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REMARKS ON THE LOCAL GEOMETRY OF ANALYTIC MAPS

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ABSTRACT. In this lecture I present recent results of mine (some obtained jointly with Piotr Tworzewski) concerning the question of how general fibres of complex maps converge to special fibres. These results include in particular: 1. a relation between the number of points in the fibre of a generically discrete map and the dimension of its singular fibres, 2. a relation between the presence of vertical components in fibred powers of a map and openness and flatness, 3. an example of an algebraic map with nonconstructible topology.

1. INTRODUCTION

We will be interested in complex analytic maps of analytic spaces $f : X \to Y$, with $\dim Y > 1$. In the case of maps with values in curves, both algebraic properties (like flatness) and topological properties (like openness or Thom's a_f condition) are relatively easy to understand. This opens the door to the study of finer algebraic and topological properties and has produced a considerable amount of results.

However, when the target space has big dimension, very few general results are available. Indeed, even such basic invariants as the dimensions of different fibres are far from being well understood. The deep reason behind this is that maps which are not Thom's a_f maps may have a wild topological behaviour (see example at the end of this article). Below, we present some recent results concerning the local topology and local algebra of maps, which are non-trivial in the case $\dim Y > 1$.

The results in sections 2 and 3 as well as Theorem 4.1 were obtained jointly with Piotr Tworzewski in [15]. Details concerning Theorems 4.2 and 5.1 can be found in the author's paper [13] and Example 7.1 - in [14].

2. FIBRES OF GENERICALLY DISCRETE MAPS

Let us start by recalling a well known result which says that if Y is smooth and if f is a *birational* map which has an *isolated* fibre of positive dimension, then that fibre is of codimension 1 in the source space. Here, "isolated" means that the neighbouring fibres are all at most zero-dimensional. Think of a blowup map as an example. Let us rephrase this result:

Consider a generically discrete map $f : X \to Y$*, with* Y *smooth. Suppose that it has an isolated fibre of positive dimension* $f^{-1}(y_0)$*. If* $\operatorname{codim}_X f^{-1}(y_0) \geq 2$ *then there are at least* 2 *points in the general fibre.*

In the above statement, two integers appear: both equal 2. Our first question will be: what if we change the first of the two integers to $3, 4, 5, \ldots$. How far can we change the second integer? The answer is the following theorem.

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Theorem 2.1. [15] *Let $f : X \to Y$ be an analytic map of analytic spaces, Y being a complex manifold and both X and Y being of pure dimension d. Assume that the image of every irreducible component of X has nonempty interior in Y. Suppose there is a point $y_0 \in Y$, such that $\dim f^{-1}(y_0) = w_0 > 0$ and $\dim f^{-1}(y) \leq 0$, for $y \neq y_0$. Then there exists an open subset U of Y, such that for all $y \in U$*

$$\# f^{-1}(y) \geq \left[\frac{d-1}{w_0}\right],$$

where the square brackets denote the integer part of a rational number.

All topological notions will always refer to the transcendental topology. An analytic space may have singularities.

In the algebraic case, when one assumes that Y is irreducible, U can be chosen to be dense in Y. In the analytic case, even $f^{-1}(U)$ need not be dense in X. This is why we do not use the word "generic" in the actual statement of the Theorem. The bound of Theorem 2.1 is sharp, as can be seen from the following example.

Example 2.2. *Consider $\mathbb{C}^d$ with coordinates $x_0, \dots, x_{d-1}$ and $\mathbb{CP}^1$ with homogeneous coordinates $(\lambda : \mu)$. Let X be the hypersurface in $\mathbb{C}^d \times \mathbb{CP}^1$ defined by the equation $x_0\lambda^{d-1} + x_1\lambda^{d-2}\mu + \dots + x_{d-1}\mu^{d-1}$ and let $f : X \to \mathbb{C}^d$ be the restriction of the first projection. Then the fibre of f at 0 is of dimension one; all the other fibres are zero-dimensional and the generic fibre consists $d-1$ points (its smallest possible cardinality by Theorem 2.1).*

To generalize Theorem 2.1 to the case of a map with non-isolated singular fibres, we introduce the following notion of an equidimensional partition, which is weaker than that of a stratification of a map.

Definition 2.3. *Let $f : X \to Y$ be an analytic map of analytic spaces. A countable partition $\{X_p\}_{p \in P}$ of X is called an* **equidimensional partition** *(for f) if for each $p \in P$:*

1. *X_p is a nonempty irreducible locally analytic subset of X,*
2. *the restriction $f|_{X_p} : X_p \to Y$ is equidimensional (i.e. all of its nonempty fibres are of pure dimension and of the same dimension).*

Standard arguments in stratification theory provide us with the following proposition.

Proposition 2.4. *For any analytic map $f : X \to Y$, there exists an equidimensional partition of X.*

Remark 2.5. *From an equidimensional partition as above, we can read off the following numerical data*

1. *w_p - the dimension of any fibre of $f|_{X_p}$,*
2. *$k_p = \dim X_p$.*

The generalization of Theorem 2.1 to the case of a generically finite map with non-isolated singular fibres of arbitrary dimensions is the following.

Theorem 2.6. [15] *Let $f : X \to Y$ be an analytic map of analytic spaces, Y being a manifold and both X and Y being of pure dimension d. Let $\{X_p\}_{p \in P}$ be an equidimensional partition of X for f. Suppose that $f^{-1}(f(x))$ is a discrete set* *for x belonging to some open dense subset of X and that f has at least one fibre of positive dimension. Then there exists an open subset U of Y, such that for all $y \in U$*

$$\#f^{-1}(y) \geq \inf\left\{\left[\frac{d-k_p+w_p-1}{w_p}\right] : p \in P,\ w_p > 0\right\}.$$

(square brackets denote the integer part).

The geometric meaning of a single fraction in the above Theorem is roughly the following: the denominator is the dimension of a special fibre and the numerator is the codimension in Y of the locus over which that dimension occurs minus one. As it turns out, the infimum also has a natural geometric meaning - even for non generically finite maps. It will be equal to the value of a topological invariant $\phi(f)$ of the map f. This invariant can be defined for any map (not just generically discrete) and proves to be useful in characterizing its topology.

3. A NEW TOPOLOGICAL INVARIANT

We shall now discuss this invariant. First we need to recall the natural concept of a vertical component. Let $g : Z \to Y$ be an analytic map of complex analytic spaces. We say that an irreducible component of Z is an **isolated vertical component** (for g) iff its image has empty interior in Y with the transcendental topology. For the time being we do not consider the embedded components of Z.

Definition 3.1. *Let $f : X \to Y$ be an analytic map. We define $\phi(f)$ as the supremum of all $i \in \mathbb{N}$, such that the natural projection of the i-fold fibred product (of f by itself)*

$$\underbrace{X \underset{Y}{\times} \cdots \underset{Y}{\times} X}_{i \text{ times}} \to Y$$

has no isolated vertical components (and as zero if no such i exists).

Theorem 3.2. *If f is a generically discrete map, then $\phi(f)$ is equal to the infimum in theorem 2.6.*

In the general case, $\phi(f)$ can also be read off data obtained from an equidimensional partition. This time however, we need some supplementary data.

Remark 3.3. *For an equidimensional partition, in addition to w_p (fibre dimension) and k_p (dimension) defined in Remark 2.5, we introduce $h_p = \min\{\dim_x X : x \in X_p\}$.*

Theorem 3.4. [15] *Let $f : X \to Y$ be an analytic map of analytic spaces, the space Y being smooth of pure dimension d. Let $\{X_p\}_{p \in P}$ be an equidimensional partition of X. Then*

$$\phi(f) = \inf\left\{\left[\frac{d-k_p+w_p-1}{w_p-(h_p-d)}\right] : p \in P,\ w_p > h_p - d\right\},$$

where the square brackets indicate the integer part of a rational number.

Again, the denominator of each fraction in the above theorem is roughly the difference between the dimension of a special fibre and the dimension of the appropriate general fibre and the numerator is the codimension in Y of the locus where that change in dimension occurs minus one.

The proof that theorem 3.4 implies theorem 2.6 essentialy relies on noticing that for generically discrete maps $\phi(f)$ is smaller than the number of points in the generic fibre. This follows from an equivalent description of the invariant ϕ as the maximal number of points of a special fibre, that can be simultaneously approached by points in one sequence of generic fibres.

The very rough idea of the proof of theorem 3.4 is as follows. Inequality "$\leq$": if some fibres have bigger dimensions than others then those dimensions grow faster in the fibred powers and eventually produce isolated vertical components. Inequality "$\geq$": Y is smooth and hence we know the number of local equations for the diagonal in $Y \times \cdots \times Y$ and thus also for $X \underset{Y}{\times} \cdots \underset{Y}{\times} X$ in $X \times \cdots \times X$. This allows us to calculate that isolated vertical components do not appear too soon, otherwise their dimensions would be too small.

4. Openness and flatness

A natural question to ask about the invariant introduced in the preceding section is when is its value infinite. Under the mild assumption that Y is locally irreducible, the asnwer is that $\phi(f)$ is infinite iff f is an open map. Without using the invariant ϕ, the statement reads as follows:

Theorem 4.1. [15] *Let $f : X \to Y$ be an analytic map of analytic spaces. Suppose that Y is locally irreducible. Then the following conditions are equivalent :*

- *f is open,*
- *for any $i \geq 1$, the canonical map $\underbrace{X \underset{Y}{\times} \cdots \underset{Y}{\times} X}_{i\ times} \to Y$ has no isolated vertical components.*

The proof of the above theorem relies mainly on the equivalence between openness and equidimensionality, stratifications and dimension counts.

The natural question now is to see what happens if in the second condition we also take into account the embedded components of the fibred powers. It turns out that we then get a characterization of flat maps :

Theorem 4.2. [13] *Let $f : X \to Y$ be an analytic map of analytic spaces. Suppose that Y is reduced and locally irreducible (i.e. every local ring $\mathcal{O}_{Y,y}$ is an integral domain). Then the following conditions are equivalent :*

- *f is flat,*
- *for any $i \geq 1$, the canonical map $\underbrace{X \underset{Y}{\times} \cdots \underset{Y}{\times} X}_{i\ times} \to Y$ has no (isolated or embedded) vertical components.*

Recall that a component of the source space of a map (be it isolated or embedded) is called a **vertical component** iff its image has empty interior in the target space with the transcendental topology. If no such component exists, we say that the map g **has no vertical components.**

Any example of an open non-flat map shows the difference between the above theorems. Also the methods of proof are quite different.

The hard part of the proof of theorem 4.2 is to show that some fibred power of a non flat map must have a vertical component (isolated or embedded). Our main tool is Hironaka's characterization of flatness ([7], 6, Proposition 10, see also

Bierstone and Milman [3]), which roughly says that a map is flat at a point of the source space if and only if a standard basis of the germ of the fibre passing through that point is a restriction of a standard basis of the germ of the source space at that point. If a map is not flat at some point $\xi \in X$, Hironaka's map κ is not injective. We pick a power series in its kernel and consider the ideal generated by the coefficients of that power series. By Noetherianity, this ideal is generated by some minimal finite set of these coefficients: $a_1, \ldots, a_i$. Now $a = a_1 \wedge \cdots \wedge a_i$ can be regarded as the germ of a function on the i-th fibred power of f at the point $(\xi, \ldots, \xi)$. The construction of a together with some calculations and standard faithful flatness arguments show that a is a nonzero torsion element over the local ring $\mathcal{O}_{Y,f(\xi)}$. Some not too difficult commutative algebra shows that the presence of such a torsion element implies the existence of a vertical component.

5. Commutative algebraic reformulations

In the standard dictionary between affine algebraic geometry and commutative algebra, the existence of vertical components (resp. isolated vertical components) of fibred powers corresponds to the existence of torsion in tensor powers (resp. tensor powers quotiented by the nilradical). Thus our theorems have commutative algebraic reformulations. In particular the following theorem is an affine, commutative algebraic analogue of theorem 4.2.

Theorem 5.1. *Let R be a finitely generated $\mathbb{C}$-algebra and a normal domain. Let A be a finitely generated R-algebra. Then the following are equivalent :*

- *A is R-flat,*
- *for any $i \geq 1$, the i-th tensor power $\underbrace{A \otimes_R \cdots \otimes_R A}_{i \text{ times}}$ is a torsion-free R-module.*

In fact, in the above theorem, "normal" can be replaced by the weaker condition that each localization at a maximal ideal is analytically irreducible.

The study of torsion in tensor products of modules (in particular in tensor powers of modules) was initiated by Auslander [2] and recently revived by Huneke and Wiegand [9]. In particular, Theorem 3.2 of [2] says that finitely generated modules with torsion-free tensor powers over unramified regular local rings are free and hence implies our theorem 5.1 in the case when A is finitely generated as an R-module. After some work it also implies theorem 4.2 for proper maps with finite fibres. Another special case of theorem 5.1, when A is a symmetric algebra, was studied by the author in [11] and applied in [12] to produce new bounds on codimensions of determinantal varieties.

6. An open problem

A natural problem for further research is to determine a value of i in theorems 4.2 and 5.1, for which it is enough to look at a fibred (resp. tensor) power in order to determine flatness. In the cases studied in [2] and in Theorem 4.1, $i = \dim Y$ is sufficient. The first question to answer would be if it is also sufficient in theorems 4.2 and 5.1.

7. A complex map with nonconstructible topology

In this section we explain why studying the topology of algebraic maps, whose target space is of big dimension, is so hard. Indeed, unlike the topology of complex spaces, where the theory of Whitney stratifications is available, we have no general theory describing the topology of maps. Stratifications of maps give us some topological triviality above each stratum, but no information at all about how the topology behaves as we approach the boundary of these strata. Stratifications satisfying Thom's a_f condition do give such information, but then not all algebraic maps have such stratifications.

Among the few *general* results describing the behaviour of fibres of analytic maps with high dimensional target space, let us cite some notable exceptions: Hironaka's flattening Theorem [7], its local version by Hironaka, Lejeune-Jalabert and Teissier [8], and analogous results by Sabbah [18] (concerning a_f instead of flatness) and Teissier [19] (concerning traingulability). Also should be cited the Relative Lefschetz Theorems of Goresky and MacPherson ([5], Part II, Chapter 1), which give bounds on the homotopy type of a complex analytic space in terms of dimensions of the singular fibres.

Below, we provide a simple example of a complex algebraic map $f : X \to Y$ which (even locally) has an infinite number of different local topological types at points of X. By local topological type, we mean the right-left topological equivalence class of a germ of f. There are related examples of Thom [20] and Nakai [16]. The difference with our's is that they treat the varying of topological type in parametrized families of maps. It is not clear if one can obtain the type of example we are looking for directly from those examples. Nevertheless, our example is inspired by Thom's (which is real and global).

Example 7.1. [14] *Let X be the hypersurface $x_1x_2 = 0$ in $\mathbb{C}^4$ with variables (x_1, x_2, z, t). Let $Y = \mathbb{C}^3$ and define the map $f : X \to Y$ by*

$$f(x_1, x_2, z, t) = (x_1 + x_2, (x_1 + tx_2)z, t) .$$

To understand the above map, keep in mind that on X either x_1 or x_2 vanishes.

Claim. [14] *In any neighbourhood of any point $(0,0,0,t) \in X$, with $|t| = 1$, the map f has infinitely many different local topological types. More precisely, suppose that $\alpha_1, \alpha_2 \in [0,1)$, α_2 is irrational, $\alpha_1 \neq \alpha_2$ and $\alpha_1 \neq 1 - \alpha_2$. Let $t_1 = e^{2\pi\alpha_1 i}$ and $t_2 = e^{2\pi\alpha_2 i}$. Then f has different topological types at $(0,0,0,t_1)$ and at $(0,0,0,t_2)$.*

The following tool is essential in the proof of the claim. It is a concrete realization of an idea of Thom.[1] On each fibre $f^{-1}(y)$, we define an **invariant relation** $\mathcal{R}(y)$ (a subset of $f^{-1}(y) \times f^{-1}(y)$). Let $U = \{x \in X \ : \ \dim_x f^{-1}(f(x)) = 0\}$.

Definition 7.2. *For $y \in Y$, let $\mathcal{R}(y) = f^{-1}(y) \times f^{-1}(y) \cap$ closure of $(U \underset{Y}{\times} U)$, where the closure is taken in the fibred product $X \underset{Y}{\times} X$ induced by the map f.*

Definition 7.3. *Let A and B be topological spaces, $a \in A$, $b \in B$. Let ρ and δ be relations on A and B respectively. We say that (ρ, a) and (δ, b) are* **topologically equivalent** *if there exists a homeomorphism of germs $g : A_a \to B_b$, such that in the induced product homeomorphism $g \times g : (A \times A)_{(a,a)} \to (B \times B)_{(b,b)}$ we have $(g \times g)^{-1}(\delta_{(b,b)}) = \rho_{(a,a)}$.*

Proposition 7.4. *If f has the same local topological type at two points a and b of X then $(\mathcal{R}(f(a)), a)$ and $(\mathcal{R}(f(b)), b)$ are topologically equivalent. (Here the relations are considered on the fibres $A = f^{-1}(f(a))$ and $B = f^{-1}(f(b))$).*

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[1] In [20] Thom writes "...dans un voisinage d'une strate éclatée, l'application opère des identifications qui se traduisent par une correspondance dans la strate (S) elle-même;...".

Since Definition 7.2 involves only local topological objects, Proposition 7.4 is obvious.

The proof of the claim then consists in determining the invariant relation $\mathcal{R}(f(a_t))$. Notice that $f^{-1}(f(a_t)) \cong \mathbb{C}$, with coordinate z. Let (z, z') be corresponding coordinates in $f^{-1}(f(a_t)) \times f^{-1}(f(a_t)) \cong \mathbb{C}^2$. Then, the relation $\mathcal{R}(f(a_t))$ is the hypersurface $H_t(z, z') = 0$, with

$$H_t(z, z') = (z - z')(z - tz')(tz - z') .$$

To establish this, remark that the set U from Definition 7.2 is in this case the complement of $\{x_1 = x_2 = 0\} \cup \{t = x_1 = 0\}$. Then, one can use computer algebra to calculate the appropriate closure or do it directly (look at limits of pairs of points of U with same image by f).

By Proposition 7.4 it is enough to show that $(\mathcal{R}(f(a_{t_1})), 0)$ and $(\mathcal{R}(f(a_{t_2})), 0)$ are not topologically equivalent. Assume the contrary. This means that there exists a homeomorphism of germs $g : \mathbb{C}_0 \to \mathbb{C}_0$, such that the product homeomorphism $g \times g : \mathbb{C}_0^2 \to \mathbb{C}_0^2$ maps the germ of the hypersurface $H_{t_1}(z, z') = 0$ to the germ of $H_{t_2}(z, z') = 0$. Of course, the diagonal is mapped to the diagonal and every other irreducible component into an irreducible component (since after removing the origin they become connected components). Suppose that the component $z' = t_1 z$ is mapped to $z' = t_2 z$ (the other case being similar). This means that g satisfies the following identity on $\mathbb{C}_0$:

$$e^{2\pi\alpha_1 i} z = g^{-1}(e^{2\pi\alpha_2 i} g(z)).$$

Replace g by a representative and let S be a small circle centered at 0. Now S is closed under multiplication by t_1 and therefore, $g(S)$ is closed under multiplication by t_2. Since α_2 is irrational, $g(S)$ is also a circle. Thus, the above identity holds after restricting g and g^{-1} to circles. This implies that rotations of the circle by $2\pi\alpha_1$ and by $2\pi\alpha_2$ have the same *rotation number* (see e.g. [1],[10],[17]) and contradicts the assumption that $\alpha_1 \neq \alpha_2$.

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