

Zygmund type results for the best approximation in Banach spaces

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1. Introduction

Let $C_{2\pi}$ denote the Banach space of all 2π -periodic, continuous functions on the real line \mathbb{R} with the norm

$$\|f\|_\infty = \max\{|f(t)| : |t| \leq \pi\}.$$

Let \mathbb{N} be the set of all positive integers, and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N}_0$, we denote by \mathcal{T}_n the set of all trigonometric polynomials of degree at most n . For a given $f \in C_{2\pi}$, we define

$$E_n(C_{2\pi}; f) = \inf\{\|f - g\|_\infty : g \in \mathcal{T}_n\},$$

which is called the best approximation of degree n to f with respect to \mathcal{T}_n . Since \mathcal{T}_n is the $2n + 1$ -dimensional Chebyshev subspace of $C_{2\pi}$, for each $f \in C_{2\pi}$, there exists a unique trigonometric polynomial $g_n \in \mathcal{T}_n$ of the best approximation of f with respect to \mathcal{T}_n , i.e., such that

$$E_n(C_{2\pi}; f) = \|f - g_n\|$$

(see, e.g., [9; Chapter 2, Theorem 6]).

The classical Weierstrass approximation theorem simply states that the sequence $\{E_n(C_{2\pi}; f)\}$ converges to zero as n tends to infinity for every $f \in C_{2\pi}$. It does not say how fast $E_n(C_{2\pi}; f)$ approaches zero. In general, the smoother the function, the faster $E_n(C_{2\pi}; f)$ tends to zero. The results that guarantee this event are sometimes called the direct theorems of Jackson-type (cf. [7]). Conversely, the inverse theorems of Bernstein-type assert that a function f has certain smoothness properties if $E_n(C_{2\pi}; f)$ tends rapidly enough to zero, then f has certain smoothness properties, which are usually given in terms of its modulus of continuity, Lipschitz classes, and differentiability properties. These results have been developed further by Zygmund [22] as follows (cf. [2; Chapter 2], [9; Chapter 4], [10; Chapters IV and V], [20; Chapters V and VI]):

Let $f \in C_{2\pi}$ and $r \in \mathbb{N}_0$. Then

$$E_n(C_{2\pi}; f) = o(n^{-r-1}) \quad (n \rightarrow \infty)$$

if and only if f is r -times continuously differentiable on \mathbb{R} and

$$\omega^*(C_{2\pi}; f^{(r)}, \delta) = o(\delta) \quad (\delta \rightarrow +0),$$

where

$$\omega^*(C_{2\pi}; f^{(r)}, \delta) = \sup\{\|f^{(r)}(\cdot + t) + f^{(r)}(\cdot - t) - 2f^{(r)}(\cdot)\|_\infty : |t| \leq \delta\}.$$

In particular,

$$E_n(C_{2\pi}; f) = o(n^{-1}) \quad (n \rightarrow \infty) \iff \omega^*(C_{2\pi}; f, \delta) = o(\delta) \quad (\delta \rightarrow +0).$$

Also, if

$$\omega^*(C_{2\pi}; f, \delta) = o(\delta) \quad (\delta \rightarrow +0),$$

then

$$\omega(C_{2\pi}; f, \delta) = o(\delta |\log \delta|) \quad (\delta \rightarrow +0),$$

where

$$\omega(C_{2\pi}; f, \delta) = \sup\{\|f(\cdot - t) - f(\cdot)\|_\infty : |t| \leq \delta\}$$

denotes the modulus of continuity of f . All the results remain true if o is replaced by O .

The statements analogous to these results also holds for the Banach space $L_{2\pi}^p$ consisting of all 2π -periodic, p th power Lebesgue integrable functions f on \mathbb{R} with the norm

$$\|f\|_p = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p dt \right)^{1/p} \quad (1 \leq p < \infty).$$

Furthermore, in [3] these results were generalized by means of the higher order moduli of continuity and consequently, a generalization of the classical theorem of de la Vallée Poussin approximation theorem (cf. [5], [17]) was obtained.

The purpose of this paper is to extend the above-mentioned results to arbitrary Banach spaces. We refer to [16] for detailed treatments (cf. [13], [14], [15]).

2. Groups of multiplier operators and moduli of continuity

Let X be a Banach space with norm $\|\cdot\|_X$, and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. Let \mathbb{Z} denote the set of all integers, and let $\{P_j : j \in \mathbb{Z}\}$ be a sequence of projection operators in $B[X]$ satisfying the following conditions:

(P-1) The projections $P_j, j \in \mathbb{Z}$, are mutually orthogonal, i.e., $P_j P_n = \delta_{j,n} P_n$ for all $j, n \in \mathbb{Z}$, where $\delta_{j,n}$ denotes the Kronecker's symbol.

(P-2) $\{P_j : j \in \mathbb{Z}\}$ is fundamental, i.e., the linear span of $\cup_{j \in \mathbb{Z}} P_j(X)$ is dense in X .

(P-3) $\{P_j : j \in \mathbb{Z}\}$ is total, i.e., if $f \in X$ and $P_j(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$. For any $f \in X$, we associate its (formal) Fourier series expansion with respect to $\{P_j : j \in \mathbb{Z}\}$

$$f \sim \sum_{j=-\infty}^{\infty} P_j(f).$$

An operator $T \in B[X]$ is called a multiplier operator on X if there exists a sequence $\{\tau_j : j \in \mathbb{Z}\}$ of scalars such that for every $f \in X$,

$$T(f) \sim \sum_{j=-\infty}^{\infty} \tau_j P_j(f),$$

and the following notation is used:

$$T \sim \sum_{j=-\infty}^{\infty} \tau_j P_j$$

Thus, this implies that $P_j T = \tau_j P_j$ for all $j \in \mathbb{Z}$ (cf. [4], [11], [12], [21]).

Let $M[X]$ denote the set of all multiplier operators on X , which is a commutative closed subalgebra of $B[X]$ containing the identity operator I . Let $\{T_t : t \in \mathbb{R}\}$ be a family of operators in $M[X]$ satisfying

$$\|T_t\|_{B[X]} \leq 1 \quad \text{for all } t \in \mathbb{R}$$

and having the expansions

$$T_t \sim \sum_{j=-\infty}^{\infty} e^{-ijt} P_j \quad (t \in \mathbb{R}).$$

Then $\{T_t : t \in \mathbb{R}\}$ becomes a strongly continuous group of operators in $B[X]$ and we have

$$G^r(P_j(g)) = (-ij)^r P_j(g) \quad (g \in X, j \in \mathbb{Z}, r \in \mathbb{N})$$

and

$$G^r(f) \sim \sum_{j=-\infty}^{\infty} (-ij)^r P_j(f) \quad (f \in D(G^r), r \in \mathbb{N}),$$

where G is the infinitesimal generator of $\{T_t : t \in \mathbb{R}\}$ with domain $D(G)$ (cf. [11, Proposition 2]). For the basic theory of semigroups of operators on Banach spaces, we refer to [1] and [6].

For each $r \in \mathbb{N}_0$ and $t \in \mathbb{R}$, we define

$$\Delta_t^0 = I, \quad \Delta_t^r = (T_t - I)^r = \sum_{m=0}^r (-1)^{r-m} \binom{r}{m} T_{mt} \quad (r \geq 1),$$

which stands for the r -th iteration of T_t . Then Δ_t^r belongs to $M[X]$ and

$$\|\Delta_t^r\|_{B[X]} \leq 2^r, \quad \Delta_t^r \sim \sum_{j=-\infty}^{\infty} (e^{-ijt} - 1) P_j.$$

If $r \in \mathbb{N}_0$, $f \in X$ and $\delta \geq 0$, then we define

$$\omega_r(X; f, \delta) = \sup\{\|\Delta_t^r(f)\|_X : |t| \leq \delta\},$$

which is called the r -th modulus of continuity of f with respect to $\{T_t : t \in \mathbb{R}\}$. This quantity has the following properties ([15; Lemma 1]):

Lemma 1. *Let $r \in \mathbb{N}$ and $f \in X$.*

- (a) $\omega_r(X; f, \delta) \leq 2^r \|f\|_X \quad (\delta \geq 0)$.
- (b) $\omega_r(X; f, \cdot)$ is a non-decreasing function on $[0, \infty)$ and $\omega_r(X; f, 0) = 0$.
- (c) $\omega_{r+s}(X; f, \delta) \leq 2^r \omega_s(X; f, \delta) \quad (s \in \mathbb{N}_0, \delta \geq 0)$. In particular, we have

$$\lim_{\delta \rightarrow +0} \omega_r(X; f, \delta) = 0.$$

- (d) $\omega_r(X; f, \xi\delta) \leq (1 + \xi)^r \omega_r(X; f, \delta) \quad (\xi, \delta \geq 0)$.

(e) If $0 < \delta \leq \xi$, then

$$\omega_r(X; f, \xi)/\xi^r \leq 2^r \omega_r(X; f, \delta)/\delta^r.$$

(f) If $f \in D(G^r)$, then

$$\omega_{r+s}(X; f, \delta) \leq \delta^r \omega_s(X; G^r(f), \delta) \quad (s \in \mathbb{N}_0, \delta \geq 0).$$

(g) $\omega_r(X; \cdot, \delta)$ is a seminorm on X for each $\delta \geq 0$.

For each non-negative integer n , let M_n be the linear span of $\{P_j(X) : |j| \leq n\}$, and we define

$$E_n(X; f) = \inf\{\|f - g\|_X : g \in M_n\},$$

which is called the best approximation of degree n to f with respect to M_n . Then we have

$$E_0(X; f) \geq E_1(X; f) \geq \cdots \geq E_n(X; f) \geq E_{n+1}(X; f) \geq \cdots \geq 0,$$

and Condition (P-2) implies that

$$\lim_{n \rightarrow \infty} E_n(X; f) = 0 \quad \text{for every } f \in X.$$

In order to achieve our purpose, we need the following Bernstein-type inequality ([16; Lemma 5]), which plays an important role in the derivation of certain smoothness properties of an element $f \in X$ from the hypothesis that the sequence $\{E_n(X; f)\}$ tends to zero with a given rapidity.

Lemma 2. *Let $n \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Then*

$$\|G^r(f)\|_X \leq (2n)^r \|f\|_X$$

holds for all $f \in M_n$.

3. The main theorem

Here we suppose that for each given $f \in X$ and each $n \in \mathbb{N}_0$, there exists an element $f_n \in M_n$ of the best approximation of f with respect to M_n , i.e., such that

$$(1) \quad E_n(X; f) = \|f - f_n\|_X.$$

Remark 1. If the dimension of M_n is finite, then every element $f \in X$ has an element of the best approximation with respect to M_n . Also, if X is a uniformly convex Banach space, then every $f \in X$ has a unique element of the best approximation with respect to M_n . In particular, if X is a Hilbert space, then for each $f \in X$ there exists a unique element of the best approximation of f with respect to M_n . For the general theory of the best approximation in normed linear spaces, we refer to [19].

Let $a \in \mathbb{N}$, $a \geq 2$ and let $\Omega \neq 0$ be a non-negative, monotone decreasing function on $[a, \infty]$ satisfying the following conditions

$$(2) \quad \lim_{x \rightarrow \infty} \Omega(x) = 0$$

and

$$(3) \quad \int_a^\infty \frac{\Omega(x)}{x} dx < \infty.$$

Let φ be a non-negative, bounded function on $[a, \infty)$ with $\lim_{x \rightarrow \infty} \varphi(x) = 0$, and we define

$$(4) \quad \varphi^*(x) = \sup\{\varphi(t) : x \leq t\} \quad (x \geq a),$$

which is a non-negative, monotone decreasing function and $\lim_{x \rightarrow \infty} \varphi^*(x) = 0$. Obviously, if $\varphi(x)$ monotonously decreases with x on $[a, \infty)$, then $\varphi^*(x) = \varphi(x)$ for all $x \geq a$.

Theorem 1. Let $f \in X$ and $r \in \mathbb{N}_0$. Suppose that

$$(5) \quad E_n(X; f) \leq \varphi(n) \frac{\Omega(n)}{n^r} \quad \text{for all } n \geq a.$$

Then $f \in D(G^r)$ and for every $k \in \mathbb{N}$,

$$(6) \quad \omega_k(X; G^r(f), \delta) = O \left(\delta^k \int_a^{a/\sqrt{\delta}} x^{k-1} \Omega(x) dx \right. \\ \left. + \varphi^* \left(\frac{a}{\sqrt{\delta}} \right) \left(\delta^k \int_a^{a/\delta} x^{k-1} \Omega(x) dx + \int_{1/\delta}^\infty \frac{\Omega(x)}{x} dx \right) \right) \quad (\delta \rightarrow +0).$$

Proof. Let f_n be an element of the best approximation of f with respect to M_n . Then by (1), (4) and (5), we have

$$(7) \quad \|f - f_{a^n}\|_X \leq \varphi(a^n) \frac{\Omega(a^n)}{a^{nr}} \leq \varphi^*(a^n) \frac{\Omega(a^n)}{a^{nr}} \quad (n \geq 1).$$

Put

$$(8) \quad g_2 = f_{a^2}, \quad g_n = f_{a^n} - f_{a^{n-1}} \quad (n \geq 3).$$

Then it follows from (7) that

$$\|g_n\|_X \leq \|f_{a^n} - f\|_X + \|f - f_{a^{n-1}}\|_X \leq (1 + a^r) \frac{\varphi^*(a^{n-1}) \Omega(a^{n-1})}{a^{nr}}$$

for all $n \geq 3$, and so Lemma 2 yields

$$(9) \quad \|G^r(g_n)\|_X \leq 2^r (1 + a^r) \varphi^*(a^{n-1}) \Omega(a^{n-1}) \quad (n \geq 3).$$

By (2), we have

$$\sum_{m=3}^{\infty} \Omega(a^{n-1}) \leq \frac{a}{a-1} \int_a^{\infty} \frac{\Omega(x)}{x} dx < \infty,$$

which together with (9) implies that there exists an element $g \in X$ such that

$$(10) \quad g = \sum_{n=2}^{\infty} G^r(g_n).$$

Also, (2), (7) and (8) imply

$$(11) \quad f = \sum_{n=2}^{\infty} g_n.$$

Since G^r is a closed linear operator, it follows from (10) and (11) that $f \in D(G^r)$ and

$$(12) \quad G^r(f) = \sum_{n=2}^{\infty} G^r(g_n).$$

Now let $0 < \delta < a^{-2}$, and we choose two numbers $m, s \in \mathbb{N}$ such that

$$m, s \geq 3, \quad a^{m-2} \leq \frac{1}{\sqrt{\delta}} < a^{m-1}, \quad a^{s-2} \leq \frac{1}{\delta} < a^{s-1}.$$

Then by (12) and Lemma 1(g), we obtain

$$\begin{aligned} \omega_k(X; G^r(f), \delta) &\leq \sum_{n=2}^m \omega_k(X; G^r(g_n), \delta) \\ &+ \left(\sum_{n=m+1}^s \omega_k(X; G^r(g_n), \delta) + \omega_k \left(X; \sum_{n=s+1}^{\infty} G^r(g_n), \delta \right) \right) \\ &= A + B, \end{aligned}$$

say. By Lemma 1(f), Lemma 2 and (9), we have

$$\begin{aligned} \omega_k(X; G^r(g_n), \delta) &\leq \delta^k \|G^k(G^r(g_n))\|_X \\ &\leq \delta^k (2a^n)^k \|G^r(g_n)\|_X \quad (n \geq 2) \\ &\leq \delta^k (2a^n)^k 2^r (1+a^r) \varphi^*(a^{n-1}) \Omega(a^{n-1}) \\ &\leq \delta^k (2a^n)^k 2^r (1+a^r) \varphi^*(a^2) \Omega(a^{n-1}) \quad (n \geq 3). \end{aligned}$$

Thus we obtain

$$A \leq \delta^k (2a^2)^k \|G^r(g_2)\|_X + 2^{k+r} (1+a^r) \varphi^*(a^2) \delta^k \sum_{n=3}^m a^{kn} \Omega(a^{n-1})$$

$$\leq C_1 \delta^k \sum_{n=2}^m \left(a^{k(n-1)} - a^{k(n-1)-1} \right) \Omega(a^{n-1}),$$

where $C_1 > 0$ is a constant independent of δ and m .

We first consider the case of $\Omega(a^2) > 0$. Then we have

$$(13) \quad \sum_{n=2}^m \left(a^{k(n-1)} - a^{k(n-1)-1} \right) \Omega(a^{n-1}) \leq (a^{2k} - a^{2k-1}) \Omega(a)$$

$$+ \sum_{n=3}^m \int_{a^{k(n-1)-1}}^{a^{k(n-1)}} \Omega(x^{1/k}) dx \leq \frac{\Omega(a)}{\Omega(a^2)} \int_{a^{2k-1}}^{a^{2k}} \Omega(x^{1/k}) dx + \int_{a^{2k-1}}^{a^{k(m-1)}} \Omega(x^{1/k}) dx$$

$$\leq \left(\frac{\Omega(a)}{\Omega(a^2)} + 1 \right) \int_{a^k}^{a^{k(m-1)}} \Omega(x^{1/k}) dx$$

$$\leq \left(\frac{\Omega(a)}{\Omega(a^2)} + 1 \right) \int_{a^k}^{(a/\sqrt{\delta})^k} \Omega(x^{1/k}) dx.$$

Therefore, putting $y = x^{1/k}$, we get

$$A \leq kC_1 \left(\frac{\Omega(a)}{\Omega(a^2)} + 1 \right) \delta^k \int_a^{a/\sqrt{\delta}} y^{k-1} \Omega(y) dy.$$

Next, let us estimate the term B . By Lemma 1(a), (f), Lemma 2 and (9) we have

$$B \leq 2^k \delta^k \sum_{n=m+1}^s a^{kn} \|G^r(g_n)\|_X + 2^k \sum_{n=s+1}^{\infty} \|G^r(g_n)\|_X$$

$$\leq 2^{k+r} (1 + a^r) \left(\delta^k \sum_{n=m+1}^s a^{kn} \varphi^*(a^{n-1}) \Omega(a^{n-1}) + \sum_{n=s+1}^{\infty} \varphi^*(a^{n-1}) \Omega(a^{n-1}) \right)$$

$$\leq 2^{k+r} (1 + a^r) \varphi^*(a^m) \left(\delta^k \sum_{n=m+1}^s a^{kn} \Omega(a^{n-1}) + \sum_{n=s+1}^{\infty} \Omega(a^{n-1}) \right)$$

$$\leq 2^{k+r} (1 + a^r) \varphi^* \left(\frac{a}{\sqrt{\delta}} \right) \left(\delta^k \sum_{n=2}^s a^{kn} \Omega(a^{n-1}) + \sum_{n=s+1}^{\infty} \Omega(a^{n-1}) \right).$$

Proceeding as in the proof of (13), we have

$$\sum_{n=2}^s a^{kn} \Omega(a^{n-1}) = \frac{a^{k+1}}{a-1} \sum_{n=2}^s \left(a^{k(n-1)} - a^{k(n-1)-1} \right) \Omega(a^{n-1})$$

$$\leq \frac{a^{k+1}}{a-1} \left(\frac{\Omega(a)}{\Omega(a^2)} + 1 \right) \int_{a^k}^{a^{k(s-1)}} \Omega(x^{1/k}) dx$$

$$\begin{aligned} &\leq \frac{a^{k+1}}{a-1} \left(\frac{\Omega(a)}{\Omega(a^2)} + 1 \right) \int_{a^k}^{(a/\delta)^k} \Omega(x^{1/k}) dx \\ &= \frac{ka^{k+1}}{a-1} \left(\frac{\Omega(a)}{\Omega(a^2)} + 1 \right) \int_a^{a/\delta} y^{k-1} \Omega(y) dy, \end{aligned}$$

and

$$\sum_{n=s+1}^{\infty} \Omega(a^{n-1}) \leq \frac{a}{a-1} \int_{1/\delta}^{\infty} \frac{\Omega(x)}{x} dx.$$

Therefore, there exists a constant $C_2 > 0$ independent of δ such that

$$B \leq C_2 \varphi^* \left(\frac{a}{\sqrt{\delta}} \right) \left(\delta^k \int_a^{a/\delta} y^{k-1} \Omega(y) dy + \int_{1/\delta}^{\infty} \frac{\Omega(x)}{x} dx \right),$$

which together with the estimate for A establishes the desired equality (6).

In case $\Omega(a^2) = 0$, (7) implies $f = f_{a^2} \in M_{a^2}$, and so Lemma 1(f) and Lemma 2 yield

$$(14) \quad \omega_k(X; G^r(f), \delta) \leq (2a^2)^k \|G^r(f)\|_X \delta^k.$$

Also, we have

$$\int_a^{a/\sqrt{\delta}} x^{k-1} \Omega(x) dx \geq \int_a^{a^{m-1}} x^{k-1} \Omega(x) dx \geq \int_a^{a^2} x^{k-1} \Omega(x) dx > 0,$$

which together with (14) clearly implies (6). The proof of the theorem is complete.

Applying Theorem 1 to the case where

$$\Omega(x) = \frac{1}{x^\alpha}, \quad \alpha > 0,$$

we have the following.

Corollary 1. *Let $\alpha > 0$, $f \in X$ and $r \in \mathbb{N}_0$. If*

$$E_n(X; f) = o\left(\frac{1}{n^{\alpha+r}}\right) \quad (n \rightarrow \infty),$$

then f belongs to $D(G^r)$ and for every $k \in \mathbb{N}$,

$$\omega_k(X; G^r(f), \delta) = \begin{cases} o(\delta^\alpha) & (\alpha < k) \\ o(\delta^k |\log \delta|) & (\alpha = k) \end{cases} \quad (\delta \rightarrow +0).$$

3 Applications to homogeneous Banach spaces

Here we restrict ourselves to the case where X is a homogeneous Banach space, i.e., X satisfies the following conditions:

(H-1) X is a linear subspace of $L_{2\pi}^1$ with a norm $\|\cdot\|_X$ under which it is a Banach space.

(H-2) X is continuously embedded in $L_{2\pi}^1$, i.e., there exists a constant $C > 0$ such that $\|f\|_1 \leq C\|f\|_X$ for all $f \in X$

(H-3) The left translation operator T_t defined by

$$T_t(f)(\cdot) = f(\cdot - t) \quad (f \in X),$$

is isometric on X for each $t \in \mathbb{R}$.

(H-4) For each $f \in X$, the mapping $t \mapsto T_t(f)$ is strongly continuous on \mathbb{R} .

Typical examples of homogeneous Banach spaces are $C_{2\pi}$ and $L_{2\pi}^p$, $1 \leq p < \infty$. For other examples, see [11] (cf. [8], [18]).

Now, we define the sequence $\{P_j : j \in \mathbb{Z}\}$ of projection operators in $B[X]$ by

$$P_j(f)(\cdot) = \hat{f} e^{ij} \quad (f \in X),$$

which satisfies Conditions (P-1), (P-2) and (P-3) just as Section 2 (cf. [8], [11]). Notice that $M_n = \mathcal{T}_n$ and we have

$$\Delta_t^0(f) = f, \quad \Delta_t^r(f)(\cdot) = \sum_{m=0}^r (-1)^{r-m} \binom{r}{m} f(\cdot - mt) \quad (f \in X, t \in \mathbb{R}, r \in \mathbb{N}).$$

Consequently, in the above setting all the results obtained in the preceding sections hold. In particular, Corollary 1 for $k = 2$ establishes the theorem of Zygmund type in arbitrary homogeneous Banach spaces.

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