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GENERALIZED FRACTIONAL PROGRAMMING

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Optimality conditions in generalized fractional programming involving nonsmooth Lipschitz functions are established. Subsequently, these optimality criteria are utilized as a basis for constructing one parametric and two other parametric-free dual models, and several duality theorems are derived.

KEY WORDS: Generalized fractional programming, invex, quasiinvex, pseudoinvex, duality.

1. INTRODUCTION

In this paper, we consider the following minimax fractional programming problem:

\[(P) \quad v^* = \min_{x \in S} \max_{1 \leq i \leq p} \left[ \frac{f_i(x)}{g_i(x)} \right],\]

where

(A1) \( S = \{ x \in \mathbb{R}^n; h_k(x) \leq 0, k = 1, 2, \ldots, m \} \) is nonempty and compact;

(A2) \( f_i : X_0 \mapsto \mathbb{R}, g_i : X_0 \mapsto \mathbb{R}, i = 1, 2, \ldots, p, \) and \( h_k : X_0 \mapsto \mathbb{R}, k = 1, 2, \ldots, m \) are locally Lipschitz continuous and \( X_0 \) is the open subset of \( \mathbb{R}^n; \)

(A3) \( g_i(x) > 0, i = 1, 2, \ldots, p, \ x \in S; \)

(A4) if \( g_i \) is not affine, then \( f_i(x) \geq 0 \) for all \( i \) and all \( x \in S. \)

Generalized fractional programming has been of much interest in the last decades; see for example [1-4, 6, 7, 10-19]. In [7], Crouzeix et al. have shown that the minimax fractional program can be derived by solving the following minimax nonlinear (nondifferentiable) parametric program:

\[(P_v) \quad \min_{x \in S} \max_{1 \leq i \leq p} (f_i(x) - v g_i(x))\]

where \( v \in \mathbb{R}_+ \equiv [0, \infty) \) is a parameter.
It is clear that \((P_v)\) is equivalent to the following problem \((EP_v)\) for a given \(v:\)

\[(EP_v) \quad \min q, \quad \begin{array}{l}
\text{subject to} \\
f_i(x) - v g_i(x) \leq q, \quad i = 1, 2, \cdots, p, \\
h_k(x) \leq 0, \quad k = 1, 2, \cdots, m.
\end{array}\]

In [2], Bector et al. employed the problem \((EP_v)\) to prove necessary and sufficient optimality conditions for problem \((P)\) and establish various duality results for problem \((EP_v)\) involving differentiable generalized convex functions (or generalized invex functions). Liu [10-12] also adapted the same approach to obtain necessary and sufficient optimality conditions; and he derived duality theorems for generalized fractional programming problems involving either nonsmooth pseudoinvex functions [11] or non-smooth \((F, \rho)\)-convex functions [10], and duality theorems for generalized fractional variational problems involving generalized \((F, \rho)\)-convex functions [12].

But, all of the above necessary optimality conditions and strong duality theorems need that the constraint of \((EP_v)\) satisfy a constraint qualification.

In order to improve this defect, we want to use problem \((P_v)\) to establish both parametric and nonparameter necessary and sufficient optimality conditions, since a constraint qualification that is imposed on the constrains of \((P)\) may not hold for \((EP_v)\) but hold for \((P_v)\). Subsequently, these optimality criteria are utilized as a basis for constructing one parametric and two other parametric-free dual models (see [13] and [16]), and some duality results for \((P)\) are established.

2. NOTATIONS AND PRELIMINARY RESULTS

Throughout this paper, let \(\mathbb{R}^n\) be the \(n\)-dimensional Euclidean space and \(\mathbb{R}^n_+\) be its non-negative orthant. Let \(X_0\) be an open subset of \(\mathbb{R}^n\).

**Definition 2.1.** The function \(\theta : X_0 \mapsto \mathbb{R}\) is said to be **Lipschitz** on \(X_0\) if there exists \(c > 0\) such that for all \(y, x \in X_0,\)

\[|\theta(y) - \theta(x)| \leq c\|y - x\|,\]

where \(\| \cdot \|\) denotes any norm in \(\mathbb{R}^n\).

For each \(d \in \mathbb{R}^n,\) \(\theta^\circ(x; d)\) is the **generalized directional derivative of Clarke** [5] defined by

\[\theta^\circ(x; d) = \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{\theta(y + td) - \theta(y)}{t}.\]

It then follows that

\[\theta^\circ(x; d) = \max \{\xi^T d \mid \xi \in \partial \theta(x)\}\]  

for any \(x\) and \(d,\)

where \(\partial \theta(\cdot)\) denotes the **Clarke's generalized gradient** [5]. The following definitions can be found in [11]:
**Definition 2.2.** The function \( \theta : \mathbb{R}^n \to \mathbb{R} \) is said to be \textbf{invex} at \( x^* \) with respect to \( \eta \) if there exists a mapping \( \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) such that, for each \( x \in \mathbb{R}^n \),

\[
\theta(x) - \theta(x^*) \geq \theta^0(x^*; \eta(x, x^*)).
\]  

(2.1)

\( \theta \) is said to be invex on \( \mathbb{R}^n \) with respect to \( \eta \) if there exists a mapping \( \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) such that, for each \( x, u \in \mathbb{R}^n \),

\[
\theta(x) - \theta(u) \geq \theta^0(u; \eta(x, u)).
\]  

(2.2)

If we have strict inequality in (2.1) and (2.2), respectively, then \( \theta \) is said to be \textbf{strictly invex} at \( x^* \) with respect to \( \eta \) and strictly invex on \( \mathbb{R}^n \) with respect to \( \eta \), respectively.

**Definition 2.3.** The function \( \theta : \mathbb{R}^n \to \mathbb{R} \) is said to be \textbf{quasiinvex} at \( x^* \) with respect to \( \eta \) if there exists a mapping \( \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) such that, for each \( x \in \mathbb{R}^n \),

\[
\theta(x) \leq \theta(x^*) \Rightarrow \theta^0(x^*; \eta(x, x^*)) \leq 0.
\]  

(2.3)

\( \theta \) is said to be quasiinvex on \( \mathbb{R}^n \) with respect to \( \eta \) if there exists a mapping \( \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) such that, for each \( x, u \in \mathbb{R}^n \),

\[
\theta(x) \leq \theta(u) \Rightarrow \theta^0(u; \eta(x, u)) \leq 0.
\]  

(2.4)

If we have strict inequality in (2.3) and (2.4), respectively, then \( \theta \) is said to be \textbf{strictly quasiinvex} at \( x^* \) with respect to \( \eta \) and strictly quasiinvex on \( \mathbb{R}^n \) with respect to \( \eta \), respectively.

**Definition 2.4.** The function \( \theta : \mathbb{R}^n \to \mathbb{R} \) is said to be \textbf{pseudoinvex} at \( x^* \) with respect to \( \eta \) if there exists a mapping \( \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) such that, for each \( x \in \mathbb{R}^n \),

\[
\theta^0(x^*; \eta(x, x^*)) \geq 0 \Rightarrow \theta(x) \geq \theta(x^*).
\]  

(2.5)

\( \theta \) is said to be pseudoinvex on \( \mathbb{R}^n \) with respect to \( \eta \) if there exists a mapping \( \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) such that, for each \( x, u \in \mathbb{R}^n \),

\[
\theta^0(u; \eta(x, u)) \geq 0 \Rightarrow \theta(x) \geq \theta(u).
\]  

(2.6)

If we have strict inequality in (2.5) and (2.6), respectively, then \( \theta \) is said to be \textbf{strictly pseudoinvex} at \( x^* \) with respect to \( \eta \) and strictly pseudoinvex on \( \mathbb{R}^n \) with respect to \( \eta \), respectively.

We need the following lemmas.

**Lemma 2.1.** [16, Lemma 3.1.] Let \( v^* \) be the optimal value of (P), and let \( V(v) \) be the optimal value of (\( P_v \)) for any fixed \( v \in \mathbb{R}_+ \) such that (\( P_v \)) has an optimal solution. Then \( x^* \) is an optimal solution of (P) if and only if \( x^* \) is an optimal solution of (\( P_{v^*} \)) with optimal value \( V(v^*) = 0 \).
Lemma 2.2. [5, Proposition 2.3.12.] Let $f_1, \cdots, f_p$ be Lipschitz functions at $x^*$ and $\alpha_i \in \mathbb{R}$ for all $i = 1, \cdots, p$. Then

(1) $\partial (\sum_{i=1}^{p} \alpha_i f_i)(x^*) \subset \sum_{i=1}^{p} \alpha_i \partial f_i(x^*)$,
(2) $\partial \{\max_{1 \leq i \leq p} f_i \}(x^*) \subset \bigcup \{ \sum_{l \in L} \alpha_l \partial f_l(lx^*) ; \alpha_l \geq 0, \sum_{l \in L} \alpha_l = 1 \}$

where $L$ is the set of indices $l$ for which $f_l(x^*) = \max_{1 \leq i \leq p} f_i(x^*)$.

Lemma 2.3. [16, Lemma 3.2.] For each $x \in S$, one has

$$\phi(x) \equiv \max_{1 \leq i \leq p} \left( \frac{f_i(x)}{g_i(x)} \right) = \max_{\beta \in U} \left( \sum_{i=1}^{p} \beta_i f_i(x) / \sum_{i=1}^{p} \beta_i g_i(x) \right)$$

where $U = \{ \beta \in \mathbb{R}_+^p | \sum_{i=1}^{p} \beta_i = 1 \}$.

For convenience, we give the scalar minimization problem as follows:

$$(SP) \quad \text{Minimize} \quad N(x),$$
subject to $h_k(x) \leq 0, \quad k = 1, 2, \cdots, m$

where $N, h_k : X_0 \mapsto \mathbb{R}, k = 1, 2, \cdots, m$, are Lipschitz on $X_0$. We need the following lemma.

Lemma 2.4. [8, Theorem 6.] If $x^* \in X_0$ is a local minimum for $(SP)$ and a constraint qualification is satisfied, then there exist $z^* = (z_1^*, \cdots, z_m^*) \in \mathbb{R}_+^m$ such that

$$0 \in \partial N(x^*) + \sum_{k=1}^{m} z_k^* \partial h_k(x^*),$$

$$z_k^* h_k(x^*) = 0, \quad \text{for all} \quad k = 1, 2, \cdots, m.$$  

For simplicity, throughout the paper we denote

$$U = \{ \alpha \in \mathbb{R}_+^p | \sum_{i=1}^{p} \alpha_i = 1 \},$$

$$F(x) = (f_1(x), \cdots, f_p(x)),$$

$$G(x) = (g_1(x), \cdots, g_p(x)), \quad \text{and}$$

$$H(x) = (h_1(x), \cdots, h_m(x)).$$

For $z \in \mathbb{R}^m$, $z^T H(x^*) = \sum_{k=1}^{m} z_k h_k(x^*)$, and $\partial (z^T H)(x^*) = \sum_{k=1}^{m} z_k \partial h_k(x^*)$.

3. NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS

In this section, we shall use Lemmas 2.1 ~ 2.4 to establish some necessary and sufficient optimality conditions for the minimax fractional programming problem (P).
**Theorem 3.1** (Necessary optimality conditions). Let \( x^* \in S \). If \( x^* \) is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification \[8\]. Then there exist \( v^* = \phi(x^*) \in \mathbb{R}_+ \), \( y^* \in U \), \( z^* \in \mathbb{R}_+^m \) such that

\[
0 \in \partial(y^* \mathrm{T} F)(x^*) - v^* \partial(y^* \mathrm{T} G)(x^*) + \partial(z^* \mathrm{T} H)(x^*), \tag{3.1}
\]

\[
y^* \mathrm{T} F(x^*) - v^* y^* \mathrm{T} G(x^*) = 0, \tag{3.2}
\]

\[
z^* \mathrm{T} H(x^*) = 0. \tag{3.3}
\]

**Proof.** If \( x^* \) is an optimal solution of (P), by Lemma 2.1, it is an optimal solution of \((P_{v^*})\) with \( v^* = \max_{1 \leq i \leq p} [f_i(x^*)/g_i(x^*)] \). Thus, by Lemma 2.4, there exist \( z^* \in \mathbb{R}_+^m \), such that

\[
0 \in \partial \left( \max_{1 \leq i \leq p} (f_i - v^* g_i) \right)(x^*) + \partial(z^* \mathrm{T} H)(x^*)
\]

and

\[
z^* \mathrm{T} H(x^*) = 0.
\]

Therefore, by Lemma 2.2, there exist \( \alpha_i \geq 0 \), \( l \in L \), \( \sum_{l \in L} \alpha_i = 1 \), such that

\[
0 \in \sum_{l \in L} \alpha_i \left( \partial f_i(x^*) + v^* \partial(-g_i(x^*)) \right) + \partial(z^* \mathrm{T} H)(x^*). \tag{3.4}
\]

It is obvious that \( v^* = \max_{1 \leq i \leq p} [f_i(x^*)/g_i(x^*)] \) if and only if \( \max_{1 \leq i \leq p} [f_i(x^*) - v^* g_i(x^*)] = 0 \). From (3.4), if we set \( y_i^* = \alpha_i \) for \( i \in L \) as well as \( y_i^* = 0 \) for \( i \in \{1, 2, \ldots, p\} \setminus L \), the expressions (3.1), (3.2) and (3.3) hold.

\[\square\]

In order to construct parameter-free duality models for problem (P), we shall formulate parameter-free versions of Theorem 3.1 as follows:

**Theorem 3.2.** Let \( x^* \in S \). If \( x^* \) is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification \[8\]. Then there exist \( y^* \in U \) and \( z^* \in \mathbb{R}_+^m \) such that

\[
0 \in y^* \mathrm{T} G(x^*) \left( \partial(y^* \mathrm{T} F)(x^*) + \partial(z^* \mathrm{T} H)(x^*) \right) - y^* \mathrm{T} F(x^*) \partial(y^* \mathrm{T} G)(x^*), \tag{3.5}
\]

\[
z^* \mathrm{T} H(x^*) = 0, \tag{3.6}
\]

and obtain the optimal value by

\[
\phi(x^*) = y^* \mathrm{T} F(x^*)/y^* \mathrm{T} G(x^*) = \max_{1 \leq i \leq p} \left( f_i(x^*)/g_i(x^*) \right). \tag{3.7}
\]

**Proof.** From (3.2) and (3.1), substituting \( y^* \mathrm{T} F(x^*)/y^* \mathrm{T} G(x^*) \) for \( v^* \), we can derive the results.

\[\square\]

The conditions (3.5) \(~ (3.7)\) will be the sufficient optimality condition which we state as the following theorem.
Theorem 3.3 (Sufficient optimality conditions). Let $x^* \in S$, and assume that there exist $y^* \in U$ and $z^* \in \mathbb{R}^m_+$, such that the conditions (3.5) \sim (3.7) hold. Let

\[
A(x) = y^{*\text{T}}G(x^*)y^{*\text{T}}F(x) - y^{*\text{T}}F(x^*)y^{*\text{T}}G(x),
\]
\[
B(x) = z^{*\text{T}}H(x),
\]
\[
C(x) = A(x) + y^{*\text{T}}G(x^*)B(x).
\]

If any one of the following conditions holds

(a) $A$ is pseudoinvex at $x^*$ with respect to $\eta$ and $B$ is quasiinvex at $x^*$ with respect to same function $\eta$,

(b) $A$ is quasiinvex at $x^*$ with respect to $\eta$ and $B$ is strictly pseudoinvex at $x^*$ with respect to same function $\eta$,

(c) $C$ is pseudoinvex at $x^*$ with respect to $\eta$.

Then $x^*$ is an optimal solution of (P).

Proof. Suppose contrary that $x^*$ were not an optimal solution of (P). Then there exists a feasible solution $x_1 \in S$ such that

\[
\phi(x^*) > \phi(x_1).
\]

From (3.7) and Lemma 2.3, we have

\[
y^{*\text{T}}F(x^*)/y^{*\text{T}}G(x^*) > \max_{\beta \in U}(\beta^{\text{T}}F(\beta \in U)x_1)/\beta^{\text{T}}G(X_1) \geq y^{*\text{T}}F(X_1)/yG*(X_1).
\]

It follows that

\[
A(x_1) = y^{*\text{T}}G(x^*)y^{*\text{T}}F(x_1) - y^{*\text{T}}F(x^*)y^{*\text{T}}G(x_1) < 0 = A(x^*). \tag{3.8}
\]

Using both the feasibility $x_1$ for (P) and the equality (3.6), we have

\[
B(x_1) \leq 0 = B(x^*). \tag{3.9}
\]

Consequently, expressions (3.8) and (3.9) yield

\[
C(x_1) < C(x^*). \tag{3.10}
\]

By (3.5), there exist $\xi \in \partial(y^{*\text{T}}F)(x^*)$, $\zeta \in \partial(z^{*\text{T}}H)(x^*)$, and $\rho \in \partial(-y^{*\text{T}}G)(x^*)$, such that

\[
y^{*\text{T}}G(x^*)(\xi + \zeta) + y^{*\text{T}}F(x^*)\rho = 0.
\]

From here it results

\[
y^{*\text{T}}G(x^*)(\xi^{\text{T}}\eta(x, x^*) + \zeta^{\text{T}}\eta(x, x^*)) + y^{*\text{T}}F(x^*)\rho^{\text{T}}\eta(x, x^*) = 0. \tag{3.11}
\]

Using the characterization of the generalized gradient of Clarke, we obtain

\[
(y^{*\text{T}}F)^{\circ}(x^*; \eta(x, x^*)) \geq \xi^{\text{T}}\eta(x, x^*), \quad \text{for all} \quad x \in S, \tag{3.12}
\]
\[
(z^{*\text{T}}H)^{\circ}(x^*; \eta(x, x^*)) \geq \zeta^{\text{T}}\eta(x, x^*), \quad \text{for all} \quad x \in S, \tag{3.13}
\]
Now, multiplying (3.12) by $y^\mathsf{T}G(x^*)$, (3.13) by $y^\mathsf{T}G(x^*)$, and (3.14) by $y^\mathsf{T}F(x^*)$, and adding the resulting inequalities and with (3.11), we obtain

$$y^\mathsf{T}G(x^*)[(y^\mathsf{T}F)(x^*;\eta(x_1,x^*)) + (z^\mathsf{T}H)(x^*;\eta(x_1,x^*))]$$

$$- y^\mathsf{T}F(x^*)(y^\mathsf{T}G)(x^*;\eta(x_1,x^*)) \geq 0, \quad \text{for all } x \in S.$$  

(3.15)

If hypothesis (a) holds, using the pseudoinvexity of $A$ at $x^*$ and the inequality (3.8), we have

$$y^\mathsf{T}G(x^*)(y^\mathsf{T}F)(x^*;\eta(x_1,x^*)) - y^\mathsf{T}F(x^*)(y^\mathsf{T}G)(x^*;\eta(x_1,x^*)) < 0.$$  

(3.16)

Consequently, the inequalities (3.15) and (3.16) yield

$$y^\mathsf{T}G(x^*)(z^\mathsf{T}H)(x^*;\eta(x_1,x^*)) > 0.$$  

Thus, we have

$$(z^\mathsf{T}H)(x^*;\eta(x_1,x^*)) > 0.$$  

(3.17)

Using the quasiinvexity of $B$ at $x^*$, we get from (3.17)

$$B(x_1) = z^\mathsf{T}H(x_1) > z^\mathsf{T}H(x^*) = B(x^*)$$

which contradicts the inequality (3.9).

Hypothesis (b) follows along with the same lines as (a).

If hypothesis (c) holds, using the pseudoinvexity of $C$ at $x^*$ and the inequality (3.10), we have

$$y^\mathsf{T}G(x^*)[(y^\mathsf{T}F)(x^*;\eta(x_1,x^*)) + (z^\mathsf{T}H)(x^*;\eta(x_1,x^*))]$$

$$- y^\mathsf{T}F(x^*)(y^\mathsf{T}G)(x^*;\eta(x_1,x^*)) < 0$$

which contradicts the inequality (3.15). Hence, the proof is complete. \square

4. THE FIRST DUAL MODEL

Utilize Theorem 3.2, in Sections 4 and 5 we shall introduce two parametric-free dual models and prove appropriate duality theorems. Indeed, we shall demonstrate that the following is dual problem for (P):

$$\text{(DI) Maximize } (y^\mathsf{T}F(u) + z^\mathsf{T}H(u))/y^\mathsf{T}G(u)$$

subject to $0 \in y^\mathsf{T}G(u)(\partial(y^\mathsf{T}F)(u) + \partial(z^\mathsf{T}H)(u))$

$$- (y^\mathsf{T}F(u) + z^\mathsf{T}H(u))\partial(y^\mathsf{T}G)(u),$$

$$y \in U, \ z \in \mathbb{R}_{+}^m.$$  

(4.1)

(4.2)

We denote by $K_1$ the set of all feasible solutions $(u,y,z) \in X_0 \times U \times \mathbb{R}_{+}^m$ of problem (DI). We assume throughout this section that $y^\mathsf{T}F(u) + z^\mathsf{T}H(u) \geq 0$ and $y^\mathsf{T}G(u) > 0.$
Theorem 4.1 (Weak Duality). Let $x \in S$ and $(u, y, z) \in K_1$ and assume that
\[ D(\cdot) = y^T G(u) [y^T F(\cdot) + z^T H(\cdot)] - y^T G(\cdot) [y^T F(u) + z^T H(u)] \]
is a pseudoinvex function with respect to $\eta$ at $u$. Then
\[ \phi(x) \geq (y^T F(u) + z^T H(u))/y^T G(u). \]

Proof. By (4.1), there exist $\xi \in \partial (y^T F)(u)$, $\zeta \in \partial (z^T H)(u)$, and $\rho \in \partial (-y^T G)(u)$, such that
\[ y^T G(u)(\xi + \zeta) + [y^T F(u) + z^T H(u)]\rho = 0. \]
From here it results
\[ y^T G(u)(\xi^T \eta(x, u) + \zeta^T \eta(x, u)) + [y^T F(u) + z^T H(u)]\rho \eta(x, u) = 0. \tag{4.3} \]
Using the characterization of the generalized gradient of Clarke, we obtain
\begin{align*}
(y^T F)^\circ(u; \eta(x, u)) &\geq \xi^T \eta(x, u), \quad \text{for all } x \in S, \tag{4.4} \\
(z^T H)^\circ(u; \eta(x, u)) &\geq \zeta^T \eta(x, u), \quad \text{for all } x \in S, \tag{4.5} \\
(-y^T G)^\circ(u; \eta(x, u)) &\geq \rho^T \eta(x, u), \quad \text{for all } x \in S. \tag{4.6}
\end{align*}
Now, multiplying (4.4) by $y^T G(u)$, (4.5) by $y^T G(u)$, and (4.6) by $y^T F(u) + z^T H(u)$, and adding the resulting inequalities and with (4.3), we obtain
\[ y^T G(u)[(y^T F)^\circ(u; \eta(x, u)) + (z^T H)^\circ(u; \eta(x, u))] - [y^T F(u) + z^T H(u)](y^T G)^\circ(u; \eta(x, u)) \geq 0, \quad \text{for all } x \in S. \tag{4.7} \]
We suppose that
\[ \phi(x) < (y^T F(u) + z^T H(u))/y^T G(u). \]
Then, by Lemma 2.3 and $y \in U$, we have
\[ y^T F(x)/y^T G(x) < (y^T F(u) + z^T H(u))/y^T G(u). \]
Thus, we have
\[ y^T G(u)y^T F(x) - y^T G(x)[y^T F(u) + z^T H(u)] < 0. \]
Hence, we have another inequality
\[ y^T G(u)[y^T F(x) + z^T H(x)] - y^T G(x)[y^T F(u) + z^T H(u)] < y^T G(u)z^T H(x). \]
Using the fact $y^T G(u) > 0$, $z^T H(x) \leq 0$, and the latest inequality, we have
\[ D(x) < 0 = D(u). \]
Using the fact that $D(\cdot)$ is a pseudoinvex function with respect to $\eta$ at $u$, we have
\begin{align*}
y^T G(u)[(y^T F)^\circ(u; \eta(x, u)) + (z^T H)^\circ(u; \eta(x, u))] - [y^T F(u) + z^T H(u)](y^T G)^\circ(u; \eta(x, u)) &< 0
\end{align*}
which contradicts the inequality (4.7). Hence, the proof is complete. $\square$
Theorem 4.2 (Strong Duality). If $x^*$ is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist $y^* \in U$ and $z^* \in \mathbb{R}_+^m$, such that $(x^*, y^*, z^*)$ is a feasible solution of (DI). Furthermore, if the conditions of Theorem 4.1 hold for all feasible solutions of (DI), then $(x^*, y^*, z^*)$ is an optimal solution of (DI) and the optimal values of (P) and (DI) are equal; that is, \( \min(P) = \max(DI) \).

Proof. By Theorem 3.2, there exist $y^* \in U$, and $z^* \in \mathbb{R}_+^m$, such that $(x^*, y^*, z^*)$ is a feasible solution of (DI). Furthermore,\[
\left( y^*^T F(x^*) + z^*^T H(x^*) \right) / y^*^T G(x^*) = y^*^T F(x^*) / y^*^T G(x^*) = \phi(x^*).
\]
Thus, optimality of $(x^*, y^*, z^*)$ for (DI) follows from Theorem 4.1. \qed

Theorem 4.3 (Strict Converse Duality). Let $x_1$ and $(x^*, y_0, z_0)$ be optimal solutions of (P) and (DI), respectively, and assume that the assumptions of Theorem 4.2 are fulfilled. If

\[
D(\cdot) = y_0^T G(x^*) [y_0^T F(\cdot) + z_0^T H(\cdot)] - y_0^T G(\cdot) [y_0^T F(x^*) + z_0^T H(x^*)]
\]

is a strictly pseudoinvex function with respect to $\eta$, then $x_1 = x^*$; that is, $x^*$ is an optimal solution of (P) with the same optimal values \( \phi(x_1) = (y_1^T F(x_1) + z_1^T H(x_1)) / y_1^T G(x_1) \).

Proof. Suppose, on the contrary, that $x_1 \neq x^*$. From Theorem 4.2 we know that there exist $y_1 \in U$ and $z_1 \in \mathbb{R}_+^m$, such that $(x_1, y_1, z_1)$ is an optimal solution of (DI) and

\[
\phi(x_1) = (y_1^T F(x_1) + z_1^T H(x_1)) / y_1^T G(x_1).
\]
Now proceeding as in the proof of Theorem 4.1 (replacing $x$ by $x_1$ and $(u, y, z)$ by $(x^*, y_0, z_0)$), we arrive at the following strict inequality:

\[
\phi(x_1) > (y_0^T F(x^*) + z_0^T H(x^*)) / y_0^T G(x^*).
\]
This contradicts the fact that

\[
\phi(x_1) = (y_1^T F(x_1) + z_1^T H(x_1)) / y_1^T G(x_1) = (y_0^T F(x^*) + z_0^T H(x^*)) / y_0^T G(x^*).
\]
Therefore, we conclude that

\[
x_1 = x^*, \quad \text{and} \quad \phi(x_1) = (y_0^T F(x^*) + z_0^T H(x^*)) / y_0^T G(x^*).
\]
\qed

5. SECOND DUAL MODEL
We shall continue our discussion of parameter-free duality model for \((P)\) in this section by showing that the following problem (DII) is also dual problem for \((P)\):

\[
\text{(DII) } \begin{array}{ll}
\text{Maximize} & y^\top F(u)/y^\top G(u) \\
\text{subject to} & 0 \in y^\top G(u) \left( \partial(y^\top F)(u) + \partial(z^\top H)(u) \right) \\
& -y^\top F(u) \partial(y^\top G)(u), \\
& z^\top H(u) \geq 0, \\
& y \in U, z \in \mathbb{R}^m_+. 
\end{array}
\]

(5.1)

We denote by \(I_f^2\) the set of all feasible solutions \((u, y, z) \in X_0 \times U \times \mathbb{R}^m_+\) of problem (DII). Throughout this section, we assume that \(y^\top F(u) \geq 0\) and \(y^\top G(u) > 0\).

Then, we can prove the following weak duality, strong duality, and strict converse duality theorems.

**Theorem 5.1 (Weak Duality).** Let \(x \in S\) and \((u, y, z) \in I_f^2\) and let

\[
E(\cdot) = y^\top G(u)y^\top F(\cdot) - y^\top F(u)y^\top G(\cdot),
\]

\[
I(\cdot) = z^\top H(\cdot), \quad \text{and} \quad J(\cdot) = E(\cdot) + y^\top G(u)I(\cdot).
\]

If any one of the following conditions holds

(a) \(E\) is a pseudoinvex function with respect to \(\eta\) at \(u\) and \(I\) is a quasiinvex function at \(u\) with respect to same function \(\eta\),

(b) \(E\) is a quasivex function with respect to \(\eta\) at \(u\) and \(I\) is a strictly pseudoinvex function at \(u\) with respect to same function \(\eta\),

(c) \(J\) is a pseudoinvex function with respect to \(\eta\) at \(u\).

Then

\[
\phi(x) \geq y^\top F(u)/y^\top G(u).
\]

**Theorem 5.2 (Strong Duality).** If \(x^*\) is an optimal solution of \((P)\) and that the constraint of \((P)\) satisfy Slater's constraint qualification [8]. Then there exist \(y^* \in U\) and \(z^* \in \mathbb{R}^m_+\), such that \((x^*, y^*, z^*)\) is a feasible solution of (DII). Furthermore, if the conditions of Theorem 5.1 hold for all feasible solutions of (DII), then \((x^*, y^*, z^*)\) is an optimal solution of (DII) and the optimal values of \((P)\) and \((DII)\) are equal; that is, \(\min(P) = \max(DII)\).

**Theorem 5.3 (Strict Converse Duality).** Let \(x_1\) and \((x^*, y_0, z_0)\) be optimal solutions of \((P)\) and (DII), respectively, and assume that the assumptions of Theorem 5.2 are fulfilled. If \(E(\cdot) = y_0^\top G(x^*)y_0^\top F(\cdot) - y_0^\top F(x^*)y_0^\top G(\cdot)\) is a strictly pseudoinvex function with respect to \(\eta\) and \(I(\cdot) = z_0^\top H(\cdot)\) is a quasiinvex function with respect to same function \(\eta\), then \(x_1 = x^*\); that is, \(x^*\) is an optimal solution of \((P)\) with the same optimal values \(\phi(x_1) = y_0^\top F(x^*)/y_0^\top G(x^*)\).

6. THE THIRD DUAL MODEL
Making use of Theorem 3.1, in this section we can formulate the following parametric dual problem:

\begin{equation}
(DIII) \text{Maximize } v \\
\text{subject to } 0 \in \partial(y^\top F)(u) - v \partial(y^\top G)(u) + \partial(z^\top H)(u),
\end{equation}

\begin{equation}
y^\top F(u) - vy^\top G(u) \geq 0,
\end{equation}

\begin{equation}
z^\top H(u) \geq 0,
\end{equation}

\begin{equation}
y \in \bar{U}, \ v \in \mathbb{R}_+, \ Z \in \mathbb{R}^m.
\end{equation}

We denote by $K_3$ the set of all feasible solutions $(u, y, z, v) \in X_0 \times U \times \mathbb{R}_+^m \times \mathbb{R}_+$ of problem (DIII). Then a weakly duality theorem is established as follows:

**Theorem 6.1 (Weak Duality).** Let $x \in S$ and $(u, y, z, v) \in K_3$, and let

\begin{align*}
L(\cdot) &= y^\top F(\cdot) - vy^\top G(\cdot), \\
I(\cdot) &= z^\top H(\cdot), \text{ and } M(\cdot) = L(\cdot) + I(\cdot).
\end{align*}

If any one of the following conditions holds

(a) $L$ is a pseudoinvex function with respect to $\eta$ at $u$ and $I$ is a quasiinvex function at $u$ with respect to same function $\eta$,

(b) $L$ is a quasiinvex function with respect to $\eta$ at $u$ and $I$ is a strictly pseudoinvex function at $u$ with respect to same function $\eta$,

(c) $M$ is a pseudoinvex function with respect to $\eta$ at $u$.

Then

\[ \phi(x) \geq v. \]

**Theorem 6.2 (Strong Duality).** If $x^*$ is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist $y^* \in U$, $z^* \in \mathbb{R}^m$, and $v^* \in \mathbb{R}_+$, such that $(x^*, y^*, z^*, v^*)$ is a feasible solution of (DIII). Furthermore, if the conditions of Theorem 6.1 hold for all feasible solutions of (DIII), then $(x^*, y^*, z^*, v^*)$ is an optimal solution of (DII) and the optimal values of (P) and (DIII) are equal; that is, $\min(P) = \max(DIII)$.

**Theorem 6.3 (Strict Converse Duality).** Let $x_1$ and $(x^*, y_0, z_0, v_0)$ be optimal solutions of (P) and (DIII), respectively, and assume that the assumptions of Theorem 6.2 are fulfilled. If $y_0^\top F(\cdot) - v_0y_0^\top G(\cdot)$ is a strictly pseudoinvex function with respect to $\eta$ and $I(\cdot) = z_0^\top H(\cdot)$ is a quasiinvex function with respect to same function $\eta$, then $x_1 = x^*$; that is, $x^*$ is an optimal solution of (P) with the same optimal values $\phi(x_1) = v_0$.

The complete proof of Theorems 5.1-5.3 and Theorems 6.1-6.3 will be appear elsewhere.

7. SOME REMARKS FOR FURTHER DEVELOPMENTS
(1) There some questions arise that whether the results develop in this paper hold in generalized \((F, \rho)\)-convex?

(2) Does the set \(I = \{1, 2, \cdots , p\}\) in the minimax fractional programming \((P)\) can be replaced by a compact subset \(Y\) of \(\mathbb{R}^m\)? that is, does one can discuss the following minimax fractional programming:

\[
\text{Minimize} \quad F(x) = \sup_{y \in Y} \frac{f(x, y)}{g(x, y)} = \sup_{y \in Y} \Psi(x, y)
\]

subject to \(h(x) \leq 0,\)

where \(Y\) is a compact subset of \(\mathbb{R}^m\).

(3) Do we can discuss this minimax fractional programming in two person game theory?

**REFERENCES**


