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Kyoto University
**Von Neumann-Jordan constant and some geometrical constants of Banach spaces**

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In this note some recent results of the authors are announced concerning von Neumann-Jordan (NJ-) constant, non-square (or James) constant, and normal structure coefficient for a Banach space.

A sequence of results on the NJ-constant of a Banach space $X$, we denote it by $c_{NJ}(X)$, has been recently obtained by the first and third authors, etc. ([8, 9, 10, 11, 12, 14]; refer to [2, 7] for classical results). Their concerns were/are as follows:

(i) Determine or estimate $c_{NJ}(X)$ for various $X$.

(ii) What informations does $c_{NJ}(X)$ give about $X$?

Here we discuss the following question raised by the second author:

(iii) What is the relation between $c_{NJ}(X)$ and some other geometrical constants of $X$?

In particular we estimate $c_{NJ}(X)$ with the non-square (or James) constant $J(X)$, and also the normal structure coefficient $N(X)$ with $c_{NJ}(X)$. An estimate for $J(X^*)$ with $J(X)$ is given as well.

The von Neumann-Jordan (NJ-) constant for a Banach space $X$ (Clarkson [2]), $c_{NJ}(X)$, is the smallest constant for which
\[
\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C \quad \forall (x, y) \neq (0, 0).
\]

The non-square (or James) constant of \(X\) (Gao-Lau [3]) is defined by

\[
J(X):= \sup_{x,y \in S_X} \min\{\|x+y\|, \|x-y\|\},
\]

where \(S_X\) stands for the unit sphere of \(X\). We recall some notions related with \(J(X)\):

(i) \(X\) is called uniformly convex ([1]) if for any \(\epsilon\) \((0 < \epsilon < 2)\) there exists a \(\delta > 0\) such that

\[
\|x-y\| \geq \epsilon \quad (x, y \in S_X) \quad \Rightarrow \quad \|(x+y)/2\| \leq 1 - \delta.
\]

(ii) \(X\) is called uniformly non-square (James [6]) if there exists a \(\delta > 0\) \((0 < \delta < 1)\) such that

\[
\|(x-y)/2\| > 1 - \delta \quad (x, y \in S_X) \quad \Rightarrow \quad \|(x+y)/2\| \leq 1 - \delta.
\]

The difference between (3) and (4) is clear: In (3) we can let \(\epsilon \to 0\). On the contrary, in (4) we cannot do it, that is, we can only get the same conclusion as (3) for \(x, y \in S_X\) apart from each other to some extent.

(iii) The modulus of convexity of \(X\) ([1]) is defined by

\[
\delta_X(\epsilon) := \inf\{1 - \|(x+y)/2\| ; \|x-y\| \geq \epsilon, x, y \in S_X\}.
\]

Now, (4) is reformulated as

\[
\min\{\|x+y\|, \|x-y\|\} \leq 2(1 - \delta);
\]

thus we understand the above definition (2) of the non-square constant \(J(X)\) as a sort of modulus of non-squareness of \(X\). Gao and Lau [3] showed that

\[
J(X) = \sup\{\epsilon > 0 ; \delta_X(\epsilon) \leq 1 - \epsilon/2\}.
\]
1. Comparison of NJ- and James constant

We compare some known facts on NJ- and James constants:

(i) For any Banach space \( X \)

\[
1 \leq c_{NJ}(X) \leq 2,
\]

\[
\sqrt{2} \leq J(X) \leq 2 \quad (\text{dim } X \geq 2)
\]

(ii) \( X \): a Hilbert space \( \iff \) \( c_{NJ}(X) = 1 \),

\( X \): a Hilbert space \( \Rightarrow \) \( J(X) = \sqrt{2} \)

(iii) \( X \): uniformly non-square \( \iff \) \( c_{NJ}(X) < 2 \) (Takahashi-Kato[14])

\( \iff \) \( J(X) < 2 \) (clear by definition)

(iv) Let \( 1 \leq p \leq 2 \), \( 1/p + 1/p' = 1 \). Then

\[
c_{NJ} (L_p) = c_{NJ} (L_{p'}) = 2^{2/p-1},
\]

\[
J(L_p) = J(L_{p'}) = 2^{1/p}.
\]

2. Relation between \( c_{NJ}(X) \) and \( J(X) \)

**Theorem 1.** For any Banach space \( X \)

\[
\frac{1}{2} J(X)^2 \leq c_{NJ}(X) \leq \frac{J(X)^2}{(J(X)-1)^2 + 1}.
\]

**Remarks.** According to the facts stated in the preceding section, equality occurs in (7) with several spaces:

(i) \( \frac{1}{2} J(L_p)^2 = c_{NJ}(L_p) \); the same is true for \( W^k_p(\Omega) \) (Sobolev space), \( c_p \) (space of \( p \)-Schatten class operators) and \( L_p(L_q) \) (\( L_q \)-valued \( L_p \)-space), etc.
(ii) For a Hilbert space $\mathbf{H}$, $\frac{1}{2} J(\mathbf{H})^2 = C_{NJ}(\mathbf{H}) = 1$.

(iii) If $\mathbf{X}$ is not uniformly non-square,

$$\frac{1}{2} J(\mathbf{X})^2 = C_{NJ}(\mathbf{X}) = \frac{J(\mathbf{X})^2}{(J(\mathbf{X}) - 1)^2 + 1} = 2.$$  

3. Relation between $J(\mathbf{X})$ and $J(\mathbf{X}^*)$

For the dual space $\mathbf{X}^*$ it is known that $C_{NJ}(\mathbf{X}^*) = C_{NJ}(\mathbf{X})$, whereas $J(\mathbf{X}^*) \neq J(\mathbf{X})$ in general. In [4] Gao and Lau ask what relation $J(\mathbf{X})$ and $J(\mathbf{X}^*)$ have. We have the following

**Theorem 2.** For any Banach space $\mathbf{X}$

$$2J(\mathbf{X}) - 2 \leq J(\mathbf{X}^*) \leq \frac{J(\mathbf{X})}{2} + 1.$$  

**Remark.** If $\mathbf{X}$ is not uniformly non-square,

$$2J(\mathbf{X}) - 2 = J(\mathbf{X}^*) = \frac{J(\mathbf{X})}{2} + 1 = 2.$$  

**Corollary.** $\mathbf{X}^*$ is uniformly non-square if and only if $\mathbf{X}$ is so.

This result seems not to have appeared in literature.

4. NJ-constant and normal structure of Banach spaces

A Banach space $\mathbf{X}$ is said to have *normal structure* provided for any bounded convex subset $\mathbf{K}$ of $\mathbf{X}$ with $\text{diam} \mathbf{K} > 0$, its radius $r(\mathbf{K})$ is less than $\text{diam} \mathbf{K}$, that is,

$$r(\mathbf{K}) < \text{diam} \mathbf{K}.$$  

If there exists some $c$ ($0 < c < 1$) such that

(8) $$r(\mathbf{K}) \leq c \cdot \text{diam} \mathbf{K},$$
X is said to have uniform normal structure. The smallest $c$ \((0 < c \leq 1)\) satisfying (8) for all $K$ (bounded convex) with $\text{diam } K > 0$, is called the normal structure coefficient of $X$ and denoted by $N(X)$. Clearly $0 \leq N(X) \leq 1$; and $X$ has uniform normal structure if and only if $N(X) < 1$. These notions are strongly connected with the fixed point property. $X$ is said to have fixed point property (FPP) (for non-expansive mappings) provided for any non-empty bounded convex subset $K$ of $X$, every non-expansive mapping $T: K \to K$ has a fixed point. It is known ([5]) that (i) if $X$ is reflexive and has the normal structure, then $X$ has FPP; (ii) if $X$ has the uniform normal structure, then $X$ is reflexive, whence $X$ has FPP.

Now, Gao and Lau [4] showed that:

If $J(X) < 3/2$, then $X$ has the uniform normal structure.

Prus [13] gave more precisely the following estimate for $N(X)$ by $J(X)$:

For any Banach space $X$,

\[
N(X) \leq \frac{1}{J(X) + 1 - \left\{ (J(X) + 1)^2 - 4 \right\}^{1/2}}.
\]

Note that the estimate (9) implies that if $J(X) < 3/2$ then $N(X) < 1$. (One should also note here that the definition of $N(X)$ in Prus [11] is the reciprocal of our $N(X)$.) We present the following estimate for $N(X)$ by NJ-constant:

**Theorem 3.** For any Banach space $X$

\[
N(X) \leq \left\{ C_{NJ}(X) - \frac{1}{4} \right\}^{1/2}.
\]

**Theorem 4.** Let $C_{NJ}(X) < 5/4$. Then $X$, as well as $X^*$, has the uniform normal structure; and hence $X (X^*)$ has the fixed point property.
Indeed, the above estimate (10) implies that if $C_{NJ}(X) < 5/4$, then $N(X) < 1$. The assertion for $X^*$ is a consequence of the fact that $C_{NJ}(X^*) = C_{NJ}(X)$.

Remarks. (i) For the spaces with $C_{NJ}(X) = \frac{1}{2} J(X)^2$ (recall Remarks after Theorem 1), Gao and Lau's condition $J(X) < 3/2$ is rewritten as $C_{NJ}(X) < 9/8$; thus our condition $C_{NJ}(X) < 5/4 = 10/8$ is weaker than theirs in this case.

(ii) The normal structure is not inherited by dual spaces ([4; esp. p. 63]).

References


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