GENERALIZED FRACTIONAL PROGRAMMING

J. C. LIU
Section of Mathematics, National Overseas Chinese Student University, PO Box 1-1337 Linkou, 24499, Taiwan.

Y. KIMURA
Department of Mathematics and Information Science, Graduate School of Science and Technology, Niigata University 950-21, Niigata, Japan.

and

K. TANAKA
Department of Mathematics, Niigata University, 950-21, Niigata, Japan.

Optimality conditions in generalized fractional programming involving nonsmooth Lipschitz functions are established. Subsequently, these optimality criteria are utilized as a basis for constructing one parametric and two other parametric-free dual models, and several duality theorems are derived.

KEY WORDS: Generalized fractional programming, invex, quasiinvex, pseudoinvex, duality.

1. INTRODUCTION

In this paper, we consider the following minimax fractional programming problem:

$$(P) \quad v^* = \min_{x \in S} \max_{1 \leq i \leq p} \left[ f_i(x)/g_i(x) \right],$$

where

(A1) $S = \{ x \in \mathbb{R}^n; h_k(x) \leq 0, k = 1, 2, \cdots, m \}$ is nonempty and compact;

(A2) $f_i : X_0 \mapsto \mathbb{R}, g_i : X_0 \mapsto \mathbb{R}, i = 1, 2, \cdots, p,$ and $h_k : X_0 \mapsto \mathbb{R}, k = 1, 2, \cdots, m$ are locally Lipschitz continuous and $X_0$ is the open subset of $\mathbb{R}^n$;

(A3) $g_i(x) > 0, i = 1, 2, \cdots, p, x \in S$;

(A4) if $g_i$ is not affine, then $f_i(x) \geq 0$ for all $i$ and all $x \in S$.

Generalized fractional programming has been of much interest in the last decades; see for example [1-4, 6, 7, 10-19]. In [7], Crouzeix et al. have shown that the minimax fractional program can be derived by solving the following minimax nonlinear (nondifferentiable) parametric program:

$$(P_v) \quad \min_{x \in S} \max_{1 \leq i \leq p} \left( f_i(x) - vg_i(x) \right)$$

where $v \in \mathbb{R}_+ \equiv [0, \infty)$ is a parameter.
It is clear that $(P_v)$ is equivalent to the following problem $(EP_v)$ for a given $v$:

$$(EP_v) \quad \min q,$$

subject to \quad \begin{align*}
    f_i(x) - vg_i(x) &\leq q, \quad i = 1, 2, \cdots, p, \\
    h_k(x) &\leq 0, \quad k = 1, 2, \cdots, m.
\end{align*}$$

In [2], Bector et al. employed the problem $(EP_v)$ to prove necessary and sufficient optimality conditions for problem (P) and establish various duality results for problem $(EP_v)$ involving differentiable generalized convex functions (or generalized invex functions). Liu [10-12] also adapted the same approach to obtain necessary and sufficient optimality conditions; and he derived duality theorems for generalized fractional programming problems involving either nonsmooth pseudoinvex functions [11] or nonsmooth $(F, \rho)$-convex functions [10], and duality theorems for generalized fractional variational problems involving generalized $(F, \rho)$-convex functions [12].

But, all of the above necessary optimality conditions and strong duality theorems need that the constraint of $(EP_v)$ satisfy a constraint qualification.

In order to improve this defect, we want to use problem $(P_v)$ to establish both parametric and nonparameter necessary and sufficient optimality conditions, since a constraint qualification that is imposed on the constrains of (P) may not hold for $(EP_v)$ but hold for $(P_v)$. Subsequently, these optimality criteria are utilized as a basis for constructing one parametric and two other parametric-free dual models (see [13] and [16]), and some duality results for (P) are established.

2. NOTATIONS AND PRELIMINARY RESULTS

Throughout this paper, let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant. Let $X_0$ be an open subset of $\mathbb{R}^n$.

**Definition 2.1.** The function $\theta : X_0 \mapsto \mathbb{R}$ is said to be **Lipschitz** on $X_0$ if there exists $c > 0$ such that for all $y, x \in X_0$,

$$|\theta(y) - \theta(x)| \leq c\|y - x\|,$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$.

For each $d$ in $\mathbb{R}^n$, $\theta^\circ(x; d)$ is the **generalized directional derivative** of Clarke [5] defined by

$$\theta^\circ(x; d) = \limsup_{\delta \to 0} \left[ \frac{\theta(y + \delta d) - \theta(y)}{\delta} \right].$$

It then follows that

$$\theta^\circ(x; d) = \max \{ \xi^T d \mid \xi \in \partial \theta(x) \} \quad \text{for any } x \text{ and } d,$$

where $\partial \theta(\cdot)$ denotes the **Clarke's generalized gradient** [5]. The following definitions can be found in [11]:
Definition 2.2. The function $\theta : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be \textbf{invex} at $x^*$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that, for each $x \in \mathbb{R}^n$,

$$\theta(x) - \theta(x^*) \geq \theta^o(x^*; \eta(x, x^*)).$$

(2.1)

$\theta$ is said to be invex on $\mathbb{R}^n$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that, for each $x, u \in \mathbb{R}^n$,

$$\theta(x) - \theta(u) \geq \theta^o(u; \eta(x, u)).$$

(2.2)

If we have strict inequality in (2.1) and (2.2), respectively, then $\theta$ is said to be \textbf{strictly invex} at $x^*$ with respect to $\eta$ and strictly invex on $\mathbb{R}^n$ with respect to $\eta$, respectively.

Definition 2.3. The function $\theta : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be \textbf{quasiinvex} at $x^*$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that, for each $x \in \mathbb{R}^n$,

$$\theta(x) \leq \theta(x^*) \Rightarrow \theta^o(x^*; \eta(x, x^*)) \leq 0.$$

(2.3)

$\theta$ is said to be quasiinvex on $\mathbb{R}^n$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that, for each $x, u \in \mathbb{R}^n$,

$$\theta(x) \leq \theta(u) \Rightarrow \theta^o(u; \eta(x, u)) \leq 0.$$

(2.4)

If we have strict inequality in (2.3) and (2.4), respectively, then $\theta$ is said to be \textbf{strictly quasiinvex} at $x^*$ with respect to $\eta$ and strictly quasiinvex on $\mathbb{R}^n$ with respect to $\eta$, respectively.

Definition 2.4. The function $\theta : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be \textbf{pseudoinvex} at $x^*$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that, for each $x \in \mathbb{R}^n$,

$$\theta^o(x^*; \eta(x, x^*)) \geq 0 \Rightarrow \theta(x) \geq \theta(x^*).$$

(2.5)

$\theta$ is said to be pseudoinvex on $\mathbb{R}^n$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that, for each $x, u \in \mathbb{R}^n$,

$$\theta^o(u; \eta(x, u)) \geq 0 \Rightarrow \theta(x) \geq \theta(u).$$

(2.6)

If we have strict inequality in (2.5) and (2.6), respectively, then $\theta$ is said to be \textbf{strictly pseudoinvex} at $x^*$ with respect to $\eta$ and strictly pseudoinvex on $\mathbb{R}^n$ with respect to $\eta$, respectively.

We need the following lemmas.

Lemma 2.1. [16, Lemma 3.1.] Let $v^*$ be the optimal value of $(P)$, and let $V(v)$ be the optimal value of $(P_v)$ for any fixed $v \in \mathbb{R}_+$ such that $(P_v)$ has an optimal solution. Then $x^*$ is an optimal solution of $(P)$ if and only if $x^*$ is an optimal solution of $(P_{v^*})$ with optimal value $V(v^*) = 0$. 

Lemma 2.2. [5, Proposition 2.3.12.] Let $f_1, \cdots, f_p$ be Lipschitz functions at $x^*$ and $\alpha_i \in \mathbb{R}$ for all $i = 1, \cdots, p$. Then

1. \[ \partial (\sum_{i=1}^{p} \alpha_i f_i)(x^*) \subset \sum_{i=1}^{p} \alpha_i \partial f_i(x^*), \]
2. \[ \partial \left[ \max_{1 \leq i \leq p} f_i \right](x^*) \subset \bigcup \left\{ \sum_{l \in L} \alpha_l \partial f_l(lx^*); \alpha_l \geq 0, \sum_{l \in L} \alpha_l = 1 \right\} \]

where $L$ is the set of indices $l$ for which

\[ f_l(x^*) = \max_{1 \leq i \leq p} f_i(x^*). \]

Lemma 2.3. [16, Lemma 3.2.] For each $x \in S$, one has

\[ \phi(x) \equiv \max_{1 \leq i \leq p} \left( \frac{f_i(x)}{g_i(x)} \right) = \max_{\beta \in U} \left( \sum_{i=1}^{p} \beta_i f_i(x) / \sum_{i=1}^{p} \beta_i g_i(x) \right) \]

where $U = \{ \beta \in \mathbb{R}_{+}^{p} | \sum_{i=1}^{p} \beta_i = 1 \}$.

For convenience, we give the scalar minimization problem as follows:

\[(SP) \quad \text{Minimize} \quad N(x), \]
subject to \[ h_k(x) \leq 0, \quad k = 1, 2, \cdots, m \]

where $N, h_k : X_0 \mapsto \mathbb{R}, k = 1, 2, \cdots, m$, are Lipschitz on $X_0$. We need the following lemma.

Lemma 2.4. [8, Theorem 6.] If $x^* \in X_0$ is a local minimum for $(SP)$ and a constraint qualification is satisfied, then there exist $z^* = (z_1^*, \cdots, z_m^*) \in \mathbb{R}_{+}^{m}$ such that

\[ 0 \in \partial N(x^*) + \sum_{k=1}^{m} z_k^* \partial h_k(x^*), \]
\[ z_k^* h_k(x^*) = 0, \quad \text{for all} \quad k = 1, 2, \cdots, m. \]

For simplicity, throughout the paper we denote

\[ U = \{ \alpha \in \mathbb{R}_{+}^{p} | \sum_{i=1}^{p} \alpha_i = 1 \}, \]
\[ F(x) = (f_1(x), \cdots, f_p(x)), \]
\[ G(x) = (g_1(x), \cdots, g_p(x)), \] and
\[ H(x) = (h_1(x), \cdots, h_m(x)). \]

For $z \in \mathbb{R}^m$, \[ z^T H(x^*) = \sum_{k=1}^{m} z_k h_k(x^*), \] and \[ \partial(z^T H)(x^*) = \sum_{k=1}^{m} z_k \partial h_k(x^*). \]

3. NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS

In this section, we shall use Lemmas 2.1 ~ 2.4 to establish some necessary and sufficient optimality conditions for the minimax fractional programming problem $(P)$. 

Theorem 3.1 (Necessary optimality conditions). Let \( x^* \in S \). If \( x^* \) is an optimal solution of (P) and that the constraint of (P) satisfy Slater’s constraint qualification [8]. Then there exist \( v^* = \phi(x^*) \in \mathbb{R}_+ \), \( y^* \in U \), \( z^* \in \mathbb{R}_+^m \) such that

\[
0 \in \partial(y^* \mathbf{T} F(x^*)) - v^* \partial(y^* \mathbf{T} G(x^*)) + \partial(z^* \mathbf{T} H(x^*)), \tag{3.1}
\]

\[
y^* \mathbf{T} F(x^*) - v^* y^* \mathbf{T} G(x^*) = 0, \tag{3.2}
\]

\[
z^* \mathbf{T} H(x^*) = 0. \tag{3.3}
\]

Proof. If \( x^* \) is an optimal solution of (P), by Lemma 2.1, it is an optimal solution of \((P_{v^*})\) with \( v^* = \max_{1 \leq i \leq p}[f_i(x^*)/g_i(x^*)] \). Thus, by Lemma 2.4, there exist \( z^* \in \mathbb{R}_+^m \), such that

\[
0 \in \partial \left( \max_{1 \leq i \leq p} (f_i - v^* g_i) \right)(x^*) + \partial(z^* \mathbf{T} H(x^*))
\]

and

\[
z^* \mathbf{T} H(x^*) = 0.
\]

Therefore, by Lemma 2.2, there exist \( \alpha_i \geq 0 \), \( l \in L \), \( \sum_{l \in L} \alpha_i = 1 \), such that

\[
0 \in \sum_{l \in L} \alpha_i (\partial f_i(x^*) + v^* \partial(-g_i(x^*))) + \partial(z^* \mathbf{T} H(x^*)), \tag{3.4}
\]

It is obvious that \( v^* = \max_{1 \leq i \leq p}[f_i(x^*)/g_i(x^*)] \) if and only if \( \max_{1 \leq i \leq p}[f_i(x^*) - v^* g_i(x^*)] = 0 \). From (3.4), if we set \( y^*_i = \alpha_i \) for \( i \in L \) as well as \( y^*_i = 0 \) for \( i \in \{1, 2, \cdots, p\} \setminus L \), the expressions (3.1), (3.2) and (3.3) hold.

In order to construct parameter-free duality models for problem (P), we shall formulate parameter-free versions of Theorem 3.1 as follows:

Theorem 3.2. Let \( x^* \in S \). If \( x^* \) is an optimal solution of (P) and that the constraint of (P) satisfy Slater’s constraint qualification [8]. Then there exist \( y^* \in U \) and \( z^* \in \mathbb{R}_+^m \) such that

\[
0 \in y^* \mathbf{T} G(x^*) \left( \partial(y^* \mathbf{T} F(x^*)) + \partial(z^* \mathbf{T} H(x^*)) \right) - y^* \mathbf{T} F(x^*) \partial(y^* \mathbf{T} G(x^*)), \tag{3.5}
\]

\[
z^* \mathbf{T} H(x^*) = 0, \tag{3.6}
\]

and obtain the optimal value by

\[
\phi(x^*) = y^* \mathbf{T} F(x^*)/y^* \mathbf{T} G(x^*) = \max_{1 \leq i \leq p} (f_i(x^*)/g_i(x^*)). \tag{3.7}
\]

Proof. From (3.2) and (3.1), substituting \( y^* \mathbf{T} F(x^*)/y^* \mathbf{T} G(x^*) \) for \( v^* \), we can derive the results.

The conditions (3.5) \( \sim \) (3.7) will be the sufficient optimality condition which we state as the following theorem.
Theorem 3.3 (Sufficient optimality conditions). Let \( x^* \in S \), and assume that there exist \( y^* \in U \) and \( z^* \in \mathbb{R}_+^m \), such that the conditions (3.5) \( \sim \) (3.7) hold. Let
\[
A(x) = y^*\mathbf{T}G(x^*) y^*\mathbf{T}F(x) - y^*\mathbf{T}F(x^*) y^*\mathbf{T}G(x),
\]
\[
B(x) = z^*\mathbf{T}H(x), \quad \text{and} \quad C(x) = A(x) + y^*\mathbf{T}G(x^*) B(x).
\]

If any one of the following conditions holds

(a) \( A \) is pseudoinvex at \( x^* \) with respect to \( \eta \) and \( B \) is quasiinvex at \( x^* \) with respect to same function \( \eta \),

(b) \( A \) is quasiinvex at \( x^* \) with respect to \( \eta \) and \( B \) is strictly pseudoinvex at \( x^* \) with respect to same function \( \eta \),

(c) \( C \) is pseudoinvex at \( x^* \) with respect to \( \eta \).

Then \( x^* \) is an optimal solution of \( (P) \).

Proof. Suppose contrary that \( x^* \) were not an optimal solution of \( (P) \). Then there exists a feasible solution \( x_1 \in S \) such that
\[
\phi(x^*) > \phi(x_1).
\]

From (3.7) and Lemma 2.3, we have
\[
y^*\mathbf{T}F(x^*)/y^*\mathbf{T}G(x^*) > \max_{\beta \in U}(\beta^\mathbf{T}F(\beta x_1)/\beta^\mathbf{T}G(x_1)) \geq y^*\mathbf{T}F(x_1)/y^*\mathbf{T}G(x_1).
\]

It follows that
\[
A(x_1) = y^*\mathbf{T}G(x^*) y^*\mathbf{T}F(x_1) - y^*\mathbf{T}F(x^*) y^*\mathbf{T}G(x_1) < 0 = A(x^*) \quad (3.8)
\]

Using both the feasibility \( x_1 \) for \( (P) \) and the equality (3.6), we have
\[
B(x_1) \leq 0 = B(x^*) \quad (3.9)
\]

Consequently, expressions (3.8) and (3.9) yield
\[
C(x_1) < C(x^*) \quad (3.10)
\]

By (3.5), there exist \( \xi \in \partial(y^*\mathbf{T}F)(x^*) \), \( \zeta \in \partial(z^*\mathbf{T}H)(x^*) \), and \( \rho \in \partial(-y^*\mathbf{T}G)(x^*) \), such that
\[
y^*\mathbf{T}G(x^*) (\xi + \zeta) + y^*\mathbf{T}F(x^*) \rho = 0.
\]

From here it results
\[
y^*\mathbf{T}G(x^*) (\xi^\mathbf{T}\eta(x, x^*) + \zeta^\mathbf{T}\eta(x, x^*)) + y^*\mathbf{T}F(x^*) \rho^\mathbf{T}\eta(x, x^*) = 0 \quad (3.11)
\]

Using the characterization of the generalized gradient of Clarke, we obtain
\[
(y^*\mathbf{T}F)^\circ(x^*; \eta(x, x^*)) \geq \xi^\mathbf{T}\eta(x, x^*), \quad \text{for all} \quad x \in S, \quad (3.12)
\]
\[
(z^*\mathbf{T}H)^\circ(x^*; \eta(x, x^*)) \geq \zeta^\mathbf{T}\eta(x, x^*), \quad \text{for all} \quad x \in S, \quad (3.13)
\]
\[ (-y^*G)^\circ(x^*; \eta(x, x^*)) \geq \rho^T \eta(x, x^*), \text{ for all } x \in S. \quad (3.14) \]

Now, multiplying (3.12) by \( y^*G(x^*), \) (3.13) by \( y^*G(x^*), \) and (3.14) by \( y^*F(x^*), \) and adding the resulting inequalities and with (3.11), we obtain
\[
y^*G(x^*)[ (y^*F)^\circ(x^*; \eta(x_1, x^*)) + (z^*H)^\circ(x^*; \eta(x_1, x^*)) ]
- y^*F(x^*)(y^*G)^\circ(x^*; \eta(x_1, x^*)) \geq 0, \text{ for all } x \in S.
\]
\[ (3.15) \]

If hypothesis (a) holds, using the pseudoinvexity of \( A \) at \( x^* \) and the inequality (3.8), we have
\[
y^*G(x^*)(y^*F)^\circ(x^*; \eta(x_1, x^*)) - y^*F(x^*)(y^*G)^\circ(x^*; \eta(x_1, x^*)) < 0.
\]
\[ (3.16) \]

Consequently, the inequalities (3.15) and (3.16) yield
\[
y^*G(x^*)(z^*H)^\circ(x^*; \eta(x_1, x^*)) > 0.
\]

Thus, we have
\[
(z^*H)^\circ(x^*; \eta(x_1, x^*)) > 0.
\]
\[ (3.17) \]

Using the quasiinvexity of \( B \) at \( x^* \), we get from (3.17)
\[
B(x_1) = z^*H(x_1) > z^*H(x^*) = B(x^*)
\]
which contradicts the inequality (3.9).

Hypothesis (b) follows along with the same lines as (a).

If hypothesis (c) holds, using the pseudoinvexity of \( C \) at \( x^* \) and the inequality (3.10), we have
\[
y^*G(x^*)[ (y^*F)^\circ(x^*; \eta(x_1, x^*)) + (z^*H)^\circ(x^*; \eta(x_1, x^*)) ]
- y^*F(x^*)(y^*G)^\circ(x^*; \eta(x_1, x^*)) < 0
\]
which contradicts the inequality (3.15). Hence, the proof is complete. \( \square \)

4. THE FIRST DUAL MODEL

Utilize Theorem 3.2, in Sections 4 and 5 we shall introduce two parametric-free dual models and prove appropriate duality theorems. Indeed, we shall demonstrate that the following is dual problem for (P):
\[
(DI) \quad \text{Maximize } \frac{(y^TF(u) + z^TH(u))}{y^TG(u)}
\]
subject to
\[
0 \in y^TG(u)(\partial(y^TF)(u) + \partial(z^TH)(u))
- (y^TF(u) + z^TH(u))\partial(y^TG)(u), \quad (4.1)
\]
y \in U, \ z \in \mathbb{R}_+^m. \quad (4.2)

We denote by \( K_1 \) the set of all feasible solutions \((u, y, z) \in X_0 \times U \times \mathbb{R}_+^m\) of problem (DI). We assume throughout this section that \( y^TF(u) + z^TH(u) \geq 0 \) and \( y^TG(u) > 0. \)
Theorem 4.1 (Weak Duality). Let $x \in S$ and $(u, y, z) \in K_1$ and assume that

$$D(\cdot) = y^T G(u)[y^T F(\cdot) + z^T H(\cdot)] - y^T G(u)[y^T F(u) + z^T H(u)]$$

is a pseudoinvex function with respect to $\eta$ at $u$. Then

$$\phi(x) \geq \left( y^T F(u) + z^T H(u) \right) / y^T G(u).$$

Proof. By (4.1), there exist $\xi \in \partial(y^T F)(u)$, $\zeta \in \partial(z^T H)(u)$, and $\rho \in \partial(-y^T G)(u)$, such that

$$y^T G(u)(\xi^T \eta(x, u) + \zeta^T \eta(x, u)) + [y^T F(u) + z^T H(u)]\rho = 0.$$ 

From here it results

$$y^T G(u)(\xi \eta(x, u) + \zeta \eta(x, u)) + [y^T F(u) + z^T H(u)]\rho \eta(x, u) = 0. \tag{4.3}$$

Using the characterization of the generalized gradient of Clarke, we obtain

$$(y^T F)^{o}(u; \eta(x, u)) \geq \xi^T \eta(x, u), \quad \text{for all } x \in S, \tag{4.4}$$

$$(z^T H)^{o}(u; \eta(x, u)) \geq \zeta^T \eta(x, u), \quad \text{for all } x \in S, \tag{4.5}$$

$$(-y^T G)^{o}(u; \eta(x, u)) \geq \rho^T \eta(x, u), \quad \text{for all } x \in S. \tag{4.6}$$

Now, multiplying (4.4) by $y^T G(u)$, (4.5) by $y^T G(u)$, and (4.6) by $y^T F(u) + z^T H(u)$, and adding the resulting inequalities and with (4.3), we obtain

$$y^T G(u)((y^T F)(u; \eta(x, u)) + (z^T H)^{o}(u; \eta(x, u))] - [y^T F(u) + z^T H(u)](y^T G)(u; \eta(x, u)) \geq 0, \quad \text{for all } x \in S. \tag{4.7}$$

We suppose that

$$\phi(x) < \left( y^T F(u) + z^T H(u) \right) / y^T G(u).$$

Then, by Lemma 2.3 and $y \in U$, we have

$$y^T F(x) / y^T G(x) < \left( y^T F(u) + z^T H(u) \right) / y^T G(u).$$

Thus, we have

$$y^T G(u)y^T F(x) - y^T G(x)[y^T F(u) + z^T H(u)] < 0.$$ 

Hence, we have another inequality

$$y^T G(u)[y^T F(x) + z^T H(x)] - y^T G(x)[y^T F(u) + z^T H(u)] < y^T G(u)z^T H(x).$$

Using the fact $y^T G(u) > 0$, $z^T H(x) \leq 0$, and the latest inequality, we have

$$D(x) < 0 = D(u).$$

Using the fact that $D(\cdot)$ is a pseudoinvex function with respect to $\eta$ at $u$, we have

$$y^T G(u)((y^T F)^{o}(u; \eta(x, u)) + (z^T H)^{o}(u; \eta(x, u))] - [y^T F(u) + z^T H(u)](y^T G)^{o}(u; \eta(x, u)) < 0$$

which contradicts the inequality (4.7). Hence, the proof is complete. $\square$
Theorem 4.2 (Strong Duality). If \(x^*\) is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification \([8]\). Then there exist \(y^* \in U\) and \(z^* \in \mathbb{R}^m_+\), such that \((x^*, y^*, z^*)\) is a feasible solution of (DI). Furthermore, if the conditions of Theorem 4.1 hold for all feasible solutions of (DI), then \((x^*, y^*, z^*)\) is an optimal solution of (DI) and the optimal values of (P) and (DI) are equal; that is, \(\min(P) = \max(DI)\).

Proof. By Theorem 3.2, there exist \(y^* \in U\), and \(z^* \in \mathbb{R}^m_+\), such that \((x^*, y^*, z^*)\) is a feasible solution of (DI). Furthermore,

\[
\left( y^*^T F(x^*) + z^*^T H(x^*) \right) / y^*^T G(x^*) = y^*^T F(x^*) / y^*^T G(x^*) = \phi(x^*).
\]

Thus, optimality of \((x^*, y^*, z^*)\) for (DI) follows from Theorem 4.1.

\[
\square
\]

Theorem 4.3 (Strict Converse Duality). Let \(x_1\) and \((x^*, y_0, z_0)\) be optimal solutions of (P) and (DI), respectively, and assume that the assumptions of Theorem 4.2 are fulfilled. If

\[
D(\cdot) = y_0^T G(x^*) [y_0^T F(\cdot) + z_0^T H(\cdot)] - y_0^T G(\cdot) [y_0^T F(x^*) + z_0^T H(x^*)]
\]

is a strictly pseudoinverse function with respect to \(\eta\), then \(x_1 = x^*\); that is, \(x^*\) is an optimal solution of (P) with the same optimal values \(\phi(x_1) = (y_1^T F(x_1) + z^T H(x_1))/y_1^T G(x_1)\).

Proof. Suppose, on the contrary, that \(x_1 \neq x^*\). From Theorem 4.2 we know that there exist \(y_1 \in U\) and \(z_1 \in \mathbb{R}^m_+\), such that \((x_1, y_1, z_1)\) is an optimal solution of (DI) and

\[
\phi(x_1) = (y_1^T F(x_1) + z_1^T H(x_1))/y_1^T G(x_1).
\]

Now proceeding as in the proof of Theorem 4.1 (replacing \(x\) by \(x_1\) and \((u, y, z)\) by \((x^*, y_0, z_0)\)), we arrive at the following strict inequality:

\[
\phi(x_1) > (y_0^T F(x^*) + z^T H(x^*)) / y_0^T G(x^*).
\]

This contradicts the fact that

\[
\phi(x_1) = (y_1^T F(x_1) + z_1^T H(x_1)) / y_1^T G(x_1) = (y_0^T F(x^*) + z_0^T H(x^*)) / y_0^T G(x^*).
\]

Therefore, we conclude that

\[
x_1 = x^*, \quad \text{and} \quad \phi(x_1) = (y_0^T F(x^*) + z_0^T H(x^*)) / y_0^T G(x^*).
\]

\[
\square
\]

5. SECOND DUAL MODEL
We shall continue our discussion of parameter-free duality model for (P) in this section by showing that the following problem (DII) is also dual problem for (P):

\[
\begin{align*}
(DII) \quad \text{Maximize} & \quad y^\top F(u)/y^\top G(u) \\
\text{subject to} & \quad 0 \in y^\top G(u) \left( \partial (y^\top F)(u) + \partial (z^\top H)(u) \right) \\
& \quad - y^\top F(u) \partial (y^\top G)(u), \\
& \quad z^\top H(u) \geq 0, \\
& \quad y \in U, \quad z \in \mathbb{R}_+^m.
\end{align*}
\]

(5.1)

We denote by \( K_2 \) the set of all feasible solutions \((u, y, z) \in X_0 \times U \times \mathbb{R}_+^m\) of problem (DII). Throughout this section, we assume that \( y^\top F(u) \geq 0 \) and \( y^\top G(u) > 0 \). Then, we can prove the following weak duality, strong duality, and strict converse duality theorems.

**Theorem 5.1 (Weak Duality).** Let \( x \in S \) and \((u, y, z) \in K_2 \) and let

\[
E(\cdot) = y^\top G(u) y^\top F(\cdot) - y^\top F(u) y^\top G(\cdot), \\
I(\cdot) = z^\top H(\cdot), \quad \text{and} \quad J(\cdot) = E(\cdot) + y^\top G(u) I(\cdot).
\]

If any one of the following conditions holds

(a) \( E \) is a pseudoinvex function with respect to \( \eta \) at \( u \) and \( I \) is a quasiinvex function at \( u \) with respect to same function \( \eta \),

(b) \( E \) is a quasiinvex function with respect to \( \eta \) at \( u \) and \( I \) is a strictly pseudoinvex function at \( u \) with respect to same function \( \eta \),

(c) \( J \) is a pseudoinvex function with respect to \( \eta \) at \( u \).

Then

\[
\phi(x) \geq y^\top F(u)/y^\top G(u).
\]

**Theorem 5.2 (Strong Duality).** If \( x^* \) is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist \( y^* \in U \) and \( z^* \in \mathbb{R}_+^m \), such that \((x^*, y^*, z^*)\) is a feasible solution of (DII). Furthermore, if the conditions of Theorem 5.1 hold for all feasible solutions of (DII), then \((x^*, y^*, z^*)\) is an optimal solution of (DII) and the optimal values of (P) and (DII) are equal; that is, \( \min(P) = \max(DII) \).

**Theorem 5.3 (Strict Converse Duality).** Let \( x_1 \) and \((x^*, y_0, z_0)\) be optimal solutions of (P) and (DII), respectively, and assume that the assumptions of Theorem 5.2 are fulfilled. If \( E(\cdot) = y_0^\top G(x^*) y_0^\top F(\cdot) - y_0^\top F(x^*) y_0^\top G(\cdot) \) is a strictly pseudoinvex function with respect to \( \eta \) and \( I(\cdot) = z_0^\top H(\cdot) \) is a quasiinvex function with respect to same function \( \eta \), then \( x_1 = x^* \); that is, \( x^* \) is an optimal solution of (P) with the same optimal values \( \phi(x_1) = y_0^\top F(x^*)/y_0^\top G(x^*) \).

6. THE THIRD DUAL MODEL
Making use of Theorem 3.1, in this section we can formulate the following parametric dual problem:

\[(DIII) \text{ Maximize } v \]

subject to \(0 \in \partial (y^T F)(u) - v \partial (y^T G)(u) + \partial (z^T H)(u),\) \hspace{1cm} (6.1)

\(y^T F(u) - vy^T G(u) \geq 0,\) \hspace{1cm} (6.2)

\(z^T H(u) \geq 0,\) \hspace{1cm} (6.3)

\(y \in U, \ v \in \mathbb{R}_+, \ z \in \mathbb{R}^m_+.\) \hspace{1cm} (6.4)

We denote by \(K_3\) the set of all feasible solutions \((u, y, z, v) \in X_0 \times U \times \mathbb{R}^m_+ \times \mathbb{R}_+\) of problem (DIII). Then a weakly duality theorem is established as follows:

**Theorem 6.1 (Weak Duality).** Let \(x \in S\) and \((u, y, z, v) \in K_3\), and let

\[L(\cdot) = y^T F(\cdot) - vy^T G(\cdot),\]

\[I(\cdot) = z^T H(\cdot),\]

and \(M(\cdot) = L(\cdot) + I(\cdot).\)

If any one of the following conditions holds

(a) \(L\) is a pseudoinvex function with respect to \(\eta\) at \(u\) and \(I\) is a quasiinvex function at \(u\) with respect to same function \(\eta\),

(b) \(L\) is a quasiinvex function with respect to \(\eta\) at \(u\) and \(I\) is a strictly pseudoinvex function at \(u\) with respect to same function \(\eta\),

(c) \(M\) is a pseudoinvex function with respect to \(\eta\) at \(u\).

Then

\[\phi(x) \geq v.\]

**Theorem 6.2 (Strong Duality).** If \(x^*\) is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist \(y^* \in U, \ z^* \in \mathbb{R}^m_+, \) and \(v^* \in \mathbb{R}_+\), such that \((x^*, y^*, z^*, v^*)\) is a feasible solution of (DIII). Furthermore, if the conditions of Theorem 6.1 hold for all feasible solutions of (DIII), then \((x^*, y^*, z^*, v^*)\) is an optimal solution of (DIIl) and the optimal values of (P) and (DIII) are equal; that is, \(\min(P) = \max(DIII)\).

**Theorem 6.3 (Strict Converse Duality).** Let \(x_1\) and \((x^*, y_0, z_0, v_0)\) be optimal solutions of (P) and (DIII), respectively, and assume that the assumptions of Theorem 6.2 are fulfilled. If \(y_0^T F(\cdot) - v_0y_0^T G(\cdot)\) is a strictly pseudoinvex function with respect to \(\eta\) and \(I(\cdot) = z_0^T H(\cdot)\) is a quasiinvex function with respect to same function \(\eta\), then \(x_1 = x^*;\) that is, \(x^*\) is an optimal solution of (P) with the same optimal values \(\phi(x_1) = v_0.\)

The complete proof of Theorems 5.1-5.3 and Theorems 6.1-6.3 will be appear elsewhere.

**7. SOME REMARKS FOR FURTHER DEVELOPMENTS**
(1) There some questions arise that whether the results develop in this paper hold in generalized $(F, \rho)$-convex?

(2) Does the set $I = \{1, 2, \cdots, p\}$ in the minimax fractional programming (P) can be replaced by a compact subset $Y$ of $\mathbb{R}^m$? that is, does one can discuss the following minimax fractional programming:

\[
\text{Minimize } \quad F(x) = \sup_{y \in Y} \frac{f(x, y)}{g(x, y)} = \sup_{y \in Y} \Psi(x, y)
\]

subject to \quad $h(x) \leq 0$,

where $Y$ is a compact subset of $\mathbb{R}^m$?

(3) Do we can discuss this minimax fractional programming in two person game theory?

REFERENCES


