle points by using fixed point theorems. Based on such results, we give a survey on

(iii) some generalized minimax theorems for vector-valued maps and an analogous idea for set-valued maps.

Key words: minimax theorem, cone saddle point, vector-valued functions, set-valued maps, cone-semicontinuity, marginal functions.

1. Introduction

Minimax theorems are concerned with various fields in mathematics, operational research, and economics. In particular, it has a strong connection with nonlinear analysis. Among many benefits of minimax theorems, most important result is as follows. The saddle point theorem in usual game theory insists that

> a real-valued payoff function possesses a saddle point if and only if the minimax value and the maximin value of the function are coincident;

and accordingly (scalar-valued) minimax theorems say:

> the minimax and maximin values are coincident under certain conditions.

We here consider two-person zero-sum games and payoff functions of players I and II are $-f(x,y)$ and $f(x,y)$ for some function $f : X \times Y \to \boldsymbol{R}$, respectively. A point (strategy pair) $(x_0, y_0) \in X \times Y$ is said to be a saddle point of f if

$$f(x_0, y) \leq f(x_0, y_0) \leq f(x, y_0)$$

for all $x \in X$, $y \in Y$. We know the minimax value is greater than or equal to the maximin value in general, and hence the insistence of minimax theorems is coincident with the following: the minimax value is less than or equal to the maximin value under some appropriate conditions. These results hold for real-valued functions, but it is not always true in the case of vector-valued payoff functions.

In the decade from 1983, some researchers have studied vector-valued minimax theorems. The common topic is whether or not games with multiple noncomparable criteria have an acceptable theory similar to standard results for scalar games, in particular, what type of

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minimax equation or inequality holds. In 1983, Nieuwenhuis gave his pioneer idea to this area, and then Corley and Ferro presented important results. The author have separately researched such minimax problems in general setting and proved minimax theorems, existence theorems for saddle points, and saddle point theorems. These results have been approached by vector optimization method.

These papers suggest interesting answers for the following questions: If we give reasonable definitions for minimax values and maximin values of a vector-valued function, what type of minimax equation or inequality holds? Also, if we give a suitable definition for saddle points of the vector-valued function, under what conditions do there exist such saddle points? What relationship holds among such minimax values and maximin values and saddle values? Moreover, this kind of research is continued for more general payoff functions, especially multi-valued functions (or set-valued maps) up to now; see papers by Luc and Vargas(1992), Ha(1995), Tan et al(1996).

On the other hand, it is also well-known that the convexity and continuity of real-valued functions play very important roles in the area of nonlinear optimization as well as in various fields of mathematics, and especially such properties are very useful for minimax theorems. Such situation remains to vector-valued minimax theory as well as vector optimization. In some papers of the author, some types of cone-convexity and (cone-)semicontinuity are introduced, and then vector-valued minimax theorems are proved for vector-valued functions which satisfy these properties.

It is, however, unfortunate that those generalizations for such relaxations and modifications into multi-valued version are incomplete, in particular, with respect to relaxations of continuity. In this paper, we consider a certain relaxation of continuity for vector-valued and multi-valued functions, which corresponds to a generalization of ordinary lower semicontinuity into vector-valued and multi-valued versions. For vector-valued functions with this generalized lower semicontinuity, we prove existence theorems for generalized saddle points (cone saddle points) of vector-valued functions, and then show some results of the author.

2. Saddle and Loose Saddle Points

We give the preliminary terminology used throughout the paper. Let Z be an ordered real topological vector space (ordered t.v.s. for short), as a range space of functions, with the vector ordering $\leq_C$ induced by a convex cone C, that is, for $x, y \in Z$, $x \leq_C y$ if $y - x \in C$. Throughout the paper, the convex cone C is assumed to be solid, that is, its topological interior **int** C is nonempty; and to be pointed, that is, $C \cap (-C) = \{0\}$. For C, an element x_0 of a subset A of Z is said to be a C-minimal point of A (or an efficient point of A with respect to C) if $\{x \in A \mid x \leq_C x_0, x \neq x_0\} = \emptyset$, which is equivalent to $A \cap (x_0 - C) = \{x_0\}$. We denote the set of all C-minimal points of A by $\mathbf{Min}A$. Also, C^0-minimal [resp., C-maximal, C^0-maximal] set of A is defined similarly, and denoted by $\mathbf{Min_w}A$ [resp. $\mathbf{Max}A$, $\mathbf{Max_w}A$]. These C^0-minimality and C^0-maximality are weaker than C-minimality and C-maximality, respectively.;

Under the previous notation, we give definitions for generalized saddle point of a vector-valued function and a set-valued map. Let $f : X \times Y \to Z$ and $F : X \times Y \rightsquigarrow Z$ be a vector-valued function and a set-valued map, respectively.

Definition 1. A point (x_0, y_0) is said to be: (i) a C-saddle point of f wrt $X \times Y$ if $f(x_0, y_0) \in \mathbf{Max}f(x_0, Y) \cap \mathbf{Min}f(X, y_0)$;

(ii) a weak C-saddle point of f wrt $X \times Y$ if $f(x_0, y_0) \in \mathbf{Max_w} f(x_0, Y) \cap \mathbf{Min_w} f(X, y_0)$.
(iii) a C-saddle point of F wrt $X \times Y$ if $F(x_0, y_0) \cap \mathbf{Max} F(x_0, Y) \cap \mathbf{Min} F(X, y_0) \neq \emptyset$;
(iv) a C-loose saddle point of F wrt $X \times Y$ if $F(x_0, y_0) \cap \mathbf{Max} F(x_0, Y) \neq \emptyset$ and $F(x_0, y_0) \cap \mathbf{Min} F(X, y_0) \neq \emptyset$.

We note that any C-saddle point of f is a weak C-saddle point of f and that any C-saddle point of F is a C-loose saddle point of F obviously. Also, in the case $C^0 = C$, the conditions (i) and (ii) are coincident. We have three types of existence theorem of weak C-saddle points for vector-valued functions, and that of C-loose saddle point for set-valued maps. For this end, we observe cone-convexity and cone-continuity of vector-valued function, and moreover cone-convexity and cone-semicontinuity of set-valued map in the following section.

3. Convexity and Semicontinuity

First, we introduce various types of convexity of vector-valued functions. Some of these convexities are collected in [7], [9], and [10]; called cone-convexity. In particular, we explicate some relationships among vector versions of quasi-convexity which correspond to generalizations of ordinary quasi-convexity for real-valued functions. We have Table 1 about the relationship among the cone-convexities; see [10, 11], and moreover we gave their extensions to set-valued functions in [2, 3].

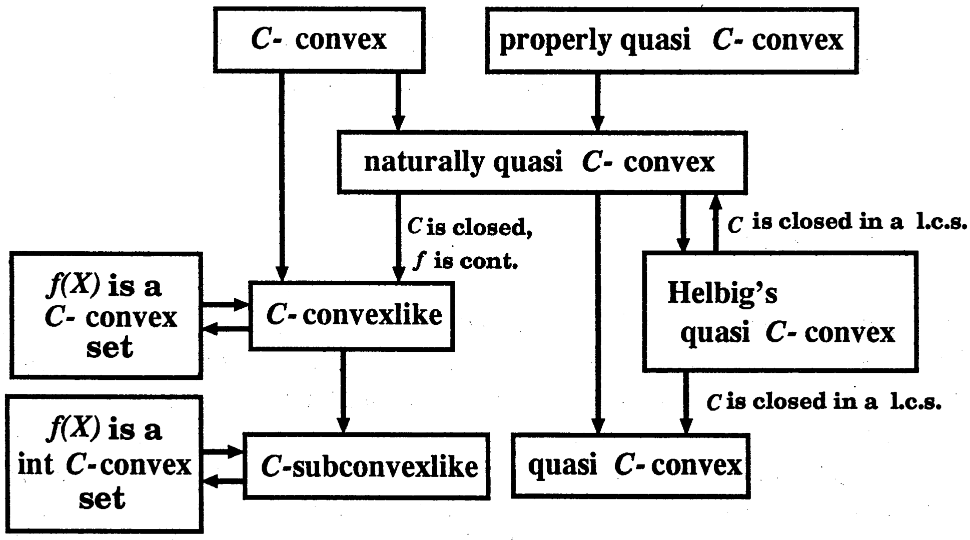

Figure 1: **Relations between cone-convexities**

Next, we propose a vector-valued version of lower semicontinuity, which is a generalization of the ordinary lower semicontinuity on real-valued functions. Then, the notion of classical upper semicontinuity of set-valued map is also generalized to cone-upper semicontinuity, and hence simultaneous semicontinuity of a real-valued function and a set-valued map is transmitted to the supremum type marginal function associated with them. Such properties of set-valued maps have been studied in [1].

Definition 2. Let X be a topological space. A vector-valued function $f : X \to Z$ is said to be C-lower semicontinuous on X if it satisfies one of the following three equivalent conditions:

(i) For all $a \in Z$, $f^{-1}(a + \textbf{int } C)$ is open;

(ii) For each $x_0 \in X$ and any open neighborhood V of $f(x_0)$, there exists an open neighborhood U of x_0 such that $f(x) \in V + C$ for all $x \in U$ ([4, Def.5.1(p.22)]);

(iii) For each $x_0 \in X$ and any $d \in \textbf{int } C$, there exists an open neighborhood U of x_0 such that $f(x) \in f(x_0) - d + \textbf{int } C$ for all $x \in U$.

Also, it is said to be C-upper semicontinuous on X if $-f$ is C-lower semicontinuous on X.

Cone-convexity and cone-semicontinuity of set-valued maps are also observed and some interesitng results are obtained. Based on these results, we study the existence of generalized saddle points for multi-valued functions. The basic idea is based on using a fixed-point theorem for the following type of set-valued map:

$$T(x_0, y_0) := \left\{ (x, y) \in A \times B \;\middle|\; \begin{array}{lcl} \min \phi \circ F(A, y_0) & \in & \phi \circ F(x, y_0) \\ \max \phi \circ F(x_0, B) & \in & \phi \circ F(x_0, y) \end{array} \right\}.$$

4. Minimax Theorems

In a few of the author's papers, he has proposed some minimax theorems for vector-valued functions. Their results are based on both existence theorems of saddle points and a saddle point theorem of a vector-valued function, which is a corollary of existence for vector-valued minimax and maximin sets. His vector-valued minimax theorems consists of three types: topological space type, topological vector space type, and locally convex space type. First one is concerned with a payoff function of the form $f(x, y) = u(x) + v(y)$ or $f(x, y) = u(x) + \beta(x)v(y)$; second one holds for a sort of vector-valued saddle function; third one is guaranteed under convexity of each weak optimal response strategies.

They are a similar statement to the ordinary minimax theorems for real-valued functions. In fact, vector-valued minimax theorems tell us that there exist some minimax strategy and maximin strategy of f such that their values are ordered by $\leq_C$ and dominated each other whenever f has a weak C-saddle point. As illustrated in Fig.2, first type minimax theorem means that minimax values and maximin values of f are entirely contained in the set of maximin values of f minus the pointed convex cone C and in the set of minimax values of f plus the pointed convex cone C, respectively. Also, as illustrated in Fig.3, second and third type minimax theorems mean that there exist some minimax values and maximin values of f such that both vectors are ordered by $\leq_C$ and dominated each other. Moreover, multi-valued version is also observed.

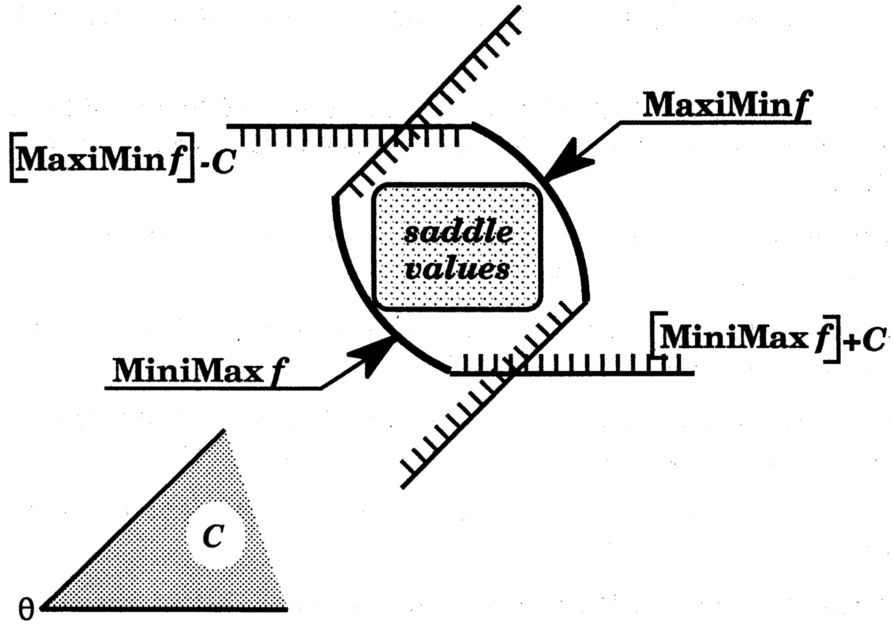

Fig.2:Minimax inequality among minimax values, maximin values, and saddle values (type I).

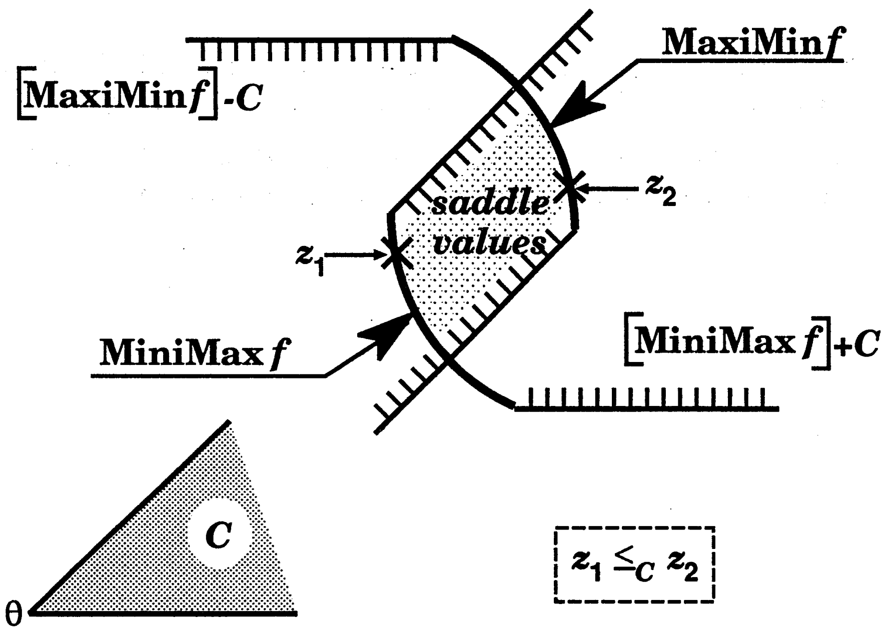

Fig.3:Minimax inequality among minimax values, maximin values, and saddle values (type II, III).

As for minimax theorems for set-valued map, it is difficult to extend vector-valued minimax theorems into set-valued ones directly, because there might be few motivation nor meaningless for game theoretic pointview. However, more advanced researches on the topics will come.

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