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Kyoto University
On the discrepancy of the $\beta$-adic van der Corput sequence

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1 Introduction

It is well known that low-discrepancy sequences and their discrepancy play essential roles in quasi-Monte Carlo methods [6]. The author constructed a new class of low-discrepancy sequences $N_\beta$ [7] by using the $\beta$-adic transformation [9][11]. Here, $\beta$ is a real number greater than 1; when $\beta$ is an integer greater than 2, $N_\beta$ becomes the classical van der Corput sequence in base $\beta$. Therefore, the class $N_\beta$ can be regarded as a generalization of the van der Corput sequence. $N_\beta$ also contains a new construction by Barat and Grabner [1] [7]. The principle of the construction of $N_\beta$ is that we can consider the van der Corput sequence to be a Kakutani adding machine [10]. Pages [8] and Hellekalek [4] also considered the van der Corput sequence from this point of view. In [7], it is shown that when $\beta$ satisfies the following two conditions:

- Markov condition: $\beta$ is simple, that is to say, for this $\beta$, the $\beta$-adic transformation becomes Markov,
- Pisot-Vijayaraghavan condition: All conjugates of $\beta$ with respect to its characteristic equation belong to $\{z \in \mathbb{C} \mid |z| < 1\}$,

the discrepancy of $N_\beta$ decreases in the fastest order $O(N^{-1}\log N)$. In this paper, we consider the case in which $\beta$ is not necessarily Markov. We introduce the function $\phi_\beta(z)$ from Ito and Takahashi [5]. It is shown that when $\beta$ satisfies the following condition:

- All zeroes of $1 - \phi_\beta(z)$ except for $z = 1$ belong to $\{z \in \mathbb{C} \mid |z| > \beta\}$,

which is a generalization of the above Pisot-Vijayaraghavan condition, the discrepancy of $N_\beta$ decreases in the order $O(N^{-1}(\log N)^2)$.

2 Low-discrepancy sequence

First, we recall the notions of a uniformly distributed sequence and the discrepancy of points [6]. A sequence $x_1, x_2, \ldots$ in the $s$-dimensional unit cube $I^s = \prod_{i=1}^{s} [0, 1)$ is said to be uniformly distributed in $I^s$ when

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_J(x_n) = \lambda_s(J)$$
holds for all subintervals $J \subset I^s$, where $c_J$ is the characteristic function of $J$ and $\lambda_s$ is the $s$-dimensional Lebesgue measure. If $x_1, x_2, \ldots \in I^s$ is a uniformly distributed sequence, the formula

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{I^s} f(x) \, dx$$

(2.1)

holds for any Riemann integrable function on $I^s$. The discrepancy of the point set $P = \{x_1, x_2, \ldots, x_N\}$ in $I^s$ is defined as follows:

$$D_N(P) = \sup_{B \subset \wp(I^s)} \left| \frac{A(B; P)}{N} - \lambda_s(B) \right|$$

(2.2)

where $B \subset \wp(I^s)$ is a non-empty family of Lebesgue measurable subsets and $A(B; P)$ is the counting function that indicates the number of $n$, where $1 \leq n \leq N$, for which $x_n \in B$. When $J^* = \{\prod_{i=1}^{l} [0, u_i), 0 \leq u_i < 1\}$, the star discrepancy $D_N^*(P)$ is defined by $D_N^*(P) = D_N(J^*; P)$. When $S = \{x_1, x_2, \ldots\}$ is a sequence in $I^s$, we define $D_N^*(S)$ as $D_N^*(S_N)$, where $S_N$ is the point set $\{x_1, x_2, \ldots, x_N\}$. Let $S$ be a sequence in $I^s$. It is known that the following two conditions are equivalent:

1. $S$ is uniformly distributed in $I^s$;
2. $\lim_{N \to \infty} D_N^*(S) = 0$.

The following classical theorem shows the importance of the notion of discrepancy:

**Theorem 2.1 (Koksma-Hlawka [6])** If $f$ has bounded variation $V(f)$ on $I^s$ in the sense of Hardy and Krause, then for any $x_1, x_2, \ldots, x_N \in I^s$, we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_{I^s} f(x) \, dx \right| \leq V(f) D_N^*(x_1, \ldots, x_N).$$

Schmidt [12] showed that, when $s = 1$ or $2$, there exists a positive constant $C$ that depends only on $s$, and the following inequality holds for an arbitrary point set $P$ consisting of $N$ elements:

$$D_N^*(P) \geq C \frac{\log N^{s-1}}{N}.$$  \hspace{1cm} (2.3)

If (2.3) holds, then there exists a positive constant $C$ that depends only on $s$, and any sequence $S \subset I^s$ satisfies

$$D_N^*(S) \geq C \frac{\log N^{s}}{N}.$$  \hspace{1cm} (2.4)

for infinitely many $N$. Taking account of (2.3) and (2.4), we define a low-discrepancy sequence for the one-dimensional case as follows:

**Definition 2.1** Let $S$ be an one-dimensional sequence in $[0, 1)$. If $D_N^*(S)$ satisfies

$$D_N^*(S) = O(N^{-1} \log N)$$

then $S$ is called a low-discrepancy sequence.

Hereafter we consider only the case where $s = 1$. We now introduce the classical van der Corput sequence [2] [6].

**Definition 2.2** Let $p \geq 2$ be an integer. Every integer $n \geq 0$ has a unique digit expansion

$$n = \sum_{j=0}^{\infty} a_j(n) p^j, \quad a_j(n) \in \{0, 1, \ldots, p-1\} \quad \text{for all} \quad j \geq 0,$$

in base $p$. Let $\tau = \{\tau_j\}_{j \geq 0}$ be a set of permutations $\tau_j$ of $\{0, 1, \ldots, p-1\}$. Then the radical-inverse function $\phi_p^\tau$ is defined by

$$\phi_p^\tau(n) = \sum_{j=0}^{\infty} \tau_j(a_j(n)) p^{-j-1} \quad \text{for all integers} \quad n \geq 0.$$  \hspace{1cm}

The van der Corput sequence in base $p$ with digit permutations $\tau$ is the sequence $\{\phi_p^\tau(n)\}_{n=0}^{\infty} \subset [0, 1)$.

**Theorem 2.2** ([2][6]) For an arbitrary integer $p \geq 2$, the van der Corput sequence in base $p$ is a low-discrepancy sequence.
3 $\beta$-adic transformation

In this section we define the fibred system and the $\beta$-adic transformation, following [5] [13].

$C, R, Z,$ and $N$ are the sets of all complex numbers, all real numbers, all integers, and all natural numbers, respectively. We also set

$$R_{>a} = \{r \in R \mid r > a\}$$
$$Z_{\geq n} = \{i \in Z \mid i \geq n\}$$

and so on. For $x \in R$, $[x]$ denotes the integer part of $x$.

Definition 3.1 Let $B$ be a set and $T : B \to B$ be a map. The pair $(B, T)$ is called a fibred system if the following conditions are satisfied:

1. There is a finite countable set $A$.
2. There is a map $k : B \to A$, and the sets

$$B(i) = k^{-1}([i]) = \{x \in B : k(x) = i\}$$

form a partition of $B$.
3. For an arbitrary $i \in A$, $T|_{B(i)}$ is injective.

Definition 3.2 Let $\Omega = A^N$ and $\sigma : \Omega \to \Omega$ be the one-sided shift operator. Let $k_j(x) = k(T^{j-1}x)$. We derive a canonical map $\varphi : B \to \Omega$ from

$$\varphi(x) = \{k_j(x)\}_{j=1}^{\infty}.$$ 

$\varphi$ is called the representation map.

We have the following commutative diagram:

$$\begin{array}{ccc}
B & \xrightarrow{T} & B \\
\downarrow\varphi & & \downarrow\varphi \\
\Omega & \xrightarrow{\sigma} & \Omega
\end{array}$$

Definition 3.3 If a representation map $\varphi$ is injective, $\varphi$ is called a valid representation.

Definition 3.4 Let $\omega \in \Omega$. If $\omega \in \text{Im}(\varphi)$, $\omega$ is called an admissible sequence.

Definition 3.5 The cylinder of rank $n$ defined by $a_1, a_2, \ldots, a_n \in A$ is the set

$$B(a_1, a_2, \ldots, a_n) = B(a_1) \cap T^{-1}B(a_2) \cap \ldots \cap T^{-n+1}B(a_n).$$

We define $B$ to be a cylinder of rank 0.

For a sequence $a \in \Omega$, we write the $i$-th element of $a$ as $a(i)$, that is, $a = (a(0), a(1), a(2), \ldots)$.

Definition 3.6 Let $\beta > 1$ and $\beta \in R$. Let $f_\beta : [0, 1) \to [0, 1)$ be the function defined by

$$f_\beta(x) = \beta x - \lfloor \beta x \rfloor.$$ 

Let $A = \mathbb{Z} \cap [0, \beta)$. Then we have the following fibred system $([0, 1), f_\beta)$:

$$\begin{array}{ccc}
[0, 1) & \xrightarrow{f_\beta} & [0, 1) \\
\downarrow\varphi & & \downarrow\varphi \\
\Omega & \xrightarrow{\sigma} & \Omega
\end{array}$$

The representation map $\varphi$ of this fibred system is defined as follows:

$$\varphi(x)(n) = k, \text{ if } \frac{k}{\beta} \leq f_\beta^n(x) < \frac{(k + 1)}{\beta}.$$
where $f^n_\beta(x) = x$, and $f^{n+1}_\beta(x) = f_\beta(f^n_\beta(x))$. Let $X_\beta$ be the closure of $\text{Im}(\varphi)$ in the product space $\Omega$ with the product topology. The lexicographical order $< (\text{resp. } >)$ is defined in $\Omega$ as follows: $\omega < \omega'$ (resp. $\omega > \omega'$) if and only if there exists an integer $n$ such that $\omega(k) = \omega'(k)$ for $k < n$ and $\omega(n) < \omega'(n)$ (resp. $\omega(n) > \omega'(n)$). We also define $\leq (\text{resp. } \geq)$ as $<$ (resp. $>$) or equal. In this situation, we set

$$f^n_\beta(1) = \lim_{n \to 1} f^n_\beta(x),$$

$$\zeta_\beta = \max\{X_\beta\} = \varphi(1),$$

and

$$\rho_\beta(a) = \sum_{n=0}^{\infty} a(n)\beta^{-n-1}.$$  

We have the following diagram:

\[
\begin{array}{ccc}
[0,1] & \xrightarrow{f_\beta} & [0,1] \\
\varphi \downarrow \rho_\beta & & \varphi \downarrow \rho_\beta \\
X_\beta & \xrightarrow{\sigma} & X_\beta
\end{array}
\]  

(3.2)

This diagram is called a $\beta$-adic transformation.

We use the following notation for periodic sequences:

$$(a_1, a_2, \ldots, a_n, \ldots) = (a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m},$$

$$a_n, a_{n+1}, \ldots, a_{n+m},$$

$$\vdots,$$

$$a_n, a_{n+1}, \ldots, a_{n+m},$$

$$\ldots)$$

We introduce the following proposition from Ito and Takahashi [5].

**Proposition 3.1** For an arbitrary $\beta \in \mathbb{R}_{>1}$ the following statements hold in (3.2).

1. $\sigma \circ \varphi = \varphi \circ f_\beta$ on $[0,1)$.
2. $\varphi : [0,1) \to X_\beta$ is an injection and is strictly order-preserving, i.e., $t < s$ implies that $\varphi(t) < \varphi(s)$.
3. $\rho_\beta \circ \varphi = \text{id}$ on $[0,1]$.
4. $\varphi \circ \sigma = f_\beta \circ \rho_\beta$ on $\text{Im}(\varphi)$.
5. $\rho_\beta : X_\beta \to [0,1]$ is a continuous surjection and is order-preserving, i.e., $\omega < \omega'$ implies that $\rho_\beta(\omega) \leq \rho_\beta(\omega')$.
6. For an arbitrary $t \in [0,1]$, $\rho_\beta^{-1}(t)$ consists either of a one point $\varphi(t)$ or of two points $\varphi(t)$ and $\sup\{\varphi(s) | s < t\}$. The latter case occurs only when $f^n_\beta(t) = (0)$ for some $n > 0$.

We also remark that the following proposition holds:

**Proposition 3.2**

$$X_\beta = \{\omega \in \Omega \mid \sigma^n \omega \leq \zeta_\beta, \text{ for all } n \geq 0\}$$

**Definition 3.7** Let $u \in X_\beta$. If there exist $n \in \mathbb{Z}_{\geq 1}$ which satisfies $u(i) = u(i + n)$ for any $i \in \mathbb{Z}$, $u$ is called a periodic sequence. When $u \in X_\beta$ is periodic, we define the period of $u$ as $\min\{n \in \mathbb{Z}_{\geq 1} \mid u(i) = u(i + n) \text{ for any } i \in \mathbb{Z}\}$.

The following definition and theorem are from Parry [9].
Definition 3.8 When $\zeta_\beta$ is periodic and its period is $m$, $\beta$ and $\beta$-adic transformation (3.2) are called Markov or simple. In this case, $\beta$ is the unique $z > 1$ solution of the following equation:

$$z^m - \sum_{i=1}^{m} a_{i-1} z^{m-i} = 0$$  (3.3)

where $\zeta_\beta = (a_0, a_1, \ldots, a_{m-2}, (a_{m-1} - 1))$. This equation is called the characteristic equation of $\beta$. When $\beta$ is Markov, $p(\beta)$ denotes the length of the period of $\zeta_\beta$.

Theorem 3.1 The conjugates of $\beta$ with respect to its characteristic equation have absolute values less than 2.

When $\beta$ is not necessarily Markov, the notion of the characteristic equation is generalized as follows. This function was first studied in Takahashi [14][15] and Ito and Takahashi [5].

Definition 3.9

$$\phi_\beta(z) = \sum_{n=0}^{\infty} \zeta_\beta(n) \left( \frac{z}{\beta} \right)^{n+1}$$

We also have the following proposition from Ito and Takahashi [5].

Proposition 3.3 $\phi_\beta(z)$ converges in a neighborhood of the unit disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ and the function $1 - \phi_\beta(z)$ has only one simple root at $z = 1$ in a neighborhood of the unit disk.

Remark 3.1 When $\beta$ is Markov, $1 - \phi_\beta(\beta/z) = 0$ becomes the characteristic equation of $\beta$.

4 Constructing the sequence

In this section, a sequence $\mathcal{N}_\beta \subset [0, 1)$ is defined by the use of $\beta$-adic transformation, following [7]. Let $\beta \in \mathbb{R}_{>1}$ and let $([0, 1], \alpha, \sigma, \phi, \rho_\beta)$ be a $\beta$-adic transformation (3.2). Let $B = [0, 1)$, and $A, \Omega, \zeta_\beta, B(a_1, \ldots, a_n)$ be the same as in the previous section.

Definition 4.1 Let $n \in \mathbb{Z}_{\geq 0}$. Define

$$X_\beta(n) = \left\{ \begin{array}{ll}
\{0\}, & n = 0 \\
\{\omega \in X_\beta \mid \sigma^{n-1} \omega \neq (0) \text{ and } \sigma^n \omega = (0)\}, & n \neq 0
\end{array} \right.$$

and

$$Y_\beta(n) = \{\omega(0), \ldots, \omega(n-1) \mid \omega \in X_\beta\},$$

and

$$Y_\beta^0(n) = \{(a_0, \ldots, a_{n-1}) \mid (a_0, \ldots, a_{n-2}, a_{n-1} + 1) \in Y_\beta(n)\}.$$

Let $k \in \mathbb{Z}_{\geq 0}$, $u \in Y_\beta(k)$, and $v \in Y_\beta(l)$. Define $Y_\beta(u; n), Y_\beta^0(u; n), Y_\beta(u; n; v), Y_\beta^1(u; n; v), G_\beta(n), G_\beta(u; n), G_\beta^0(n), G_\beta^0(u; n),$ and $G_\beta^1(u; n; v)$ as follows:

$$Y_\beta(u; n) = \{u \cdot \omega \mid u \cdot \omega \in Y_\beta(k + n)\}$$

$$Y_\beta^0(u; n) = \{u \cdot \omega \mid u \cdot \omega \in Y_\beta^0(k + n)\}$$

$$Y_\beta(u; n; v) = \{u \cdot \omega \cdot v \mid u \cdot \omega \cdot v \in Y_\beta(k + n + l)\}$$

$$Y_\beta^1(u; n; v) = \{u \cdot \omega \cdot v \mid u \cdot \omega \cdot v \in Y_\beta^1(k + n + l)\}$$

$$G_\beta(n) = \# Y_\beta(n)$$

$$G_\beta^0(n) = \# Y_\beta^0(n)$$

$$G_\beta(u; n) = \# Y_\beta(u; n)$$

$$G_\beta^0(u; n) = \# Y_\beta^0(u; n)$$

$$G_\beta(u; n; v) = \# Y_\beta(u; n; v)$$

$$G_\beta^1(u; n; v) = \# Y_\beta^1(u; n; v)$$

where $u \cdot v$ means the concatenation of $u$ and $v$, that is to say,

$$u \cdot v = (u(0), \ldots, u(n-1), v(0), v(1), \ldots).$$

Finally we set $Y_\beta(0) = Y_\beta^0(0) = \{\epsilon\}$ where $\epsilon$ is the empty word and satisfies $\epsilon \cdot u = u \cdot \epsilon = u$ for any $u \in Y_\beta(n)$. 

Definition 4.2 Define the right-to-left lexicographical order $\preceq_\beta$ in $\bigcup_{n=0}^\infty X_\beta(n)$ as follows: $\omega \preceq_\beta \omega'$ if and only if $(\omega(n-1), \ldots, \omega(0)) < (\omega'(n-1), \ldots, \omega'(0))$ where $\omega \in X_\beta(n)$ and $\omega' \in X_\beta(m)$.

Definition 4.3 ($N_\beta$ [7]) Define $L_\beta = \{\omega_i\}_{i=0}^\infty$ as $\bigcup_{n=0}^\infty X_\beta(n)$ ordered in right-to-left lexicographical order, that is, $L_\beta$ is $\bigcup_{n=0}^\infty X_\beta(n)$ as a set and $\omega_i \preceq_\beta \omega_j$ holds for all $i < j$. Then, the sequence $N_\beta$ is defined as follows:

$$N_\beta = \{\rho_\beta(\omega_i)\}_{i=0}^\infty.$$

Example 4.1 If $\beta = \frac{1+\sqrt{5}}{2}$, then $B = (1, 0)$ and elements of $N_\beta$ are calculated as follows:

$$
\begin{align*}
N_\beta(0) &= \rho_\beta(0) = 0 \\
N_\beta(1) &= \rho_\beta(1) = 0.618033988749895 \ldots \\
N_\beta(2) &= \rho_\beta(01) = 0.381966011250106 \ldots \\
N_\beta(3) &= \rho_\beta(001) = 0.236067987749979 \ldots \\
N_\beta(4) &= \rho_\beta(101) = 0.854101996249686 \ldots \\
N_\beta(5) &= \rho_\beta(0001) = 0.145898033750316 \ldots \\
N_\beta(6) &= \rho_\beta(1001) = 0.76393202250212 \ldots \\
N_\beta(7) &= \rho_\beta(0101) = 0.527864045000422 \ldots \\
N_\beta(8) &= \rho_\beta(00001) = 0.090169943749474 \ldots \\
N_\beta(9) &= \rho_\beta(10001) = 0.70820393249937 \ldots \\
N_\beta(10) &= \rho_\beta(01001) = 0.47213595499581 \ldots \\
N_\beta(11) &= \rho_\beta(00101) = 0.326237921249265 \ldots \\
N_\beta(12) &= \rho_\beta(10101) = 0.94427190999161 \ldots \\
N_\beta(13) &= \rho_\beta(000001) = 0.055728090008416 \ldots \\
N_\beta(14) &= \rho_\beta(100001) = 0.673762078750737 \ldots \\
N_\beta(15) &= \rho_\beta(010001) = 0.437694101250947 \ldots \\
N_\beta(16) &= \rho_\beta(0100001) = 0.437694101125094 \ldots \\
\vdots
\end{align*}
$$

From this definition, we immediately have the following proposition:

Proposition 4.1 If $\beta$ is an integer greater than 2 then $N_\beta$ is the van der Corput sequence in base $\beta$ with all digit permutations $\tau_2 = \text{id}$.

From Theorem 2.2 and Proposition 4.1, we see that if $\beta \in \mathbb{Z}_{\geq 2}$ then $N_\beta$ is a low-discrepancy sequence, that is to say, $D_M^*(N_\beta) = O(M^{-1} \log M)$ holds for all $\beta \in \mathbb{Z}_{\geq 2}$. We also have the following theorem:

Theorem 4.1 Let $\beta$ be a real number greater than 1, and let the following condition (PV) hold:

(PV) All zeroes of $1 - \phi_\beta(z)$ except for $z = 1$ belong to $\{z \in \mathbb{C} \mid |z| > \beta\}$.

Then,

$$D_M^*(N_\beta) = O\left(\frac{(\log M)^2}{M}\right)$$

holds. Moreover, if $\beta$ is Markov, then

$$D_M(N_\beta) = O\left(\frac{\log M}{M}\right)$$

holds.

Remark 4.1 When $\beta$ is Markov, the condition (PV) is equivalent to the condition that all conjugates of $\beta$ with respect to its characteristic equation (3.3) belong to $\{z \in \mathbb{C} \mid |z| < 1\}$.

Remark 4.2 In [7], the case in which $\beta$ is Markov is proved.
To prove this theorem, we provide lemmas and definitions. We use the following notations:

$$\omega[i,j] = \begin{cases} (\omega(i), \ldots, \omega(j-1)), & i < j \\ \epsilon, & i = j \end{cases}$$

where \(\omega \in \mathcal{X}_\beta\) and \(i, j \in \mathbb{Z}_{\geq 0}\). \(R_\beta(u) = \lambda(B(u))\) where, \(\lambda\) is the one-dimensional Lebesgue measure, \(u \in \mathcal{X}_\beta(n)\), and \(B(u)\) is the cylinder (3.5). For a sequence \(S, S[N]\) denotes the point set consisting of the first \(N\) elements of \(S\), and \(S[N; M] = S[N + M] \setminus S[N]\).

**Definition 4.4** For any \(k \geq 0\) and \(u \in \mathcal{Y}_\beta(k)\), define

$$
eq \{i \in \mathbb{Z}_{\geq 0} \mid \zeta_\beta[0, i + 1) \cdot u \notin \mathcal{Y}_\beta(k + i + 1)\}.$$

**Lemma 4.1 ([5])** For an arbitrary \(k \geq 0\) and \(u \in \mathcal{Y}_\beta(k)\), we have the following partitioning of \(\mathcal{Y}_\beta(u; n)\):

$$\mathcal{Y}_\beta(u; n) = \bigcup_{j=1}^{n} \mathcal{Y}_\beta^0(u; j) \cdot \zeta_\beta[0, n - j) \bigcup \max\{\mathcal{Y}_\beta(u; n)\}$$

**Proof.** It is trivial to show that the left-hand side includes the right-hand side.

If \(v = (a_1, \ldots, a_{n+k}) \in \mathcal{Y}_\beta(u; n) \setminus \mathcal{Y}_\beta^0(u; n)\) and \(v \neq \max\{\mathcal{Y}_\beta(u; n)\}\), then there exists an integer \(l\) that satisfies

$$k + 1 \leq l \leq n + k$$

and

$$\min\{w \in \mathcal{Y}_\beta(u; n) \mid w > v\} = (a_1, \ldots, a_{l} + 1, 0, \ldots, 0).$$

This means that

$$(a_{l+1}, \ldots, a_{n+k}) = \zeta_\beta[0, n - l)$$

and

$$(a_1, \ldots, a_{l-1}, a_{l} + 1) \in \mathcal{Y}_\beta^0(u; l - k)$$

hold. \(\square\)

Taking account of Lemma 4.1, we give the following definition:

**Definition 4.5** For an arbitrary \(u \in \mathcal{Y}_\beta(n)\), define an integer \(d(u)\) as follows: \(d(u) = k\) if

$$u \in \mathcal{Y}_\beta^0(k) \cdot \zeta_\beta[0, n - k)$$

holds. Remark that \(\max\{\mathcal{Y}_\beta(n)\} = \zeta_\beta[0, n)\).

From Lemma 4.1, Definition 4.4, and Definition 4.5 we have the following lemma:

**Lemma 4.2** For any \(k, l, n \geq 0, u \in \mathcal{Y}_\beta(k),\) and \(v \in \mathcal{Y}_\beta(l)\), we have the following partitioning of \(\mathcal{Y}_\beta(u; n; v)\):

$$\mathcal{Y}_\beta(u; n; v) \cong \left\{ \begin{array}{ll}
\bigcup_{1 \leq j \leq n} \mathcal{Y}_\beta^0(u; j) \cdot \zeta_\beta[0, n - j), & \text{if} \quad n + k - d(\max\{\mathcal{Y}_\beta(u; n)\}) - 1 \in \mathcal{e}(v) \\
\bigcup_{1 \leq j \leq n} \mathcal{Y}_\beta^0(u; j) \cdot \zeta_\beta[0, n - j) \bigcup \max\{\mathcal{Y}_\beta(u; n)\}, & \text{otherwise.}
\end{array} \right.$$}

**Lemma 4.3** For any \(n \geq 0\) and \(u \in \mathcal{Y}_\beta(n)\),

$$R_\beta(u) = \frac{1}{\beta^{d(u)}} \left( 1 - \sum_{i=0}^{n-d(u)-1} \zeta_\beta(i) \right)$$

holds.
Proof. Let \( u = u^0 \cdot \zeta_{\rho}(0, n - d(u)) \) where \( u^0 \in Y_{\rho}^0(d(u)) \). From Definition 3.6,

\[
R_{\rho}(u^0) = \rho_{\rho}((u^0(0), \ldots, u^0(d(u) - 1) + 1) - \rho_{\rho}((u^0(0), \ldots, u^0(d(u) - 1)) = \frac{1}{\rho_\rho(u)}
\]

and

\[
R_{\rho}(\zeta_{\rho}(0, n - d(u))) = 1 - \frac{1}{\rho_{\rho}^{(1)}}.
\]

When \( v \cdot w \in Y_{\rho}(m) \), it follows that \( R_{\rho}(v \cdot w) = R_{\rho}(v)R_{\rho}(w) \). Then, the lemma holds. \( \square \)

Remark 4.3 From Definition 3.6, it follows that

\[
f_{\rho}^n(x) = \beta^n \left( x - \sum_{i=0}^{n-1} \frac{\varphi(x)(i)}{\beta^{i+1}} \right)
\]

for any \( x \in [0, 1] \) and \( n \geq 0 \). Then, we have

\[
R_{\rho}(u) = \frac{1}{\beta^n} f_{\rho}^{n-d(u)}(1)
\]

for any \( u \in Y_{\rho}(n) \) and \( n \geq 0 \), from Lemma 4.3.

Lemma 4.4 ([3]) Let \( r \) be the absolute value of the second smallest zero of \( 1 - \varphi_{\rho}(x) \), that is, \( r = \min\{|z| \mid z \in \mathbb{C}, \ z \neq 1\} \). Then for any small \( \varepsilon > 0 \), there exists a constant \( C_{\varepsilon} > 0 \) and

\[
\left| G_{\rho}^0(u; n) - \frac{\beta^n + 1 R_{\rho}(u)}{\rho_{\rho}^{(1)}} \right| \leq \frac{C_{\varepsilon}}{n} \left( \frac{\beta}{r - \varepsilon} \right)^n
\]

holds for any \( n \geq 0, k \geq 0 \) and \( u \in Y_{\rho}(k) \).

Proof. Let \( k \geq 0 \) and \( u \in Y_{\rho}(k) \). Remark that

\[
R_{\rho}(u) = \sum_{u \cdot v \in Y_{\rho}(u; n)} R_{\rho}(u \cdot v)
\]

holds. From (4.1), Lemma 4.1, and Remark 4.3, we have

\[
\beta^{n+k} R_{\rho}(u) = \sum_{j=0}^{n-1} f_{\rho}^{j}(1) G_{\rho}^0(u; n-j) + f_{\rho}^{n+1}(1)
\]

where \( l = k - d(\max\{Y_{\rho}(u; n)\}) \geq 0 \). Remark that the formal power series

\[
\sum_{n \geq 1} z^n \sum_{j=0}^{n-1} f_{\rho}^{j}(1) G_{\rho}^0(u; n-j) \beta^{-(n+k)}
\]

converges for \( |z| < 1 \). We have the following equality from (4.2):

\[
\beta^k \sum_{n \geq 1} z^n R_{\rho}(u) = \sum_{n \geq 1} \left( \frac{z}{\beta} \right)^n \sum_{j=0}^{n-1} f_{\rho}^{j}(1) G_{\rho}^0(u; n-j) + \sum_{n \geq 1} \left( \frac{z}{\beta} \right)^n f_{\rho}^{n+1}(1)
\]

(4.3)

We also have

\[
\sum \left( \frac{z}{\beta} \right)^n \sum_{j=0}^{n-1} f_{\rho}^{j}(1) G_{\rho}^0(u; n-j) = \sum \sum \sum \sum f_{\rho}^{j-1}(1) G_{\rho}^0(u; n-j+1) \left( \frac{z}{\beta} \right)^n
\]

\[
= \sum_{j \geq 1} \sum_{n \geq j} f_{\rho}^{j}(1) \left( \frac{z}{\beta} \right)^j \sum_{n \geq 1} G_{\rho}^0(u; n) \left( \frac{z}{\beta} \right)^n
\]
and, from Remark 4.3,

\[
(1 - z) \sum_{n \geq 0} f_\beta^n(1) \left( \frac{z}{\beta} \right)^n = (1 - z) + (1 - z) \sum_{n \geq 1} \left( 1 - \sum_{i=0}^{\beta^{n-1}} \zeta_\beta(i) \right) (z/\beta)^n
\]

\[
= 1 - \frac{\zeta_\beta(0)}{\beta} + \sum_{n \geq 1} (1 - z) \left( 1 - \sum_{i=0}^{\beta^{n-1}} \zeta_\beta(i) \right) (z/\beta)^n
\]

\[
= 1 - \frac{\zeta_\beta(n)}{\beta} = 1 - \frac{\zeta_\beta(n)}{\beta} = 1 - \phi_\beta(z).
\]

By using these two equalities, we obtain from (4.3) that

\[
\sum_{n \geq 1} G_\beta^n(u; n) \left( \frac{z}{\beta} \right)^n = \frac{z \beta^k R_\beta(u)}{1 - \phi_\beta(z)} - \frac{(1 - z) \sum_{n \geq 1} f_\beta^n(1)(z/\beta)^n}{1 - \phi_\beta(z)}.
\]

(4.4)

Consider the function

\[
h_u(z) = \sum_{n \geq 1} \left( G_\beta^n(u; n) \left( \frac{z}{\beta} \right)^n - \frac{\beta^k R_\beta(u)}{\phi_\beta(1)} (z/\beta)^n \right)
\]

\[
= \frac{z \beta^k R_\beta(u)}{1 - \phi_\beta(z)} - \frac{(1 - z) \sum_{n \geq 1} f_\beta^n(1)(z/\beta)^n}{1 - \phi_\beta(z)} - \frac{z \beta^k R_\beta(u)}{(1 - z) \phi_\beta(1)}.
\]

(4.5)

The second equality comes from (4.4). From Proposition 3.3, we see that \( h_u(z) \) is analytic in a neighborhood of \( \{z \in \mathbb{C} | |z| \leq r - \epsilon, z \neq 1\} \). We also see from (4.5) that \( \lim_{z \rightarrow 1} (1 - z) h_u(z) = 0 \). Considering the fact that \( \beta^k R_\beta(u) \leq 1 \) for any \( u \in Y_\beta(k), k \geq 1 \) and that the second term of the right-hand side of (4.4) and its derivative are bounded uniformly in \( l \), we see that there exists a constant \( C_\epsilon \) and

\[
\sup_{k \geq 1, u \in Y_\beta(k)} |h'_u(z)| < C_\epsilon
\]

(4.6)

holds. Then we have

\[
n! \left| \frac{G_\beta^n(u; n) \beta^n}{\phi_\beta(1)} - \frac{\beta^k R_\beta(u)}{\phi_\beta(1)} \right| = \left| h_u^{(n)}(0) \right| = \left| \frac{d^{n-1}h'_u(0)}{dz^{n-1}} \right| = \left| \frac{(n - 1)!}{2\pi(r - \epsilon)^n} \int_{|z|=r-\epsilon} h'_u(z) \frac{dz}{z} \right| \leq \frac{(n - 1)!}{2\pi(r - \epsilon)^n} \frac{C_\epsilon}{(r - \epsilon)^n}
\]

and the lemma follows.

Lemma 4.5 If \( \beta \in \mathbb{R}_{>1} \) is Markov and \( \zeta_\beta = (a_0, \ldots, a_{m-2}, (a_{m-1} - 1)) \), where \( m = p(\beta) \), then we have the following statements:

1. For an arbitrary \( v \in X_\beta \), \( \{G_\beta^n(n)\}_{n=0}^\infty \) and \( \{G_\beta^n(n)\}_{n=0}^\infty \), satisfy the following linear recurrent equation:

\[
G_\beta(\epsilon; n + m; v) - \sum_{i=0}^{m-1} a_i G_\beta(\epsilon; n + m - i - 1; v) = 0.
\]

(4.7)

2. For arbitrary \( u \in Y_\beta(k), k \geq m \) and \( v \in X_\beta \), the following equation holds for any \( n \geq m - k + d \):

\[
G_\beta(u; n; v) = \left\{ \begin{array}{ll}
\sum_{i=1}^{m-k+d} a_{k-d+i} G_\beta(\epsilon; n - i; v) & \text{when } d > k - m \\
G_\beta(\epsilon; n; v) & \text{when } d = k - m
\end{array} \right.
\]

(4.8)

where \( d = d(u[\max(0, k - m + 1), k + 1]) + k - m \).
Proof. From Proposition 3.2, we have the following partitioning:

\[ Y_{\beta}(\epsilon; n + m; v) = \bigcup_{j=0}^{m-1} \bigcup_{i=0}^{j-1} \zeta_{\beta}[0, j] \cdot i \cdot Y_{\beta}(\epsilon; n + m - j - 1; v). \]

When \( d = k - m \), it is trivial to obtain this partitioning from Proposition 3.2. When \( d > k - m \), we obtain the following partitioning from the same proposition.

\[ Y_{\beta}(u; n; v) = \bigcup_{j=1}^{m-k+d} \bigcup_{i=0}^{j-1} u \cdot i \cdot Y_{\beta}(\epsilon; n - j; v) \]

The lemma follows from these partitionings. \( \square \)

Proof of Theorem 4.1. Let \( k > 0 \), \( u \in Y_{\beta}(k) \). Let \( M \in \mathbb{N} \) and \( b = (b_0, b_1, \ldots, b_{m-1}) = L_{\beta}(M) \). We assume \( M \) to satisfy \( m > k \). Define

\[ \Delta(I; P) = A(I; P) - M\lambda(I), \]

where \( I \) is an interval in \([0, 1)\) and \( P = \{x_1, x_2, \ldots, x_M\} \subset [0, 1) \). For any finite sets of points \( P, P' \) in \([0, 1)\) and any intervals \( I, I' \subset [0, 1), I \cap I' = \emptyset \),

\[ \Delta(I; P \cup P') = \Delta(I; P) + \Delta(I'; P') \]

holds. Here, \( P \cup P' \) is the disjoint union of \( P \) and \( P' \) or the union of \( P \) and \( P' \) with multiplicity. From Definition 4.3 and (4.9), we have

\[ \Delta(B(u); N_{\beta}[M]) = \sum_{j=0}^{m-1} \sum_{i=0}^{j-1} \Delta(B(u); Y_{\beta}(\epsilon; j; v_{ij})) \]

where \( v_{ij} = i \cdot b(j + 1, m) \). Consider the \( 0 \leq j \leq k \) part of the right hand side of (4.10).

\[ \sum_{j=0}^{k} \sum_{i=0}^{j-1} |\Delta(B(u); Y_{\beta}(\epsilon; j; v_{ij}))| \leq \sum_{j=0}^{k} (|\beta| + 1)G_{\beta}(j)R_{\beta}(u) \]

(4.11)

holds from the definition of \( \Delta \). Since \( R_{\beta}(u) \leq \beta^{-k} \) and \( G_{\beta}(j) \leq (|\beta| + 1)^j \), there exists a constant \( C_0 \), and

\[ \sum_{j=0}^{k} (|\beta| + 1)G_{\beta}(j)R_{\beta}(u) < C_0 \]

is satisfied for any \( k \). Then, from (4.10) and (4.11), we have

\[ \Delta(B(u); N_{\beta}[M]) \leq C_0 + \sum_{j=k+1}^{m-1} \sum_{i=0}^{j-1} |\Delta(B(u); Y_{\beta}(\epsilon; j; v_{ij}))|. \]

(4.12)

Define

\[ \delta(u; n) = G_{\beta}^0(u; n) - \frac{\beta^{n+k}R_{\beta}(u)}{\phi_{\beta}'(1)} \]

\[ \delta(n) = G_{\beta}^0(n) - \frac{\beta^n}{\phi_{\beta}'(1)} \]

for \( u \in Y_{\beta}(k) \) and \( k, n \geq 0 \). From this definition,

\[ |\Delta(B(u); Y_{\beta}^0(n))| = |G_{\beta}^0(u; n) - R_{\beta}(u)G_{\beta}^0(k + n)| \]

(4.13)
holds. Then, from Lemma 4.2 we have
\[
\sum_{j=k+1}^{m-1} \sum_{i=0}^{b_j-1} |\Delta(B(u); Y_\beta(c, j, v_d))| \leq \sum_{j=k+1}^{m-1} \sum_{i=0}^{b_j-1} \left( \sum_{l=1}^{t_{j+1}} |\Delta(B(u); Y_\beta^{(l)}(\cdot \cdot \cdot j - 1) \cdot \not\in (v_u) \sum_{=0}^{r} \lambda(B(u); Y_\beta^{(l)})| + 1 \right).
\]
(4.14)

From the (PV) condition and Lemma 4.4, there exist \( r > \beta \) and a constant \( C_r \) that satisfies
\[
|\Delta(B(u); N_{\beta}[M])| \leq C_0 + C_r([\beta] + 1) \sum_{j=k+1}^{m-1} \left( \sum_{l=1}^{t_{j+1}} \left( \frac{1}{l} \left( \frac{\beta}{r} \right)^l \frac{1}{k+l} \left( \frac{\beta}{r} \right)^{k+l} R_\beta(u) \right) + 1 \right) = O(M) = O(\log M)
\]
holds.

Choose an arbitrary \( t \in [0, 1) \). Let \( M \in \mathbb{N} \) and \( L_{\beta}(M) = (b_0, \ldots, b_{m-1}) \). Let \( B(t_0, \ldots, t_{m-1}) \) be a cylinder of rank \( m \) that satisfies \( t \in B(t_0, \ldots, t_{m-1}) \). Then we have
\[
[0, t) = B_{t_1} \cup B_{t_2} \cup \ldots \cup B_{t_k} \cup R,
\]
where \( 0 \leq s_1 < s_2 < \ldots < s_k = m - 1 \), \( B_{s_i} \) is a cylinder of rank \( s_i \) and \( \lambda(R) < \beta^{-m+1} \). Then from (4.9) and (4.16), we have
\[
|\Delta([0, t); N_{\beta}[M])| = O((\log M)^2),
\]
and therefore
\[
D^*_M(N_{\beta}) = O \left( \frac{\log M^2}{M} \right).
\]

In the following part, we consider the case in which \( \beta \) is Markov. Let \( l = p(\beta) \) and \( \zeta_\beta = (\alpha_0, \ldots, \alpha_{l-2}, (\alpha_{l-1} - 1)) \). Then, \( \beta \) is the unique \( z > 1 \) solution of
\[
z^l - \sum_{i=0}^{l-1} a_i z^{l-1-i} = 0.
\]
(4.17)

Let \( \alpha_1, \ldots, \alpha_q \) be the conjugates of \( \beta \) with respect to the equation (4.17), that is,
\[
z^l - \sum_{i=0}^{l-1} a_i z^{l-1-i} = (z - \beta) \prod_{i=1}^{q} (z - \alpha_i)^{l_i}
\]
where \( l_i \geq 1, \alpha_i \neq \alpha_j \) for all \( i \neq j \) and \( \sum_{i=1}^{q} l_i = l - 1 \). We also have
\[
|\alpha_i| < 1, \quad \text{for all } i \in \{1, \ldots, q\}
\]
(4.18)
from the (PV) condition. Let \( v \in X_\beta \). From Lemma 4.5, there exist complex numbers \( c, c_{ij} \) (\( i = 1, \ldots, q, \ j = 0, \ldots, l_i - 1 \)) that satisfy the following equation:
\[
G_\beta(c; n; v) = c\beta^n + \sum_{i=1}^{q} \sum_{j=0}^{l_i-1} c_{ij} n^j \alpha_i^n \quad \text{for all } n \in \mathbb{N}.
\]
(4.19)
From Lemma 4.3, Lemma 4.5, and (4.19), we have

$$
\Delta(B(u); N_\beta[G_\beta(c; k + n; v)])
= \left\{ \begin{array}{ll}
\sum_{h=1}^{k} \sum_{j=0}^{l} c_{\beta^j} \left( n^2 \alpha_h^{n} - \frac{1}{\beta^k} (k + n)^2 \alpha_h^{k+n} \right), & \text{when } d = k - l \\
\sum_{i=k-d}^{l-1} \sum_{h=1}^{k} \sum_{j=0}^{l} c_{\beta^j} \left( (k + n - d)^2 \alpha_h^{k+n-d-i} - \frac{1}{\beta^{k+i}} (k + n)^2 \alpha_h^{k+n+i} \right), & \text{when } d > k - l
\end{array} \right.
$$

(4.20)

where $u \in Y_\beta(k)$, $n \in \mathbb{N}$, and $d = d(u[\max(0, k - l + 1), k + 1]) + k - l$. From (4.9), (4.12), (4.14), (4.18), and (4.20), there exists a constant $C$ that satisfies the following inequality (4.21) for any cylinder $B(u)$ of any rank $k$ and $M > G_\beta(l + d)$.

$$
|\Delta(B(u); N_\beta[M])| < C
$$

(4.21)

Then, we obtain

$$
D_M(N_\beta) = O \left( \frac{\log M}{M} \right)
$$

by the above reasoning.

参考文献


