数理解析研究所講究録
1033 巻 1998 年 119-130

MILNOR NUMBERS FOR LOCALLY COMPLETE INTERSECTIONS WITH NON-ISOLATED SINGULARITIES

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Let V be complex variety of complex dimension n. When V is non-singular and compact, let us recall 2 very well known formulas:

1) the Gauss-Bonnet theorem: $\chi(V) = c_n(V) \frown [V]$, where $\chi(V)$ denotes the Euler-Poincaré characteristic of V,

2) the Poincaré-Hopf theorem: $\chi(V) = \sum_\alpha \mathrm{PH}(X, S_\alpha)$, where$X$ denotes a vector field X on V, $(S_\alpha)_\alpha$ the connected components of the singular set of X, and $\mathrm{PH}(X, S_\alpha)$ the (generalized) Poincaré-Hopf index of X at S_α (the usual index when S_α is a point) which depends only on the local behavior of X near (but away from) S_α.

The aim of our work is to understand what become these formulas when V may have singularities. The principle of our method is based on generalizing a formula given in [P, PP] for hypersurfaces and in [SS2] for (strong) local complete intersections with isolated singularities: for an analytic variety V which is locally a set-theoretic complete intersection (see the precise definition below), we consider some global topological invariant representing a kind of obstruction for the Gauss-Bonnet theorem to be true. This obstruction is in fact "localized" at the singular set $\mathrm{Sing}(V)$ of V and the Milnor number $\mu_\alpha(V)$ associated with each connected component S_α of $\mathrm{Sing}(V)$ is then the contribution of S_α to the obstruction. It coïncides with the usual Milnor number defined by J. Milnor in [M] in case of isolated singularities of complex hypersurfaces, and more generally by Hamm ([H1]) for locally complete intersections with isolated singularities (cf. also calculations in Greuel [G] and Lê Dung Tran [Lê]). It coïncides also with the Milnor number defined by A. Parusiński [P] for hypersurfaces possibly with non-isolated

(1) The matter of my talk at the RIMS conference is a report on a joint work with J.Seade and T.Suwa ([LS'S]). This article will be finally included in a more general framework, with the further cooperation of J.P.Brasselet (under progress).

singularities. Notice that none of the methods used in these particular cases may generalize to the situation that we wish to look on. Furthermore, our method may also be efficient for computing new examples, even in the previous situations already known (see for instance example 1).

In the regular case, $c_n(V)$ denotes the n^{th} Chern class of the (complex) tangent bundle to V. Before to know wether the Gauss-Bonnet theorem is true or not in the singular case, it is necessary to extend the definition of $c_n(V)$ in our situation: it is the reason why we shall assume that V is a "locally set-theoretic complete intersection". This means that are given a holomorphic vector bundle $E \longrightarrow W$ of rank $q = dim(W) - n$ over a complex (non singular) manifold W, and a holomorphic section s of E generically transverse to the zero section, such that $V = s^{-1}(0)$: using a local trivialization of E, it is clear that V is locally defined by q equations in W; furthermore, it is easy to prove that the restriction $E|_{V_0}$ of E to the regular part V_0 of V may be naturally identified with the normal (complex) bundle $N(V_0)$ of V_0 in W. Examples of this situation are:
- hypersurfaces (E is then the line bundle associated to the divisor defined by V),
- set-theoretic complete intersections (defined by q global equations in W: E is there the trivial bundle of rank q),
- and set-theoretic (projective algebraic) complete intersections in a complex projective space $\mathbf{CP(n+q)}$: if V is the intersection of q algebraic hypersurfaces H_λ $(1 \leq \lambda \leq q)$ of respective degree d_λ, we may take $E = \oplus_{\lambda=1}^{q} L^{\otimes d_\lambda}$, where L denotes the hyperplane line bundle , dual of the tautological line bundle on $\mathbf{CP(n+q)}$.

Thus, the restriction $N = E|_V$ of E to V is an extension of the normal bundle of V_0 in W which will be called "normal bundle " to V, and the difference $\tau = TW|_V - N$ in $KU(V)$ the "virtual tangent bundle" to V. Its (total) Chern class [(1)] $c(\tau)$ reduces to the usual Chern class $c(V)$ when V is non-singular. We call "total Milnor number" the integer $\mu(V) = (-1)^n \Big[c_n(\tau) \frown [V] - \chi(V) \Big]$.

(1) Let us remark that τ, as well as $c(\tau)$ and and the Milnor number that we wish to define, depend on the choice of $E|_V$. However, if we assume furthermore that s is a "regular" section, i.e. that the components of s with respect to any local trivialization of $E|_U$ generate the ideal $I(V \cap U)$ of (local) holomorphic functions on U vanishing on V, then N is well defined (see[LS]), and will be called "the reduced extension" of $N(V_0)$. The usual Milnor number refers to this reduced extension.

Let now S be a compact subset of V which we assume furthermore to be either a connected component of $Sing(V)$ or included in V_0. For a continuous vector field X defined and non vanishing near but away from S in V_0, we define [1], as generalizations of the Poincaré-Hopf index, two indices of X at S, which are called below the "generalized Schwartz index" $\mathrm{Sch}(X, S)$ and the "virtual index" $\mathrm{Vir}(X, S)$, which are localizations of $\chi(V)$ and $c_n(\tau) \frown [V]$ respectively, in the sense of parts (i) and (ii) of theorem 2 below. [The Schwartz index depends only on X and V, while the virtual index also takes into account the way how V is embedded in W and depends on the choice of E].

I Definition of the virtual index:

We first need some definitions. Let ∇ and ∇' be connections for TW and E, respectively, defined on some submanifold Ω of W. Denoting by $\nabla^\bullet$ the pair (∇, ∇'), we set

$$c_n(\nabla^\bullet) = \sum_\ell \varphi_\ell(\nabla) \cdot \psi_\ell(\nabla'),$$

where the product is the exterior product. Then $c_n(\nabla^\bullet)$ is a closed $2n$-form and defines the class $c_n(TW - E)$ on Ω. If $\nabla_1^\bullet = (\nabla_1, \nabla_1')$ and $\nabla_2^\bullet = (\nabla_2, \nabla_2')$ are two such pairs, we set:

$$c_n(\nabla_1^\bullet, \nabla_2^\bullet) = \sum_\ell \left(\psi_\ell(\nabla_1') \cdot \varphi_\ell(\nabla_1, \nabla_2) + \psi_\ell(\nabla_1', \nabla_2') \cdot \varphi_\ell(\nabla_2)\right).$$

Then we have:

Lemma

$$d\, c_n(\nabla_1^\bullet, \nabla_2^\bullet) = c_n(\nabla_2^\bullet) - c_n(\nabla_1^\bullet).$$

Recall that there is an exact sequence of vector bundles on V_0:

$$0 \to TV_0 \to TW|_{V_0} \xrightarrow{\pi} N_{V_0} \to 0.$$

Let Ω_0 be a subset in $V_0 \cap \Omega$. The pair $\nabla^\bullet = (\nabla, \nabla')$ will be said to be "compatible" on Ω_0 if, on Ω_0, the connection ∇' is obtained from ∇ by passing to the quotient:

(1) Most of our constructions and results, except the integrality of the virtual indices and the Milnor numbers, would still be valid under the weaker following assumption on V: there exists a C^∞ vector bundle E on a neighborhood of V in W which extends the normal bundle of the regular part V_0 of V in W; if it is just for defining the Milnor number, we do not need really V to be defined as the zero set of a holomorphic section of E, not even E to be holomorphic. (See example 4 below).

$\pi \circ \nabla = \nabla' \circ \pi$. This implies that ∇ preserves the subbundle $TV_0|_{\Omega_0}$ of TW. The induced connection for TV_0 will be denoted by ∇^V. Thus the triple $(\nabla^V, \nabla, \nabla')$ is compatible with (2.3) in the sense of [BB] 4.16.

Lemma

(i) *If $\nabla^\bullet$ is a compatible pair on Ω_0, then $c_n(\nabla^\bullet) = c_n(\nabla^V)$ on Ω_0.*

(ii) *If $\nabla_1^\bullet$ and $\nabla_2^\bullet$ are two compatible pairs on Ω_0, then $c_n(\nabla_1^\bullet, \nabla_2^\bullet) = c_n(\nabla_1^V, \nabla_2^V)$ on Ω_0.*

Let now V be as above, and let S be either a compact connected set in V_0 or a compact connected component of $\mathrm{Sing}(V)$. Also let $\tilde{U}$ be a neighborhood of S in W such that $U - S$ is in V_0, $U = \tilde{U} \cap V$. For a C^∞ vector field X non-singular on $U - S$, we define the *virtual index* $\mathrm{Vir}(X, S)$ of X at S as follows. First, we take a compact real $2(n+k)$-dimensional manifold $\tilde{\mathcal{T}}$ with C^∞ boundary $\partial\tilde{\mathcal{T}}$ in $\tilde{U}$ such that S is in the interior of $\tilde{\mathcal{T}}$ and that $\partial\tilde{\mathcal{T}}$ is transverse to V. We set $\mathcal{T} = \tilde{\mathcal{T}} \cap V$ and $\partial\mathcal{T} = \partial\tilde{\mathcal{T}} \cap V$. We set

$$\mathrm{Vir}(X, S) = \int_{\mathcal{T}} c_n(\nabla_0^\bullet) + \int_{\partial\mathcal{T}} c_n(\nabla_0^\bullet, \nabla^\bullet).$$

This definition depends only of the local behavior of X near S, but not on the various choices used in the formula.

This virtual index has been introduced in [LSS]. If the singularity S is an isolated point and if V is a complete intersection near S, then the virtual index coincides with the "GSV-index" of [Se, GSV, SS1], which is closely related to the Milnor fiber and the (usual) Milnor number. We may also interpret the virtual index in terms of "smoothing" of V, proving by the way the integrality of the virtual index, thus of the generalized Milnor number.

II Difference of two vector fields near S:

For 2 C^∞ vector fields X_1 and X_2, both non-singular on $U - S$, we define the difference $d_S(X_1, X_2)$ of the vector fields near S by the formula

$$d_S(X_1, X_2) = \int_{\partial\mathcal{T}} c_n(\nabla_1, \nabla_2),$$

where ∇_1 and ∇_2 denote connections on $T(V_0)$ defined near $\partial\mathcal{T}$, and preserving respectively X_1 and X_2 (X_1trivial, X_2trivial). Then we have:

Lemma

(i) $\mathrm{Vir}(X_2, S) - \mathrm{Vir}(X_1, S) = d_S(X_1, X_2)$.

(ii) $d_S(X_1, X_3) = d_S(X_1, X_2) + d_S(X_2, X_3)$, for any 3 vector fields X_1, X_2 and X_3 non-singular on $U - S$.

There is also a topological definition of this difference, proving in particular that it is always an integer.

III Definition of the Schwartz index index:

Let X_0 be a radial vector field (outbound) from S), that is smooth and non vanishing near (but off) S, and transverse out bound from $\partial \mathcal{T}$, where $\tilde{\mathcal{T}}$ has been chosen so that S be a deformation retract of $\mathcal{T}$. (Such vector fields always exist after [SS_2]) . We define the Schwartz index as

$$\mathrm{Sch}(X, S) = \chi(S) + d_S(X_0, X).$$

The generalized Schwartz index is introduced in [SS2] when the singularity S is an isolated point. Here we generalize it to the case of non-isolated singularities using radial vector fields as our basic vector fields. Let us only say that it is equal to $\chi(S)$ in case of a radial vector field outbound from S. (There is another generalization in [KT] of the Schwartz index for stratified vector fields which are possibly not radial. We follow however the point of view given in [SS2]).

IV Results

We may now summarize our results in 3 theorems:

Theorem 1.

Let V be an analytic variety satisfying the above assumption and let S and X be as above.

(i) *The numbers* $\mathrm{Sch}(X, S)$ *and* $\mathrm{Vir}(X, S)$ *are integers.*

(ii) *We have* $\mathrm{Sch}(X, S) = \mathrm{Vir}(X, S) = \mathrm{PH}(X, S)$, *if S is in V_0.*

(iii) *The difference* $\mathrm{Sch}(X, S) - \mathrm{Vir}(X, S)$ *does not depend on the vector field X.*

In view of the above, we define, for a compact component S of $\mathrm{Sing}(V)$, a generalized Milnor number $\mu_S(V)$ as being the integer

$$\mu_S(V) = (-1)^n \Big[\mathrm{Vir}(X, S) - \mathrm{Sch}(X, S) \Big],$$

which is an integer, independent of the choosen vector field X (non-singular near but away from S). We remark that there is always such a vector field, e.g., a radial vector field of M.-H. Schwartz [Sc, BS].

Assume now V to be compact, and let X be a continuous vector field defined on a part of V_0. Denote by $Sing_0(X)$ the set of singular points of X, i.e. the set of points in V_0 where X either vanishes or is not defined. Let $(S_\alpha)_\alpha$ be the family of connected components of the compact set $Sing(X) = Sing_0(X) \cup Sing(V)$, and assume that each S_α is either included in V_0 or is a connected component of $Sing(V)$.

Theorem 2.

Assuming V to be compact and X as above, we have the formulas:

(i) $\sum_\alpha \mathrm{Sch}_\alpha(X, S_\alpha) = \chi(V)$.

(ii) $\sum_\alpha \mathrm{Vir}_\alpha(X, S_\alpha) = c_n(\tau) \frown [V]$.

(iii) $c_n(\tau) \frown [V] - \chi(V) = (-1)^n \sum_\alpha \mu_\alpha(V)$.

where we have written respectively $\mu_\alpha(V)$, $\mathrm{Vir}_\alpha(X)$ and $\mathrm{Sch}_\alpha(X)$ instead of $\mathrm{Vir}(X, S_\alpha)$, $\mathrm{Sch}(X, S_\alpha)$ and $\mu_{S_\alpha}(V)$.

Remark that (i) and (ii) become both the Poincaré-Hopf theorem when V is non singular, while (iii) becomes the Gauss-Bonnet theorem.

The formula (iii) generalizes the one for hypersurfaces in [P] and the one for "strong" local complete intersections with isolated singuralities in [SS2] (see also [D1,2, PP]). As noted in [SS2], this formula reduces to the classical adjunction formula when V is a compact (singular) complex curve in a complex surface W.

Theorem 3.

(i) *If S consists of a point p and if V is a complete intersection near p, then $\mu_p(V)$ coincides with the usual Milnor number of* [M, H1, Lê, G, Lo].

(ii) *If V is a hypersurface, $\mu_S(V)$ coincides with the generalized Milnor number of Parusiński* [P].

V Examples

Example 1: Let $F = (f_1, f_2, \cdots, f_q)$ be a family of q quasi-homogeneous polynomials in $n+q$ variables of the same weights $(d_1, \cdots, d_{n+q})$ and respective

weighted degree $r_1, \cdots, r_q$: this means that
$X.f_\lambda = r_\lambda f_\lambda$, $(\lambda = 1, \cdots, q)$, where $X = \sum_{i=1}^{n+q} \frac{z_i}{d_i} \frac{\partial}{\partial z_i}$ on $\mathbf{C}^{n+q}$. Assume furthermore:

(i) The point $0 \in \mathbf{C}^{n+q}$ is an isolated singularity of $V = F^{-1}(0)$,

(ii) the sequence $(z_1, \ldots, z_n, f_1, \ldots, f_q)$ is regular,

(iii) the natural projection $(z_1, \cdots, z_n, z_{n+1}, \cdots, z_{n+q}) \to (z_1, \cdots, z_n)$ induces by restriction to $F^{-1}(0) - \{0\}$ an N-fold covering, where $N = \prod_{\lambda=1}^{q} r_\lambda d_{n+\lambda}$.

After [LSS], $Vir(X, 0) = \left[\frac{\prod_{i=1}^{n+q}(t+d_i))}{\prod_{\lambda=1}^{q}(t+\frac{1}{r_\lambda})}\right]_n$, where $\left[\cdots\right]_n$ denotes the coefficient of t^n in the power series expansion of $\left[\cdots\right]$ in t. Since X is radial outbound from 0, the Schwartz index $\mathrm{Sch}(X, 0)$ is equal to 1, and the Milnor number of V at 0 is given by

$$\mu_0(V) = (-1)^n \left(\left[\frac{\prod_{i=1}^{n+q}(t + d_i))}{\prod_{\lambda=1}^{q}(t + \frac{1}{r_\lambda})} \right]_n - 1 \right).$$

This formula certainly belongs to the folklore for the specialists. Here are some particular cases:

a) Assume that all r_λ are equal to 1. Denoting by σ_i the i-th elementary symmetric function of $n + k$ variables, the Milnor number is still equal to

$$\mu_0(V) = \sum_{i=n+1}^{n+q} \binom{i-1}{n} \sigma_i(d_1 - 1, \cdots, d_{n+q} - 1).$$

In fact, we have $\mathrm{Vir}(X, 0) = \frac{\Phi^{(n)}(0)}{n!}$ with $\Phi(t) = \frac{\prod_{i=1}^{n+q}(t+d_i)}{(1+t)^q}$. Writing further $s = 1 + t$ and $\Psi(s) = \Phi(t)$, we have $\mathrm{Vir}(X, 0) = \frac{\Psi^{(n)}(1)}{n!}$. If we set $\sigma_i = \sigma_i(d_1 - 1, \cdots, d_{n+q} - 1)$, we get $\Psi(s) = \sum_{j=0}^{n+q} \sigma_j s^{n-j}$ and $\Psi^{(n)}(s) = n! + \sum_{j=1}^{q} \sigma_{n+j} \left(s^{-j}\right)^{(n)}$. Since the value for $s = 1$ of the n-th derivative of the function s^{-j} is equal to $(-1)^n j(j+1) \cdots (j+n-1)$, we get the formula.

We remark that:

1) For $q = 1$, we recover the usual formula for the Milnor number of quasi-homogeneous functions ([MO]).

2) In the particular case of functions given by

$$f_\lambda(z_1, \cdots, z_{n+q}) = \sum_{i=1}^{n+q} a_{\lambda i}\, z_i^{d_i},$$

such that all the q-minors of the $q \times (n+q)$ matrix $(a_{\lambda i})$ are non-zero, this formula has been proved by very different methods, computing the homology of the Milnor fiber in [H2], and using methods of local algebra in [G].

b) Assume that $q = 2$ and that P and Q are homogeneous polynomials of respective degree k and l. According to [LSS] section 4, we have:

$$\mathrm{Vir}(H, p_0) = \ell m \sum_{j=0}^{n} (-1)^j \tbinom{n+2}{n-j} \frac{\ell^{j+1} - m^{j+1}}{\ell - m},$$

while $\mathrm{Sch}(H, p_0)$ is equal to 1 (since H is radial outbound from p_0), hence the Milnor number

$$\mu_{p_0}(V) = (-1)^n \left(\ell m \sum_{j=0}^{n} (-1)^j \tbinom{n+2}{n-j} \frac{\ell^{j+1} - m^{j+1}}{\ell - m} - 1 \right).$$

In particular, for $\ell = m$, we get:

$$\mu_{p_0}(V) = (\ell - 1)^{n+1} \left(\ell(n+1) + 1 \right).$$

In fact, if we write $\Phi(t) = \sum_{i=2}^{n+2} \binom{n+2}{i} (i-1) t^{i-2}$, then $\Phi(-\ell) = \frac{1}{\ell^2} \left((-1)^n \mu_{p_0}(V) + 1 \right)$. It is easy to check that $\Phi(t) = \frac{d}{dt} \left(\frac{(1+t)^{n+2} - 1}{t} \right)$. Thus we deduce: $t^2 \Phi(t) = (1+t)^{n+1} \left(t(n+1) - 1 \right) + 1$, hence from the value of $\Phi(-\ell)$, we get the above formula for $\mu_{p_0}(V)$. In particular, for $\ell = 2$, we recover the value $\mu_{p_0}(V) = 2n + 3$ given in [Lo] p.78, for $P(z_1, \ldots, z_{n+2}) = \sum_{i=1}^{n+2} z_i^2$ and $Q(z_1, \ldots, z_{n+2}) = \sum_{i=1}^{n+2} \lambda_i z_i^2$, the λ_i's being distinct complex numbers.

Application to the computation of $\chi(V)$: If γ denotes the Chern class $c_1(L)$ of the hyperplane bundle L (the dual to the tautological line bundle on $\mathbf{CP}^{n+2}$), the virtual tangent bundle τ of V is equal to the restriction to V of $(n+3)L - L^\ell - L^m$, so that

$$c_n(\tau) \frown [V] = \ell m \left[\frac{(1+\gamma)^{n+3}}{(1+\ell\gamma)(1+m\gamma)} \right]_n,$$

hence $\chi(V) = c_n(\tau) \frown [V] + (-1)^{n+1} \mu_{p_0}(V)$.

Taking for instance $n = 2$, we get:

$$\mu_{p_0}(V) = -1 + \ell m \left(6 - 4(\ell + m) + (\ell^2 + \ell m + m^2) \right),$$

while $c_n(\tau) \frown [V] = \ell m \left(10 - 5(\ell + m) + (\ell^2 + \ell m + m^2) \right)$,

$$\text{hence} \;\; \chi(V) = 1 + \ell m \left(4 - (\ell + m) \right).$$

Example 2: Take for W the projective space $\mathbf{CP}^4$ with homogeneous coordinates $[X, Y, Z, T, U]$, and let V be the cone defined by $X^2 - YT = 0$ and $Z^2 - XY = 0$ in $\mathbf{CP}^4$. It is easy to check that the singular set S of V is the (T, U)-axis $X = Y = Z = 0$.

For any complex number a, the vector field

$$R_a = (2+a)x\frac{\partial}{\partial x} + (4+a)y\frac{\partial}{\partial y} + (3+a)z\frac{\partial}{\partial z} + at\frac{\partial}{\partial t}$$

(with respect to the affine coordinates $(x, y, z, t) = \left(\frac{X}{U}, \frac{Y}{U}, \frac{Z}{U}, \frac{T}{U}\right)$ in the affine space $U \neq 0$) is tangent to V, and extends naturally to the hyperplane at infinity $U = 0$.

For $a = -4$, R_a vanishes along the (Y, U)-axis $X = Z = T = 0$, which is included into V and is not included into S while intersecting it. Thus, it does not satisfy the required assumption of the article.

For all other values of a, the only singular point of R_a on $V - S$ is the isolated regular point $p = [0, 1, 0, 0, 0]$. Thus $\mathrm{Sing}(R_a)$ has two components which are S and $\{p\}$.

All R_a $(a \neq -4)$ are radial outbound from p, while all R_a such that $a \neq -2, -3, -4$ are radial outbound from S. Thus $\chi(V) = \chi(S) + \chi(p) = 2 + 1 = 3$, $\mathrm{Sch}(R_a, S) = 2$ and $\mathrm{Sch}(R_a, p) = 1$.

On the other hand the virtual tangent bundle τ to V is equal to the restriction to V of $5L - L^2 - L^2$, hence $c_2(\tau) \frown [V] = 4\left[\frac{(1+t)^5}{(1+2t)^2}\right]_2 = 8$. Since the point p is regular, $\mathrm{Vir}(R_a, p) = \mathrm{Sch}(R_a, p) = 1$ for $a \neq -4$ (this can be easily checked by a direct computation). We deduce therefore $\mathrm{Vir}(R_a, S) = 8 - 1 = 7$, and $\mu_S(V) = 7 - 2 = 5$

Example 3: Take for W the projective space $\mathbf{CP}^4$ with homogeneous coordinates $[X_0, \ldots, X_4]$ and for V the algebraic set of pure dimension two defined by

$$\begin{cases} (a_1X_1^2 + a_2X_2^2)X_0^2 + a_3X_3^4 + a_4X_4^4 = 0, \\ (b_1X_1^2 + b_2X_2^2)X_0^2 + b_3X_3^4 + b_4X_4^4 = 0. \end{cases}$$

First, we have:

$$c_2(\tau) \frown [V] = 4 \cdot 4\left[\frac{(1+t)^5}{(1+4t)^2}\right]_2 = 288.$$

Now we assume that all numbers $D_{i,j} = a_ib_j - a_jb_i$ $(i < j)$ are different from zero. Denote by p_i the point $[X_j = 0,\ \forall j, j \neq i]$. Since $D_{3,4} \neq 0$, the set $V \cap (X_0 = 0)$ of points "at infinity" is the projective line $L_{12} = (p_1p_2)$ joining p_1 and p_2. Since $D_{i,j} \neq 0$ $(i < j)$, $\mathrm{Sing}(V)$ has two components, which are p_0 and L_{12}.

The vector field

$$v = \frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\right) + \frac{1}{4}\left(z_3\frac{\partial}{\partial z_3} + z_4\frac{\partial}{\partial z_4}\right),$$

defined for $X_0 \neq 0$ (with $z_i = \frac{X_i}{X_0}, i \neq 0$), extends at infinity, and is tangent to V. It is expressed as

$$v = -\frac{1}{2}z_0'\frac{\partial}{\partial z_0'} - \frac{1}{4}\left(z_3'\frac{\partial}{\partial z_3'} + z_4'\frac{\partial}{\partial z_4'}\right),$$

for $X_1 \neq 0$ (with $z_i' = \frac{X_i}{X_1}$, $i \neq 1$), and similarly for $X_2 \neq 0$. The restriction to V of this vector field does not vanish off $\mathrm{Sing}(V)$. Since this vector field is radial outbound from p_0, and radial inbound to L_{12}, we get $\mathrm{Sch}(v, p_0) = \chi(p_0) = 1$ and $\mathrm{Sch}(v, L_{12}) = \chi(L_{12}) = 2$. Thus we get:

$$\chi(V) = 1 + 2 = 3.$$

By example 1 (a), we have

$$\mu_{p_0}(V) = 3^1(4+4) + 3^2(4-1) = 51,$$

hence $\mathrm{Vir}(v, p_0) = \mu_{p_0}(V) + 1 = 52$.
Thus we have $\mathrm{Vir}(v, L_{12}) = c_2(\tau) \frown [V] - \mathrm{Vir}(v, p_0) = 236$
and $\mu_{L_{12}}(V) = \mathrm{Vir}(v, L_{12}) - \mathrm{Sch}(v, L_{12}) = 234$.

Example 4: Take for V the curve $X^3 - Y^2Z = 0$ in the space $W = \mathbf{CP}^2$ with homogeneous coordinates $[X, Y, Z]$. This curve V is an irreducible component of V' defined by $Y(X^3 - Y^2Z) = 0$. The origin $[0, 0, 1]$ is the only singular point of both V and V'. Thus, the normal bundle of the regular part V_0 of V coincides with the restriction to V_0 of the normal bundle to the regular part of V'. It may therefore extend to W as L^3 (the reduced extension) and as L^4. Thus we get two possible virtual tangent bundles τ, and two possible values for the Milnor number which are respectively equal to $\chi(V)$ for the reduced Milnor number, and $\chi(V) + 3$ for the other one. Note that $\chi(V) = 2$, since the map $[u, v] \to [u^2v, u^3, v^3]$ from $\mathbf{CP}^1$ into $\mathbf{CP}^2$ is a homeomorphism from $\mathbf{CP}^1$ onto V. Thus, the reduced Milnor number is 2, and we can check that it coincides with the usual Milnor number, which is also given as the dimension of $\mathcal{O}\{x, y\}/J_f$ with J_f the jacobian ideal of the function $f(x, y) = x^3 - y^2$ in the ring $\mathcal{O}\{x, y\}$ of convergent power series in (x, y).

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