# A localization lemma and its applications

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A localization lemma and its applications*

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Abstract

In this article, we give alternative proofs of two famous facts, the Poincaré-Hopf index theorem and the compatibility of two definitions of the degree of a divisor on a compact Riemann surface, and define a generalization of the tangential index [Br] and [Ho] and prove its index theorem by the method of the localization of the Chern class of a virtual bundle. The tangential index and its index formula was ordinary defined and proved by M.Brunella [Br] for a curve and a singular foliation on a compact complex surface and the author reproved it for a compact curve and a singular foliation on a complex surface [Ho].

1 Introduction

Let $X$ be a $C^\infty$ manifold of dimension $m$ and $E$ a $C^\infty$ complex vector bundle of rank $n$. We consider the Chern class $c(E) \in H^*(X; \mathbb{C})$ of $E$. Note that we use the complex number field $\mathbb{C}$ as the coefficient of the cohomology groups although in fact $c(E)$ itself is in $H^*(X; \mathbb{Z})$, since we use the Chern-Weil theory for the construction of Chern classes. If $E$ has a global section $s : X \to E$, which is not identically zero, we can make a frame, including $s$, of the restriction of $E$ to the complement of the zero set of $s$. Therefore the top Chern class can be localized to the neighborhood of the zero set of $s$. This fact have many applications. In this article, we consider a simple generalization of this fact.

Let $\mathcal{V} = \{V_\alpha\}$ be an open covering of $X$ such that the vector bundle $E$ has a section $s_\alpha : V_\alpha \to E$ on an open set $V_\alpha$, which is not a zero section. Assume that there exist non-vanishing functions $f_{\alpha\beta}$ on $V_\alpha \cap V_\beta$ such that

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*This article is based on the author's talk "A vanishing lemma and some indices" at RIMS. The title was changed.
\( s_\beta = s_\alpha f_{\alpha \beta} \) and the system \( \{ f_{\alpha \beta} \} \) is a cocycle. We denote by \( F \) the line bundle defined by \( \{ f_{\alpha \beta} \} \). Then we consider the Chern class of the virtual bundle \( E - F \). It is localized to the neighborhood of each connected component of the union of the zero set of each \( s_\alpha \). Then we can define the index of \( E \) by \( F \) and get its index formula.

In section 2, we consider a localization lemma and, as examples, the Poincaré-Hopf index formula and the compatibility of two definitions of the degree of a divisor on a compact Riemann surface. Although the Čech-de Rham cohomology theory and its integration theory play important roles in this article, we refer to [BT], [Leh1], [Leh2], [LS] and [Su] for the details of these theories. In section 3, a generalization of the tangential index [Br] and [Ho] are defined and we prove its index formula. This index can be considered to represent how a variety and a one dimensional singular foliation intersect, and it is a kind of indices relative not only to a singular foliation but also to a variety. The tangential index is defined by M. Brunella [Br] for a curve and a singular foliation on a compact complex surface. We generalize it for a variety and a dimension one singular foliation on \( X \).

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2 Localization lemma

Let \( X \) be a \( C^\infty \) manifold of dimension \( m \), \( E \) a complex vector bundle of rank \( n \), \( \mathcal{V} = \{ V_\alpha \}_{\alpha \in A} \) an open covering of \( X \) and \( s_\alpha : V_\alpha \rightarrow E \) a \( C^\infty \) section of \( E \) on each \( V_\alpha \). We can assume that \( E \) is trivial on each \( V_\alpha \) if necessary taking a refinement of \( \mathcal{V} \). Moreover we assume the following condition.

**Assumption 2.1** For any \( \alpha, \beta \in A \) such that \( V_\alpha \cap V_\beta \neq \emptyset \), there exists a non-vanishing \( C^\infty \) function \( f_{\alpha \beta} : V_\alpha \cap V_\beta \rightarrow \mathbb{C}^* \) such that \( s_\beta = s_\alpha f_{\alpha \beta} \) on \( V_\alpha \cap V_\beta \) and the system \( \{ f_{\alpha \beta} \} \) forms a cocycle.

We denote by \( F \) the line bundle which is defined by this cocycle \( \{ f_{\alpha \beta} \} \). Then there exists a bundle map \( f : F \rightarrow E \) such that

1. \( f(F_p) \subset E_p \) for all \( p \in E \)
2. there exist a subset \( S \subset X \) such that \( f_p : F_p \rightarrow E_p \) is injective for \( p \in X - S \).

We call \( S \) the set of singularities of the line bundle \( F \). Let \( S = \bigcup_{\lambda \in \Lambda} T_\lambda \) be the decomposition to connected components. We assume that each \( T_\lambda \) is compact. Take an open set \( U_\lambda \) for each \( \lambda \) such that \( U_\lambda \supset T_\lambda \) and \( U_\lambda \cap U_\mu = \emptyset \)
for \( \lambda \neq \mu \). Then \( \mathcal{U} = \{U_0, (U_\lambda)_{\lambda \in \Lambda}\} \), where \( U_0 = X - S \), is an open covering of \( X \).

We consider the Čech-de Rham cohomology group \( H^*(A^*(\mathcal{U})) \) associated with this open covering \( \mathcal{U} \). Note that this cohomology is isomorphic to the de Rham cohomology (see [BT]). The \( n \)-th Chern class \( c_n(E - F) \) of the virtual bundle \( E - F \) has a representative \((\sigma_n^0, (\sigma_n^\lambda)_{\lambda}, (\sigma_n^{0\lambda})_{\lambda})\) in the Čech-de Rham cohomology group \( H^{2n}(A^*(\mathcal{U})) \) of degree \( 2n \), where \( \sigma_n^0 \) and \( \sigma_n^\lambda \) are \( 2n \)-closed forms which are representatives of \( c_n(E - F) \) on \( U_0 \) and \( U_\lambda \), respectively, in the de Rham cohomology group and \( \sigma_n^{0\lambda} \) is a \((2n - 1)\)-form on \( U_0 \cap U_\lambda \) such that \( d\sigma_n^{0\lambda} = \sigma_n^\lambda - \sigma_n^0 \). Note that we can construct \( \sigma_n^0, \sigma_n^\lambda \) and \( \sigma_n^{0\lambda} \) from connections of \( E \) and \( F \), using the Chern-Weil theory, and the Čech-de Rham cohomology class represented by these forms is independent on the choice of the connections.

**Lemma 2.2 (localization)** Let \( j^*: H^{2n}(X, X - S; \mathbb{C}) \rightarrow H^{2n}(X; \mathbb{C}) \) be natural map. Then there exists \( c \in H^{2n}(X, X - S; \mathbb{C}) \) such that \( j^*(c) = c_n(E - F) \).

**Proof.** Since \( F \) can be considered a subbundle of \( E \) on \( U_0 \), there exists the decomposition \( F \oplus E' \) of \( E \). The system \( \{s_\alpha\} \) forms a frame of \( F \). Let \( \nabla^F_0 \) be a trivial connection of \( F \) respect to the frame, \( \nabla^{E'}_0 \) a connection of \( E' \) on \( U_0 \) and \( \kappa \) the curvature matrix of the connection \( \nabla_0 = \nabla^F_0 \oplus \nabla^{E'}_0 \) of \( E \) on \( U_0 \). Then \( \sigma_n^0 = \det \kappa = 0 \). We can construct \( \sigma_n^\lambda \) and \( \sigma_n^{0\lambda} \) from \( \nabla_0 \) and connections of \( E \) and \( F \) on \( U_\lambda \).

Therefore the representative of \( c_n(E - F) \) in the Čech-de Rham cohomology is \( \sigma = (0, (\sigma_n^\lambda)_\lambda, (\sigma_n^{0\lambda})_\lambda) \). This is a \( 2n \)-cocycle in the Čech-de Rham complex relative to \( X - S \). Let \( \tau \) be a \( 2n \)-form on \( X \) corresponding to \( \sigma \). Then \( c = [\tau] \in H^{2n}(X, X - S; \mathbb{C}) \) and \( j^*(c) = c_n(E - F) \).

We denote \( c \in H^{2n}(X, X - S; \mathbb{C}) \) by \( c_n(E; F) \). This is a localization of \( c_n(E - F) \).

If \( X \) is compact, there exists following commutative diagram.

\[
\begin{array}{ccc}
H^{2n}(X, X - S; \mathbb{C}) & \xrightarrow{A} & H_{m-2n}(S; \mathbb{C}) = \bigoplus_{\lambda \in \Lambda} H_{m-2n}(T_\lambda; \mathbb{C}) \\
\downarrow{j^*} & & \downarrow{i_*} \\
H^{2n}(X; \mathbb{C}) & \xrightarrow{[X]} & H_{m-2n}(X; \mathbb{C}),
\end{array}
\]

where \( A \) is the Alexander duality, \( i \) the natural inclusion and \([X]\) the fundamental class of \( X \).
Definition 2.3 We define an index $I(E, F; T_\lambda) \in H_{m-2n}(T_\lambda; \mathbb{C})$ of $E$ by $F$ at $T_\lambda$ by
\[
A(c) = (I(E, F; T_\lambda))_{\lambda \in \Lambda}.
\]

Remark 2.4 We can define the index $I(E, F; T_\lambda)$ if $S$ is compact.

From the commutativity of the above diagram, we have following theorem.

Theorem 2.5 If $X$ is compact, we have
\[
\sum_{\lambda \in \Lambda} \dot{i}_* I(E, F; T_\lambda) = c_n(E - F) \sim [X].
\]

In the rest of this section, we assume that $X$ is compact and $S$ consists only of isolated points. Since each $T_\lambda$ consists of a point $p_\lambda$ under this assumption, we can take a sufficiently small open neighborhood $U_\lambda$ of $p_\lambda$. Then we can assume each $\sigma^n_\lambda$ is 0 without loosing generalities. Hence the localized Chern class $c_n(E; F)$ has a representative $(0, 0, (\sigma^n_\lambda)_\lambda)$. So it is important to write the $(2n-1)$-form $\sigma^n_\lambda$ explicitly. We have to mention the Bochner-Martinelli kernel for the purpose of writing $\sigma^n_\lambda$ clearly.

Definition 2.6 We call following $(n, n-1)$-form $\beta_n$ on $\mathbb{C}^n$ the Bochner-Martinelli kernel;
\[
\beta_n = C_n \sum_{i=1}^{n} (-1)^{i-1} \overline{z_i} dz_1 \wedge \cdots \wedge \overline{d z_i} \wedge \cdots \wedge d z_n \wedge d z_1 \wedge \cdots \wedge d z_n \left/ ||z||^{2n} \right.,
\]
where
\[
C_n = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi \sqrt{-1})^n}.
\]

Remark 2.7 Let $S^{2n-1} \subset \mathbb{C}^n$ be a $(2n-1)$-sphere centered at the origin 0. Then the Bochner-Martinelli kernel $\beta_n$ is real on $S^{2n-1}$ and a generator of the cohomology group $H^{2n-1}(S^{2n-1}; \mathbb{C})$:
\[
\int_{S^{2n-1}} \beta_n = 1
\]

Theorem 2.8 Assume that $p_\lambda \in V_\alpha$ for some $\alpha$. Then we have
\[
\sigma^n_{0\lambda} = -s^*_\alpha \beta_n,
\]

To prove this theorem, the Chen-Weil theory and the integration along the fiber are needed. Here the proof is omitted.
Hereafter we assume that $X$ is oriented. We introduce the integration on the Čech-de Rham cohomology group.

Let $R_{\lambda}$ be a closed neighborhood of $p_{\lambda}$ such that $R_{\lambda} \subset U_{\lambda}$ for each $\lambda$. Put $R_0 = X - \bigcup_{\lambda \in \Lambda} \text{int} R_{\lambda}$ and $R_{0\lambda} = R_0 \cap R_{\lambda}$. $R_0$ and $R_{\lambda}$ are oriented as submanifolds of $X$ for each $\lambda$ and $R_{0\lambda}$ as the boundary of $R_0$; $R_{0\lambda} = \partial R_0 = -\partial R_{\lambda}$. We call a family $\mathcal{R} = \{R_0, (R_{\lambda})_{\lambda}, (R_{0\lambda})_{\lambda}\}$ a system of honey-comb cells adapted to the open covering $\mathcal{U}$.

Then we can define the integration on the Čech-de Rham cohomology group $H^m(A^*(\mathcal{U}))$ associated with $\mathcal{U}$ when $X$ is compact. For any $\sigma = [(\sigma_0, (\sigma_{\lambda})_{\lambda}, (\sigma_{0\lambda})_{\lambda})] \in H^m(A^*(\mathcal{U}))$, we define the integration by

$$\int_X \sigma = \int_{R_0} \sigma_0 + \sum_{\lambda \in \Lambda} \int_{R_{\lambda}} \sigma_{\lambda} + \sum_{\lambda \in \Lambda} \int_{R_{0\lambda}} \sigma_{0\lambda}.$$ 

This definition is well-defined and compatible with the integration on the de Rham cohomology group;

$$\int_X \sigma = \int_X \tau,$$

where $\tau$ is a $2n$-form on $X$ corresponding to $\sigma$. See [Leh1], [Leh2], [LS] and [Su] for the details and more general definitions.

Then we describe examples.

**Corollary 2.9** Let $C$ be a compact Riemann surface, $D = \{(U_i, f_i)\}$ a Cartier divisor on $C$, $D' = \sum_{i=1}^{n} n_i p_i$ the Weil divisor corresponding to $D$. Then

$$\int_C c_1([D]) = \sum_{i=1}^{n} n_i,$$

where $[D]$ is the line bundle associated with $D$.

**Proof.** We can assume that each point $p_i$ in $D'$ is contained in $U_i$ and not contained in other $U_j$. There exists a coordinate $z_i$ on each $U_i$ such that $z_i(p_i) = 0$ and $f_i(z_i) = z_i^{n_i}$. Then $\mathcal{U} = \{U_0, (U_i)\}$, where $U_0 = C - \{p_1, p_2, \cdots, p_n\}$ is an open covering of $C$. Let $\mathcal{R}$ be a system of honeycomb cell adapted to $\mathcal{U}$. Note that each $f_i$ is a section of $[D]$ on $U_i$. From theorem(2.5) and (2.8), we have

$$\int_C c_1([D]) = \sum_{i=1}^{n} \int_{R_{0i}} -f_i^* \beta_1 = \sum_{i=1}^{n} \frac{1}{2\pi \sqrt{-1}} \int_{S^1_{p_i}} \frac{df_i}{f_i} = \sum_{i=1}^{n} \frac{1}{2\pi \sqrt{-1}} \int_{S^1_{p_i}} \frac{n_i dz}{z} = \sum_{i=1}^{n} n_i.$$
where $S^1_{p_i}$ is a 1-sphere in $\mathbb{C}$ centered at $p_i$ and oriented naturally.

As the second example, we consider the Poincaré-Hopf index formula for a dimension one reduced singular foliation.

**Definition 2.10** A dimension one singular foliation $\mathcal{F}$ on a complex manifold $X$ is determined by a triple $(\{V_\alpha\}, v_\alpha, e_{\alpha\beta})$ such that

1. $\{V_\alpha\}$ is an open covering of $X$ and, for each $\alpha$, $v_\alpha$ is a holomorphic vector field on $V_\alpha$,

2. for each pair $(\alpha, \beta)$, $e_{\alpha\beta}$ is a non-vanishing holomorphic function on $V_\alpha \cap V_\beta$ which satisfies the cocyle condition, $e_{\alpha\gamma} = e_{\alpha\beta}e_{\beta\gamma}$ on $V_\alpha \cap V_\beta \cap V_\gamma$,

3. $v_\beta = v_\alpha e_{\alpha\beta}$ on $V_\alpha \cap V_\beta$.

The cocycle $\{e_{\alpha\beta}\}$ defines a line bundle which is called the holomorphic tangent bundle of $\mathcal{F}$.

Note that this definition is adapted to the assumption (2.1) if we regard the holomorphic tangent bundle $TX$ as a $C^\infty$ complex vector bundle $E$. The singular set of a foliation is defined similarly. A dimension one singular foliation is said to be reduced if its singular set consists only of isolated points.

**Corollary 2.11 (Poincaré-Hopf)** Let $X$ be a compact complex manifold of complex dimension $n$, $\mathcal{F} = (\{V_\alpha\}, v_\alpha, e_{\alpha\beta})$ a reduced dimension one singular foliation and $F$ a holomorphic tangent bundle of $\mathcal{F}$. Then we have

$$\sum_{p \in S} PH(v, p) = \int_X c_n(TX - F),$$

where $S$ is the singular set of $\mathcal{F}$ and $PH(v, p)$ is the Poincaré-Hopf index of $v$ at $p$.

**Proof.** Note that the Poincaré-Hopf index $PH(v, p)$ is written as

$$\int_{S^2_\alpha} v^* \beta_n = \int_{R_{0\lambda}} \sigma_n^{0\lambda},$$

for some $\lambda$. Hence this formula is an obvious corollary of the theorem (2.5) and (2.8).
Remark 2.12 The original Poincaré-Hopf index formula is

$$\sum_{p \in S} PH(v, p) = \chi(X),$$

where $v$ is a vector field on $X$. If there exists a global vector field with only isolated zero points, the tangent bundle $F$ is trivial and we get the classical formula, using the fact $\int_X c_n(X) = \chi(X)$, from the theorem (2.11). This formula is a special case of the Baum-Bott residue theorem [BB].

Note that some other formulas, for example, the Riemann-Hurwitz formula, can be proved in this way.

3 Tangential index

Let $X$ be a complex manifold of dimension $n + k$, $V \subset X$ a strong locally complete intersection (SLCI) of dimension $n$ (See [LS] for the definition of SLCI), \( V = \{ V_\alpha \} \) an open covering of $X$ and $V' = \text{Reg}(V) = V - \text{Sing}(V)$ a regular part of $V$. Since $V$ is an SLCI, there exists a $C^\infty$ complex vector bundle $\tilde{N}$ on a neighborhood $U$ of $V$ in $X$ such that the restriction $\tilde{N}|_{V'}$ is the normal bundle $N_{V'}$ of $V'$.

Assumption 3.1 There exists a bundle map $\tilde{\pi} : TX|_U \rightarrow \tilde{N}$ such that a diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & TV' \\
\downarrow i & & \downarrow i \\
TX|_{V'} & \longrightarrow & N_{V'} \\
\pi & & \pi \\
TX|_U & \longrightarrow & \tilde{N}
\end{array}
$$

is commutative.

The above assumption (3.1) is satisfied, for example, when an SLCI $V$ is defined by a holomorphic section $s$ of a holomorphic vector bundle $E$; $V = s^{-1}(0)$. In this case, $E$ is isomorphic to $\tilde{N}$.

Let $f_1^\alpha, f_2^\alpha, \cdots, f_k^\alpha$ be defining functions of $V$ on $V_\alpha$;

$$V \cap V_\alpha = \{ f_1^\alpha = f_2^\alpha = \cdots = f_k^\alpha = 0 \}.$$

We can take coordinates $(x_1^\alpha, x_2^\alpha, \cdots, x_{n+k}^\alpha)$ on $V_\alpha$ such that $x_{n+i}^\alpha = f_i^\alpha$ for $i = 1, 2, \cdots, k$. Then

$$\pi \frac{\partial}{\partial x_{n+1}^\alpha}, \pi \frac{\partial}{\partial x_{n+2}^\alpha}, \cdots, \pi \frac{\partial}{\partial x_{n+k}^\alpha}$$
form a frame of $N_{V'} = (TX|_{V'})/TV'$. We can assume $\tilde{N}$ is trivial on each $V_\alpha \cap U$ and there exists a frame \{${e}_1^\alpha, {e}_2^\alpha, \cdots, {e}_k^\alpha$\} of $\tilde{N}$ such that

$$e_i^\alpha|_{V'} = \pi \frac{\partial}{\partial x_{n+i}}$$

for each $i$. This frame is said to be associated with \{${f}_1^\alpha, {f}_2^\alpha, \cdots, f_k^\alpha$\}.

Let $F = \{(V_\alpha, v_\alpha, e_{\alpha\beta})\}$ be a dimension one singular foliation and $F$ the holomorphic tangent bundle of $F$.

**Assumption 3.2** The SLCI $V$ is not invariant by $F$, i.e. $v_\alpha(f_{\alpha,i}) \not\in I(V \cap V_\alpha)$, where $I(V \cap V_\alpha)$ is the ideal of holomorphic functions vanishing on $V \cap V_\alpha$ and generated by the defining functions of $V$ on $V_\alpha$.

Take a frame \{${e}_i^\alpha$\} of $\tilde{N}$ associated with \{${f}_i^\alpha$\}. Then we get

$$\tilde{\pi}(v_\alpha)|_V = \sum_{i=1}^k v_\alpha(f_{\alpha,i}) {e}_i^\alpha.$$

Let

$$T_\alpha = \text{Sing}V \cup \{p \in V' \cap V_\alpha \mid v_\alpha(p) \in T_p V'\}.$$

Then we have

$$\tilde{\pi}(v_\alpha)|_V = \tilde{\pi}(v_\beta)|_V e_\beta$$

$$T_\alpha = \{p \in V \cap V_\alpha \mid \tilde{\pi}(v_\alpha)(p) = 0\}.$$

So $T = \bigcup_\alpha T_\alpha$ is well-defined. $T$ is the set of tangential points of $F$ and $V$

Let $T = \coprod_{\lambda \in \Lambda} T_\lambda$ be the decomposition to connected components and we assume that $T$ is compact. Then there exists generalized tangential index.

**Theorem 3.3 (tangential index)** There exists index

$$I(N, F; T_\lambda) \in H_{2(n-k)}(T_\lambda; \mathbb{C})$$

of $N$ by $F$ at $T_\lambda$. Moreover if $V$ is compact,

$$\sum_{\lambda \in \Lambda} i_* I(N, F; T_\lambda) = c_k(N - F) \wedge [V]$$

**Theorem 3.4** If $n = k$ and $T$ consists only of isolated points, then

$$I(N, F; p) = \int_L (\pi(v))^* \beta_k,$$

where $p \in T$ and $L$ is a link of $V$ at $p$ with a usual orientation; $L = \{f_1 = f_2 = \cdots = f_k = 0, |v(f_1)|^2 + |v(f_2)|^2 + \cdots + |v(f_k)|^2 = \epsilon\}$ for a sufficient small $\epsilon > 0$ and $d \arg v(f_1) \wedge d \arg v(f_2) \wedge \cdots \wedge d \arg v(f_k) > 0$. 

These two theorems are corollaries of theorem (2.5) and (2.8), respectively. Apply these theorems to a virtual bundle $\tilde{N} - F$ and $V$.

This index can be considered to represent how tangent a variety and on dimensional singular foliation. If $n = k = 1$ then we have

$$I(N, F; p) = \frac{1}{2\pi \sqrt{-1}} \int_L \frac{dv(f)}{v(f)}.$$ 

This coincides with an intersection number $(v(f), f)_p$ at $p$ (See [GH] Chapter 5) and the original tangential index [Br] and [Ho].

References


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