GENERALIZED FRACTIONAL PROGRAMMING

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Optimality conditions in generalized fractional programming involving nonsmooth Lipschitz functions are established. Subsequently, these optimality criteria are utilized as a basis for constructing one parametric and two other parametric-free dual models, and several duality theorems are derived.

KEY WORDS: Generalized fractional programming, invex, quasiinvex, pesudoinvex, duality.

1. INTRODUCTION

In this paper, we consider the following minimax fractional programming problem:

\[
(P) \quad v^* = \min_{x \in S} \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)},
\]

where

(A1) \( S = \{ x \in \mathbb{R}^n; h_k(x) \leq 0, k = 1, 2, \cdots, m \} \) is nonempty and compact;
(A2) \( f_i : X_0 \mapsto \mathbb{R}, g_i : X_0 \mapsto \mathbb{R}, i = 1, 2, \cdots, p, \) and \( h_k : X_0 \mapsto \mathbb{R}, k = 1, 2, \cdots, m \) are locally Lipschitz continuous and \( X_0 \) is the open subset of \( \mathbb{R}^n; \)
(A3) \( g_i(x) > 0, i = 1, 2, \cdots, p, \) \( x \in S; \)
(A4) if \( g_i \) is not affine, then \( f_i(x) \geq 0 \) for all \( i \) and all \( x \in S. \)

Generalized fractional programming has been of much interest in the last decades; see for example [1-4, 6, 7, 10-19]. In [7], Crouzeix et al. have shown that the minimax fractional program can be derived by solving the following minimax nonlinear (nondifferentiable) parametric program:

\[
(P_v) \quad \min_{x \in S} \max_{1 \leq i \leq p} (f_i(x) - vg_i(x))
\]

where \( v \in \mathbb{R}_+ = [0, \infty) \) is a parameter.
It is clear that $(P_v)$ is equivalent to the following problem $(EP_v)$ for a given $v$:

$$(EP_v) \quad \min q,$$

subject to

$$f_i(x) - vg_i(x) \leq q, \quad i = 1, 2, \ldots, p,$$

$$h_k(x) \leq 0, \quad k = 1, 2, \ldots, m.$$  

In [2], Bector et al. employed the problem $(EP_v)$ to prove necessary and sufficient optimality conditions for problem $(P)$ and establish various duality results for problem $(EP_v)$ involving differentiable generalized convex functions (or generalized invex functions). Liu [10-12] also adapted the same approach to obtain necessary and sufficient optimality conditions; and he derived duality theorems for generalized fractional programming problems involving either nonsmooth pseudoinvex functions [11] or nonsmooth $(F, \rho)$-convex functions [10], and duality theorems for generalized fractional variational problems involving generalized $(F, \rho)$-convex functions [12].

But, all of the above necessary optimality conditions and strong duality theorems need that the constraint of $(EP_v)$ satisfy a constraint qualification.

In order to improve this defect, we want to use problem $(P_v)$ to establish both parametric and nonparameter necessary and sufficient optimality conditions, since a constraint qualification that is imposed on the constrains of $(P)$ may not hold for $(EP_v)$ but hold for $(P_v)$. Subsequently, these optimality criteria are utilized as a basis for constructing one parametric and two other parametric-free dual models (see [13] and [16]), and some duality results for $(P)$ are established.

2. NOTATIONS AND PRELIMINARY RESULTS

Throughout this paper, let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $\mathbb{R}_+^n$ be its non-negative orthant. Let $X_0$ be an open subset of $\mathbb{R}^n$.

**Definition 2.1.** The function $\theta : X_0 \rightarrow \mathbb{R}$ is said to be Lipschitz on $X_0$ if there exists $c > 0$ such that for all $y, x \in X_0$,

$$|\theta(y) - \theta(x)| \leq c \|y - x\|,$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$.

For each $d$ in $\mathbb{R}^n$, $\theta^o(x; d)$ is the generalized directional derivative of Clarke [5] defined by

$$\theta^o(x; d) = \limsup_{t \downarrow 0} \frac{\theta(y + td) - \theta(y)}{t}.$$

It then follows that

$$\theta^o(x; d) = \max \{\xi^T d \mid \xi \in \partial \theta(x)\} \quad \text{for any } x \text{ and } d,$$

where $\partial \theta(\cdot)$ denotes the Clarke’s generalized gradient [5]. The following definitions can be found in [11]:

(Continued on the next page...)
**Definition 2.2.** The function $\theta : \mathbb{R}^n \to \mathbb{R}$ is said to be **invex** at $x^*$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that, for each $x \in \mathbb{R}^n$,

$$\theta(x) - \theta(x^*) \geq \theta^o(x^*; \eta(x, x^*)).$$

(2.1)

$\theta$ is said to be invex on $\mathbb{R}^n$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that, for each $x, u \in \mathbb{R}^n$,

$$\theta(x) - \theta(u) \geq \theta^o(u; \eta(x, u)).$$

(2.2)

If we have strict inequality in (2.1) and (2.2), respectively, then $\theta$ is said to be **strictly invex** at $x^*$ with respect to $\eta$ and strictly invex on $\mathbb{R}^n$ with respect to $\eta$, respectively.

**Definition 2.3.** The function $\theta : \mathbb{R}^n \to \mathbb{R}$ is said to be **quasiinvex** at $x^*$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that, for each $x \in \mathbb{R}^n$,

$$\theta(x) \leq \theta(x^*) \Rightarrow \theta^o(x^*; \eta(x, x^*)) \leq 0.$$

(2.3)

$\theta$ is said to be quasiinvex on $\mathbb{R}^n$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that, for each $x, u \in \mathbb{R}^n$,

$$\theta(x) \leq \theta(u) \Rightarrow \theta^o(u; \eta(x, u)) \leq 0.$$

(2.4)

If we have strict inequality in (2.3) and (2.4), respectively, then $\theta$ is said to be **strictly quasiinvex** at $x^*$ with respect to $\eta$ and strictly quasiinvex on $\mathbb{R}^n$ with respect to $\eta$, respectively.

**Definition 2.4.** The function $\theta : \mathbb{R}^n \to \mathbb{R}$ is said to be **pseudoinvex** at $x^*$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that, for each $x \in \mathbb{R}^n$,

$$\theta^o(x^*; \eta(x, x^*)) \geq 0 \Rightarrow \theta(x) \geq \theta(x^*).$$

(2.5)

$\theta$ is said to be pseudoinvex on $\mathbb{R}^n$ with respect to $\eta$ if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that, for each $x, u \in \mathbb{R}^n$,

$$\theta^o(u; \eta(x, u)) \geq 0 \Rightarrow \theta(x) \geq \theta(u).$$

(2.6)

If we have strict inequality in (2.5) and (2.6), respectively, then $\theta$ is said to be **strictly pseudoinvex** at $x^*$ with respect to $\eta$ and strictly pseudoinvex on $\mathbb{R}^n$ with respect to $\eta$, respectively.

We need the following lemmas.

**Lemma 2.1.** [16, Lemma 3.1.] Let $v^*$ be the optimal value of $(P)$, and let $V(v)$ be the optimal value of $(P_v)$ for any fixed $v \in \mathbb{R}_+$ such that $(P_v)$ has an optimal solution. Then $x^*$ is an optimal solution of $(P)$ if and only if $x^*$ is an optimal solution of $(P_{v^*})$ with optimal value $V(v^*) = 0$. 
Lemma 2.2. [5, Proposition 2.3.12.] Let $f_1, \cdots, f_p$ be Lipschitz functions at $x^*$ and $\alpha_i \in \mathbb{R}$ for all $i = 1, \cdots, p$. Then

1. $\partial \left(\sum_{i=1}^{p} \alpha_i f_i(x^*)\right) \subset \sum_{i=1}^{p} \alpha_i \partial f_i(x^*)$,
2. $\partial \left[\max_{1 \leq i \leq p} f_i(x^*)\right] \subset \bigcup \{ \sum_{i \in L} \alpha_i \partial f_i(x^*) \mid \alpha_i \geq 0, \sum_{i \in L} \alpha_i = 1 \}$ where $L$ is the set of indices $l$ for which $f_l(x^*) = \max_{1 \leq i \leq p} f_i(x^*)$.

Lemma 2.3. [16, Lemma 3.2.] For each $x \in S$, one has

$$\phi(x) = \max_{1 \leq i \leq p} \left(\frac{f_i(x)}{g_i(x)}\right) = \max_{\beta \in U} \left(\sum_{i=1}^{p} \beta_i f_i(x)/\sum_{i=1}^{p} \beta_i g_i(x)\right)$$

where $U = \{ \beta \in \mathbb{R}_+^p \mid \sum_{i=1}^{p} \beta_i = 1 \}$.

For convenience, we give the scalar minimization problem as follows:

$$(SP) \quad \text{Minimize} \quad N(x),$$
subject to $h_k(x) \leq 0, \quad k = 1, 2, \cdots, m$

where $N, h_k : X_0 \mapsto \mathbb{R}, k = 1, 2, \cdots, m$, are Lipschitz on $X_0$. We need the following lemma.

Lemma 2.4. [8, Theorem 6.] If $x^* \in X_0$ is a local minimum for $(SP)$ and a constraint qualification is satisfied, then there exist $z^* = (z_1^*, \cdots, z_m^*) \in \mathbb{R}_+^m$ such that

$$0 \in \partial N(x^*) + \sum_{k=1}^{m} z_k^* \partial h_k(x^*),$$
$$z_k^* h_k(x^*) = 0, \quad \text{for all} \quad k = 1, 2, \cdots, m.$$

For simplicity, throughout the paper we denote

$$U = \{ \alpha \in \mathbb{R}_+^p \mid \sum_{i=1}^{p} \alpha_i = 1 \},$$
$$F(x) = (f_1(x), \cdots, f_p(x)), \quad G(x) = (g_1(x), \cdots, g_p(x)), \quad \text{and} \quad H(x) = (h_1(x), \cdots, h_m(x)).$$

For $z \in \mathbb{R}^m$, $z^T H(x^*) = \sum_{k=1}^{m} z_k h_k(x^*)$, and $\partial (z^T H)(x^*) = \sum_{k=1}^{m} z_k \partial h_k(x^*)$.

3. NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS

In this section, we shall use Lemmas 2.1 ~ 2.4 to establish some necessary and sufficient optimality conditions for the minimax fractional programming problem (P).
**Theorem 3.1** (Necessary optimality conditions). Let $x^* \in S$. If $x^*$ is an optimal solution of (P) and that the constraint of (P) satisfy Slater’s constraint qualification [8]. Then there exist $v^* = \phi(x^*) \in \mathbb{R}_+$, $y^* \in U$, $z^* \in \mathbb{R}_+^m$ such that

\begin{align*}
0 &\in \partial(y^*^T F)(x^*) - y^* \partial(y^*^T G)(x^*) + \partial(z^*^T H)(x^*), \quad (3.1) \\
y^*^T F(x^*) - v^* y^*^T G(x^*) & = 0, \quad (3.2) \\
z^*^T H(x^*) & = 0. \quad (3.3)
\end{align*}

**Proof.** If $x^*$ is an optimal solution of (P), by Lemma 2.1, it is an optimal solution of $(P_{v^*})$ with $v^* = \max_{1 \leq i \leq p}[f_i(x^*)/g_i(x^*)]$. Thus, by Lemma 2.4, there exist $z^* \in \mathbb{R}_+^m$, such that

$$0 \in \partial \left( \max_{1 \leq i \leq p} (f_i - v^* g_i) \right)(x^*) + \partial(z^*^T H)(x^*)$$

and

$$z^*^T H(x^*) = 0.$$ 

Therefore, by Lemma 2.2, there exist $\alpha_i \geq 0$, $l \in L$, $\sum_{l \in L} \alpha_i = 1$, such that

$$0 \in \sum_{l \in L} \alpha_i \left( \partial f_i(x^*) + v^* \partial(-g_i(x^*)) + \partial(z^*^T H)(x^*) \right). \quad (3.4)$$

It is obvious that $v^* = \max_{1 \leq i \leq p}[f_i(x^*)/g_i(x^*)]$ if and only if $\max_{1 \leq i \leq p}[f_i(x^*) - v^* g_i(x^*)] = 0$. From (3.4), if we set $y_i^* = \alpha_i$ for $i \in L$ as well as $y_i^* = 0$ for $i \in \{1, 2, \cdots, p\} \setminus L$, the expressions (3.1), (3.2) and (3.3) hold.

In order to construct parameter-free duality models for problem (P), we shall formulate parameter-free versions of Theorem 3.1 as follows:

**Theorem 3.2.** Let $x^* \in S$. If $x^*$ is an optimal solution of (P) and that the constraint of (P) satisfy Slater’s constraint qualification [8]. Then there exist $y^* \in U$ and $z^* \in \mathbb{R}_+^m$ such that

\begin{align*}
0 &\in y^*^T G(x^*) \left( \partial(y^*^T F)(x^*) + \partial(z^*^T H)(x^*) \right) - y^*^T F(x^*) \partial(y^*^T G)(x^*), \quad (3.5) \\
z^*^T H(x^*) & = 0, \quad (3.6)
\end{align*}

and obtain the optimal value by

$$\phi(x^*) = y^*^T F(x^*)/y^*^T G(x^*) = \max_{1 \leq i \leq p} (f_i(x^*)/g_i(x^*)). \quad (3.7)$$

**Proof.** From (3.2) and (3.1), substituting $y^*^T F(x^*)/y^*^T G(x^*)$ for $v^*$, we can derive the results.
Theorem 3.3 (Sufficient optimality conditions). Let \( x^* \in S \), and assume that there exist \( y^* \in U \) and \( z^* \in \mathbb{R}_+^m \), such that the conditions (3.5) \( \sim \) (3.7) hold. Let
\[
A(x) = y^* \mathbf{T} G(x^*) y^* \mathbf{T} F(x) - y^* \mathbf{T} F(x^*) y^* \mathbf{T} G(x),
\]
\[
B(x) = z^* \mathbf{T} H(x),
\]
\[
C(x) = A(x) + y^* \mathbf{T} G(x^*) B(x).
\]
If any one of the following conditions holds
(a) \( A \) is pseudoinvex at \( x^* \) with respect to \( \eta \) and \( B \) is quasiinvex at \( x^* \) with respect to same function \( \eta \),
(b) \( A \) is quasiinvex at \( x^* \) with respect to \( \eta \) and \( B \) is strictly pseudoinvex at \( x^* \) with respect to same function \( \eta \),
(c) \( C \) is pseudoinvex at \( x^* \) with respect to \( \eta \).
Then \( x^* \) is an optimal solution of (P).

**Proof.** Suppose contrary that \( x^* \) were not an optimal solution of (P). Then there exists a feasible solution \( x_1 \in S \) such that
\[
\phi(x^*) > \phi(x_1).
\]
From (3.7) and Lemma 2.3, we have
\[
y^* \mathbf{T} F(x^*) / y^* \mathbf{T} G(x^*) > \max_{\beta \in U} (\beta \mathbf{T} F(x^*) / \beta \mathbf{T} G(x^*)) \geq y^* \mathbf{T} F(x_1) / y^* \mathbf{T} G(x_1).
\]
It follows that
\[
A(x_1) = y^* \mathbf{T} G(x^*) y^* \mathbf{T} F(x_1) - y^* \mathbf{T} F(x^*) y^* \mathbf{T} G(x_1) < 0 = A(x^*). \tag{3.8}
\]
Using both the feasibility \( x_1 \) for (P) and the equality (3.6), we have
\[
B(x_1) \leq 0 = B(x^*). \tag{3.9}
\]
Consequently, expressions (3.8) and (3.9) yield
\[
C(x_1) < C(x^*). \tag{3.10}
\]
By (3.5), there exist \( \xi \in \partial (y^* \mathbf{T} F)(x^*) \), \( \zeta \in \partial (z^* \mathbf{T} H)(x^*) \), and \( \rho \in \partial (-y^* \mathbf{T} G)(x^*) \), such that
\[
y^* \mathbf{T} G(x^*)(\xi + \zeta) + y^* \mathbf{T} F(x^*) \rho = 0.
\]
From here it results
\[
y^* \mathbf{T} G(x^*)(\xi \mathbf{T} \eta(x, x^*) + \zeta \mathbf{T} \eta(x, x^*)) + y^* \mathbf{T} F(x^*) \rho \mathbf{T} \eta(x, x^*) = 0. \tag{3.11}
\]
Using the characterization of the generalized gradient of Clarke, we obtain
\[
(y^* \mathbf{T} F)^{\circ}(x^*; \eta(x, x^*)) \geq \xi \mathbf{T} \eta(x, x^*), \quad \text{for all} \quad x \in S, \tag{3.12}
\]
\[
(z^* \mathbf{T} H)^{\circ}(x^*; \eta(x, x^*)) \geq \zeta \mathbf{T} \eta(x, x^*), \quad \text{for all} \quad x \in S. \tag{3.13}
\]
\[-y^{*\top}G(x^{*};\eta(x,x^{*})) \geq \rho^{\top}\eta(x,x^{*}), \quad \text{for all } x \in S. \tag{3.14}\]

Now, multiplying (3.12) by \(y^{*\top}G(x^{*})\), (3.13) by \(y^{*\top}G(x^{*})\), and (3.14) by \(y^{*\top}F(x^{*})\), and adding the resulting inequalities and with (3.11), we obtain
\[
y^{*\top}G(x^{*})[(y^{*\top}F)(x^{*};\eta(x_{1},x^{*})) + (z^{*\top}H)(x^{*};\eta(x_{1},x^{*}))] \\
- y^{*\top}F(x^{*})(y^{*\top}G)(x^{*};\eta(x_{1},x^{*})) \geq 0, \quad \text{for all } x \in S. \tag{3.15}\]

If hypothesis (a) holds, using the pseudoinvexity of \(A\) at \(x^{*}\) and the inequality (3.8), we have
\[
y^{*\top}G(x^{*})(y^{*\top}F)(x^{*};\eta(x_{1},x^{*})) - y^{*\top}F(x^{*})(y^{*\top}G)(x^{*};\eta(x_{1},x^{*})) < 0. \tag{3.16}\]

Consequently, the inequalities (3.15) and (3.16) yield
\[
y^{*\top}G(x^{*})(z^{*\top}H)(x^{*};\eta(x_{1},x^{*})) > 0.
\]
Thus, we have
\[
(z^{*\top}H)(x^{*};\eta(x_{1},x^{*})) > 0. \tag{3.17}\]
Using the quasiinvexity of \(B\) at \(x^{*}\), we get from (3.17)
\[
B(x_{1}) = z^{*\top}H(x_{1}) > z^{*\top}H(x^{*}) = B(x^{*})
\]
which contradicts the inequality (3.9).

Hypothesis (b) follows along with the same lines as (a).

If hypothesis (c) holds, using the pseudoinvexity of \(C\) at \(x^{*}\) and the inequality (3.10), we have
\[
y^{*\top}G(x^{*})[(y^{*\top}F)(x^{*};\eta(x_{1},x^{*})) + (z^{*\top}H)(x^{*};\eta(x_{1},x^{*}))] \\
- y^{*\top}F(x^{*})(y^{*\top}G)(x^{*};\eta(x_{1},x^{*})) < 0
\]
which contradicts the inequality (3.15). Hence, the proof is complete. \(\square\)

4. THE FIRST DUAL MODEL

Utilize Theorem 3.2, in Sections 4 and 5 we shall introduce two parametric-free dual models and prove appropriate duality theorems. Indeed, we shall demonstrate that the following is dual problem for (P):

\[
\begin{align*}
\text{(DI)} & \quad \text{Maximize } \frac{(y^{\top}F(u) + z^{\top}H(u))/y^{\top}G(u)}{y^{\top}G(u)} \\
\text{subject to } & \quad 0 \in y^{\top}G(u)(\partial(y^{\top}F)(u) + \partial(z^{\top}H)(u)) \\
& \quad - (y^{\top}F(u) + z^{\top}H(u))\partial(y^{\top}G)(u), \\
& \quad y \in U, \quad z \in \mathbb{R}^{m}. \tag{4.1}
\end{align*}
\]

We denote by \(K_{1}\) the set of all feasible solutions \((u,y,z) \in X_{0} \times U \times \mathbb{R}^{m}\) of problem (DI). We assume throughout this section that \(y^{\top}F(u) + z^{\top}H(u) \geq 0\) and \(y^{\top}G(u) > 0\).
**Theorem 4.1 (Weak Duality).** Let \( x \in S \) and \((u, y, z) \in K_1 \) and assume that
\[
D(\cdot) = y^\top G(u)[y^\top F(\cdot) + z^\top H(\cdot)] - y^\top G(u)[y^\top F(u) + z^\top H(u)]
\]
is a pseudoinvex function with respect to \( \eta \) at \( u \). Then
\[
\phi(x) \geq (y^\top F(u) + z^\top H(u))/y^\top G(u).
\]

**Proof.** By (4.1), there exist \( \xi \in \partial(y^\top F)(u) \), \( \zeta \in \partial(z^\top H)(u) \), and \( \rho \in \partial(-y^\top G)(u) \), such that
\[
y^\top G(u)(\xi^\top \eta(x, u) + \zeta^\top \eta(x, u)) + [y^\top F(u) + z^\top H(u)]\rho \eta^\top(x, u) = 0.
\]
From here it results
\[
y^\top G(u)(\xi^\top \eta(x, u) + \zeta^\top \eta(x, u)) + [y^\top F(u) + z^\top H(u)]\rho \eta^\top(x, u) = 0. \tag{4.3}
\]
Using the characterization of the generalized gradient of Clarke, we obtain
\[
(y^\top F)^o(u; \eta(x, u)) \geq \xi^\top \eta(x, u), \quad \text{for all } x \in S, \tag{4.4}
\]
\[
(z^\top H)^o(u; \eta(x, u)) \geq \zeta^\top \eta(x, u), \quad \text{for all } x \in S, \tag{4.5}
\]
\[
(-y^\top G)^o(u; \eta(x, u)) \geq \rho^\top \eta(x, u), \quad \text{for all } x \in S. \tag{4.6}
\]
Now, multiplying (4.4) by \( y^\top G(u) \), (4.5) by \( y^\top G(u) \), and (4.6) by \( y^\top F(u) + z^\top H(u) \), and adding the resulting inequalities and with (4.3), we obtain
\[
y^\top G(u)[(y^\top F)^o(u; \eta(x, u)) + (z^\top H)^o(u; \eta(x, u))] - [y^\top F(u) + z^\top H(u)](y^\top G)^o(u; \eta(x, u)) \geq 0, \quad \text{for all } x \in S. \tag{4.7}
\]
We suppose that
\[
\phi(x) < (y^\top F(u) + z^\top H(u))/y^\top G(u).
\]
Then, by Lemma 2.3 and \( y \in U \), we have
\[
y^\top F(x)/y^\top G(x) < (y^\top F(u) + z^\top H(u))/y^\top G(u).
\]
Thus, we have
\[
y^\top G(u)y^\top F(x) - y^\top G(x)[y^\top F(u) + z^\top H(u)] < 0.
\]
Hence, we have another inequality
\[
y^\top G(u)[y^\top F(x) + z^\top H(x)] - y^\top G(x)[y^\top F(u) + z^\top H(u)] < y^\top G(u)z^\top H(x).
\]
Using the fact \( y^\top G(u) > 0, \ z^\top H(x) \leq 0, \) and the latest inequality, we have
\[
D(x) < 0 = D(u).
\]
Using the fact that \( D(\cdot) \) is a pseudoinvex function with respect to \( \eta \) at \( u \), we have
\[
y^\top G(u)[(y^\top F)^o(u; \eta(x, u)) + (z^\top H)^o(u; \eta(x, u))] - [y^\top F(u) + z^\top H(u)](y^\top G)^o(u; \eta(x, u)) < 0
\]
which contradicts the inequality (4.7). Hence, the proof is complete. \( \square \)
Theorem 4.2 (Strong Duality). If $x^*$ is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist $y^* \in U$ and $z^* \in \mathbb{R}_+^m$, such that $(x^*, y^*, z^*)$ is a feasible solution of (DI). Furthermore, if the conditions of Theorem 4.1 hold for all feasible solutions of (DI), then $(x^*, y^*, z^*)$ is an optimal solution of (DI) and the optimal values of (P) and (DI) are equal; that is, \( \min(P) = \max(DI) \).

Proof. By Theorem 3.2, there exist $y \in U$, and $z \in \mathbb{R}_+^m$, such that $(x^*, y^*, z^*)$ is a feasible solution of (DI). Furthermore,

\[
\left( y^T F(x^*) + z^T H(x^*) \right) / y^T G(x^*) = y^T F(x^*) / y^T G(x^*) = \phi(x^*).
\]

Thus, optimality of $(x^*, y^*, z^*)$ for (DI) follows from Theorem 4.1.

\( \square \)

Theorem 4.3 (Strict Converse Duality). Let $x_1$ and $(x^*, y_0, z_0)$ be optimal solutions of (P) and (DI), respectively, and assume that the assumptions of Theorem 4.2 are fulfilled. If

\[
D(\cdot) = y_0^T G(x^*)[y_0^T F(\cdot) + z_0^T H(\cdot)] - y_0^T G(\cdot)[y_0^T F(x^*) + z_0^T H(x^*)]
\]

is a strictly pseudoinvex function with respect to $\eta$, then $x_1 = x^*$; that is, $x^*$ is an optimal solution of (P) with the same optimal values $\phi(x_1) = (y_0^T F(x^*) + z_0^T H(x^*)) / y_0^T G(x^*)$.

Proof. Suppose, on the contrary, that $x_1 \neq x^*$. From Theorem 4.2 we know that there exist $y_1 \in U$ and $z_1 \in \mathbb{R}_+^m$, such that $(x_1, y_1, z_1)$ is an optimal solution of (DI) and

\[
\phi(x_1) = \left( y_1^T F(x_1) + z_1^T H(x_1) \right) / y_1^T G(x_1).
\]

Now proceeding as in the proof of Theorem 4.1 (replacing $x$ by $x_1$ and $(u, y, z)$ by $(x^*, y_0, z_0)$), we arrive at the following strict inequality:

\[
\phi(x_1) > \left( y_0^T F(x^*) + z_0^T H(x^*) \right) / y_0^T G(x^*).
\]

This contradicts the fact that

\[
\phi(x_1) = \left( y_1^T F(x_1) + z_1^T H(x_1) \right) / y_1^T G(x_1) = \left( y_0^T F(x^*) + z_0^T H(x^*) \right) / y_0^T G(x^*).
\]

Therefore, we conclude that

\[
x_1 = x^*, \quad \text{and} \quad \phi(x_1) = \left( y_0^T F(x^*) + z_0^T H(x^*) \right) / y_0^T G(x^*).
\]

\( \square \)

5. SECOND DUAL MODEL
We shall continue our discussion of parameter-free duality model for \((P)\) in this section by showing that the following problem \((\text{DII})\) is also dual problem for \((P):\)

\[
\text{(DII)} \quad \text{Maximize} \quad y^T F(u)/y^T G(u) \\
\text{subject to} \quad 0 \in y^T G(u) \left( \partial (y^T F)(u) + \partial (z^T H)(u) \right) \\
- y^T F(u) \partial (y^T G)(u), \quad (5.1) \\
z^T H(u) \geq 0, \quad (5.2) \\
y \in U, \quad z \in \mathbb{R}_+^m. \quad (5.3)
\]

We denote by \(K_2\) the set of all feasible solutions \((u, y, z) \in X_0 \times U \times \mathbb{R}_+^m\) of problem \((\text{DII}).\) Throughout this section, we assume that \(y^T F(u) \geq 0\) and \(y^T G(u) > 0.\) Then, we can prove the following weak duality, strong duality, and strict converse duality theorems.

**Theorem 5.1 (Weak Duality).** Let \(x \in S\) and \((u, y, z) \in K_2\) and let

\[
E(\cdot) = y^T G(u) y^T F(\cdot) - y^T F(u) y^T G(\cdot), \quad I(\cdot) = z^T H(\cdot), \quad \text{and} \quad J(\cdot) = E(\cdot) + y^T G(u) I(\cdot).
\]

If any one of the following conditions holds

(a) \(E\) is a pseudoinvex function with respect to \(\eta\) at \(u\) and \(I\) is a quasiinvex function at \(u\) with respect to same function \(\eta,

(b) \(E\) is a quasivex function with respect to \(\eta\) at \(u\) and \(I\) is a strictly pseudoinvex function at \(u\) with respect to same function \(\eta,

(c) \(J\) is a pseudoinvex function with respect to \(\eta\) at \(u\).

Then

\[
\phi(x) \geq y^T F(u)/y^T G(u).
\]

**Theorem 5.2 (Strong Duality).** If \(x^*\) is an optimal solution of \((P)\) and that the constraint of \((P)\) satisfy Slater’s constraint qualification [8]. Then there exist \(y^* \in U\) and \(z^* \in \mathbb{R}_+^m\), such that \((x^*, y^*, z^*)\) is a feasible solution of \((\text{DII}).\) Furthermore, if the conditions of Theorem 5.1 hold for all feasible solutions of \((\text{DII}),\) then \((x^*, y^*, z^*)\) is an optimal solution of \((\text{DII})\) and the optimal values of \((P)\) and \((\text{DII})\) are equal; that is,

\[
\min(P) = \max(\text{DII}).
\]

**Theorem 5.3 (Strict Converse Duality).** Let \(x_1\) and \((x^*, y_0, z_0)\) be optimal solutions of \((P)\) and \((\text{DII}),\) respectively, and assume that the assumptions of Theorem 5.2 are fulfilled. If \(E(\cdot) = y_0^T G(x^*) y_0^T F(\cdot) - y_0^T F(x^*) y_0^T G(\cdot)\) is a strictly pseudoinvex function with respect to \(\eta\) and \(I(\cdot) = z_0^T H(\cdot)\) is a quasiinvex function with respect to same function \(\eta,\) then \(x_1 = x^*;\) that is, \(x^*\) is an optimal solution of \((P)\) with the same optimal values \(\phi(x_1) = y_0^T F(x^*)/y_0^T G(x^*).\)

6. **THE THIRD DUAL MODEL**
Making use of Theorem 3.1, in this section we can formulate the following parametric dual problem:

\[(DIII) \text{ Maximize } v\]
\[\text{subject to } 0 \in \partial(y^\top F)(u) - v \partial(y^\top G)(u) + \partial(z^\top H)(u), \quad (6.1)\]
\[y^\top F(u) - vy^\top G(u) \geq 0, \quad (6.2)\]
\[z^\top H(u) \geq 0, \quad (6.3)\]
\[y \in U, \; v \in \mathbb{R}_+, \; z \in \mathbb{R}_+^m. \quad (6.4)\]

We denote by $K_3$ the set of all feasible solutions $(u, y, z, v) \in X_0 \times U \times \mathbb{R}_+^m \times \mathbb{R}_+$ of problem (DIII). Then a weakly duality theorem is established as follows:

**Theorem 6.1 (Weak Duality).** Let $x \in S$ and $(u, y, z, v) \in K_3$, and let

\[L(\cdot) = y^\top F(\cdot) - vy^\top G(\cdot), \quad I(\cdot) = z^\top H(\cdot), \quad \text{and} \quad M(\cdot) = L(\cdot) + I(\cdot).\]

If any one of the following conditions holds

(a) $L$ is a pseudoinvex function with respect to $\eta$ at $u$ and $I$ is a quasiinvex function at $u$ with respect to same function $\eta$,

(b) $L$ is a quasiinvex function with respect to $\eta$ at $u$ and $I$ is a strictly pseudoinvex function at $u$ with respect to same function $\eta$,

(c) $M$ is a pseudoinvex function with respect to $\eta$ at $u$.

Then

\[\phi(x) \geq v.\]

**Theorem 6.2 (Strong Duality).** If $x^*$ is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist $y^* \in U$, $z^* \in \mathbb{R}_+^m$, and $v^* \in \mathbb{R}_+$, such that $(x^*, y^*, z^*, v^*)$ is a feasible solution of (DIII). Furthermore, if the conditions of Theorem 6.1 hold for all feasible solutions of (DIII), then $(x^*, y^*, z^*, v^*)$ is an optimal solution of (DIII) and the optimal values of (P) and (DIII) are equal; that is, $\min(P) = \max(DIII)$.

**Theorem 6.3 (Strict Converse Duality).** Let $x_1$ and $(x^*, y_0, z_0, v_0)$ be optimal solutions of (P) and (DIII), respectively, and assume that the assumptions of Theorem 6.2 are fulfilled. If $y_0^\top F(\cdot) - v_0y_0^\top G(\cdot)$ is a strictly pseudoinvex function with respect to $\eta$ and $I(\cdot) = z_0^\top H(\cdot)$ is a quasiinvex function with respect to same function $\eta$, then $x_1 = x^*$; that is, $x^*$ is an optimal solution of (P) with the same optimal values $\phi(x_1) = v_0$.

The complete proof of Theorems 5.1-5.3 and Theorems 6.1-6.3 will be appear elsewhere.

7. SOME REMARKS FOR FURTHER DEVELOPMENTS
(1) There some questions arise that whether the results develop in this paper hold in generalized $(F, \rho)$-convex?

(2) Does the set $I = \{1, 2, \cdots, p\}$ in the minimax fractional programming $(P)$ can be replaced by a compact subset $Y$ of $\mathbb{R}^m$? that is, does one can discuss the following minimax fractional programming:

$$\text{Minimize} \quad F(x) = \sup_{y \in Y} \frac{f(x, y)}{g(x, y)} = \sup_{y \in Y} \Psi(x, y)$$

subject to \quad $h(x) \leq 0$,

where $Y$ is a compact subset of $\mathbb{R}^m$?

(3) Do we can discuss this minimax fractional programming in two person game theory?

REFERENCES


