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# QUANTIFICATION OF THE SINGULARITIES OF OSGOOD AND WHITNEY

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We introduce a few quantities and functions for germs of complex spaces or their morphisms. Some are related to zero estimate developed in transcendence theory and others to Chevalley's theorem on homomorphisms of local rings. These allow us to measure how singular are the classical examples of singularities, Osgood's dimension raising morphism and Whitney's nonalgebraic singularities. Outline of the part concerning transcendence is already announced in [$I_4$]. This time we add a few new facts and lay emphasis on the examples. The detailed proofs will be given in [$I_5$].

Let $X_\xi$ be a germ of a complex space and $\Phi := \{\Phi_1, ..., \Phi_n\}$ a finite subset of its local ring $(\mathcal{O}_{X,\xi}, \mathfrak{m})$ (we call this *analytic local algebra*). If $\Phi$ is considered mapping components, it determines a germ of a holomorphic map $X_\xi \longrightarrow (\mathbb{C}^n)_{z_0}$ with $\Phi(\xi) = z_0$ and the associated homomorphism $\varphi : \mathbb{C}\{z - z_0\} \longrightarrow \mathcal{O}_{X,\xi}$ such that $\varphi(f) = f(\Phi_1, ..., \Phi_n)$. Let us define

- *order* $\nu_\mathfrak{m} : \mathcal{O}_{X,\xi} \longrightarrow \mathbb{N} := \{0, 1, ...\}$
  by $\nu_\mathfrak{m}(f) := \sup\{p : f \in \mathfrak{m}^p\}$;
- *growth function of order* $\theta_\Phi : \mathbb{N} \longrightarrow \mathbb{N}$
  by $\theta_\Phi(k) := \sup\{\nu_\mathfrak{m}(f) : f \neq 0 \text{ is a polynomial in } \Phi \text{ of degree} \leq k\}$;
- *growth exponent of order* $\alpha(\Phi) \in [0, \infty]$
  by $\alpha(\Phi) := \varlimsup_{k\to\infty} \log_k \theta_\Phi(k)$.

The growth function and the growth exponent are not determined only by the map germ (or homomorphism). They depend upon the choice of the affine coordinates of the target space $\mathbb{C}^n$ of the holomorphic map. A value $a$ of $\alpha(\Phi)$ is called to be *attained* if there exist $c,\ c' > 0$ such that $\theta_\Phi(k) \leq ck^a$ holds for sufficiently large $k$ and if $\theta_\Phi(k) \geq c'k^a$ holds for an infinite number of $k$.

The followings are fundamental.

**Fact I** (cf. [$P_1$], [$P_2$], [P-W]). Let $L_1, ..., L_p$ be independent $\mathbb{C}$-linear forms in $x := (x_1, ..., x_m)$ and put

$$\Phi := \{\lambda_{11} \exp L_1, ..., \lambda_{1i_1} \exp L_1, ..., \lambda_{p1} \exp L_p, ..., \lambda_{pi_p} \exp L_p\},$$

$$\Psi := \{x\} \cup \Phi \subset \mathbb{C}\{x\}.$$

Then we have

$$\alpha(\Phi) = \max_k \dim_\mathbb{Q} \sum_{j=1}^{i_k} \mathbb{Q}\lambda_{kj}. \quad \alpha(\Psi) = 1 + \max_k \dim_\mathbb{Q} \sum_{j=1}^{i_k} \mathbb{Q}\lambda_{kj}.$$

These values are attained.

**Fact II** (lower estimate, [$I_5$]). Let $(R, \mathfrak{m})$ be an analytic local algebra and $\Phi \subset \mathfrak{m}$ a subset. Then $\alpha(\Phi) \geq \operatorname{trdeg}_\mathbb{C} \mathbb{C}(\Phi)/\dim R$.

**Fact III** (upper estimate, [$I_5$]). If the elements of $\Psi$ are algebraic over $\mathbb{C}(\Phi)$, then $\theta_\Psi$

is estimated from above by a linear function of $\theta_\Phi$ and hence $\alpha(\Psi) \le \alpha(\Phi)$.

We define other invariants for a germ of a holomorphic map $\Phi : X_\xi \longrightarrow Y_\eta$ or for the associated homomorphism $\varphi : (S := \mathcal{O}_{Y,\eta}, \mathfrak{n}) \longrightarrow (R := \mathcal{O}_{X,\xi}, \mathfrak{m})$ as follows.

- *Chevalley function* $\kappa_\varphi \equiv \kappa_\Phi : \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$
  by $\kappa_\varphi(t) \equiv \kappa_\Phi(t) := \sup\{\nu_\mathfrak{m}(\varphi(f)) : \nu_\mathfrak{n}(f) \le t\}$;
- *Chevalley exponent* $\beta(\varphi) \equiv \beta(\Phi)$
  by $\beta(\varphi) \equiv \beta(\Phi) := \overline{\lim_{t\to\infty}} \log_t \kappa_\varphi(t)$.

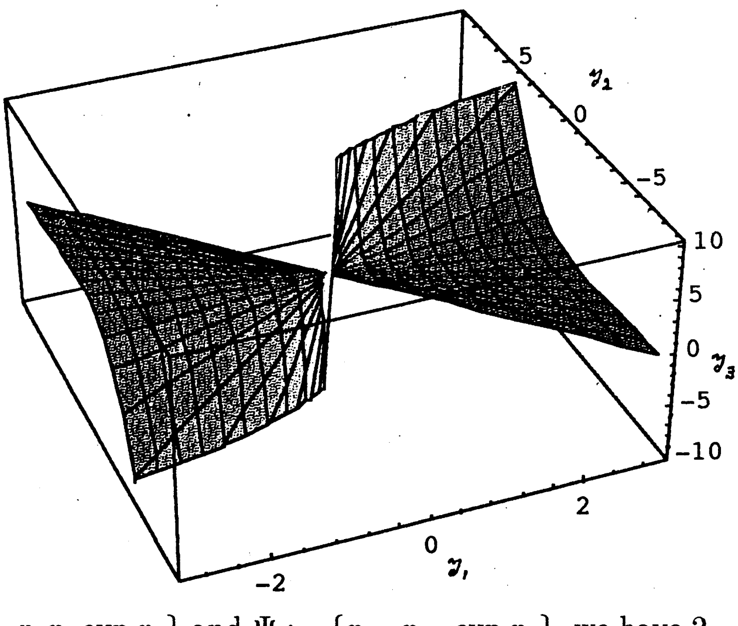
**Fact IV** ([C]). $\kappa_\varphi(t) < \infty$ for all $t \in \mathbb{N}$ if and only if the completion $\hat{\varphi}$ is injective.

**Fact V** ([$\mathrm{I}_5$]). Let $\varphi : \mathbb{C}\{y\} \longrightarrow R$ $(y := (y_1, ..., y_n))$ be the homomorphism of analytic local algebras determined by $\Phi := \{\Phi_1, ..., \Phi_n\} \subset \mathfrak{m}$ $(\varphi(y_i) = \Phi_i)$, where $\mathfrak{m}$ denotes the maximal ideal of $R$. Then $\alpha(\Phi) \le \beta(\varphi)$.

**Fact VI** ([$\mathrm{I}_1$]). Suppose that the integral analytic local algebras, $R$ and $S$, are integral domains in the above. Then $\kappa_\varphi(t)$ for the associated homomorphism $\varphi : S \to R$ is majorized by a linear function of $t$ if and only if the generic rank of $\Phi$ is equal to $\dim Y_\eta$. Here the generic rank is a half of the topological dimension of the image by $\Phi$ of a sufficiently small neighbourhood of $\xi$ (cf. [$\mathrm{G}_2$]). If this is the case, we have $\beta(\varphi) = 1$ and $\alpha(\Psi) = \alpha(\varphi(\Psi))$ for any subset $\Psi \subset S$.

**Osgood's example** ([G-R]). Let $\varphi : \mathbb{C}\{y_1, y_2, y_3\} \to \mathbb{C}\{x_1, x_2\}$ denote the homomorphism with $\varphi(y_1) := x_1$, $\varphi(y_2) := x_1x_2$, $\varphi(y_3) := x_1x_2 \exp x_2$. If we put $\Phi :=$

The image of Osgood map




$\{x_1,\ x_1x_2,\ x_1x_2 \exp x_2\}$ and $\Psi := \{x_1,\ x_2,\ \exp x_2\}$, we have $2 = \alpha(\Psi) = \alpha(\Phi) \le \beta(\varphi)$ by Fact I, Fact III and Fact V.

Now suppose that $\nu_{(y)}(f) = t$. Let $f_t$ denote the $t$-th homogeneous part of $f$. Then the general form of $\varphi(f_t)$ is $x_1^t g_t(x_2)$ with $g_t \in \mathbb{C}[x_2, \exp x_2]$ of degree not greater than $2t$. If $t \ne t'$, then $\varphi(f_t)$ and $\varphi(f_{t'})$ have no common monomial in $\{x_1, x_2\}$. By Fact I,

$\nu_{(x)}(g_t) \leq at^2$ for some $a \in \mathbb{R}$ and for all $t \in \mathbb{N}$. Hence, $\nu_{(x)}(\varphi(f_t)) \leq at^2 + t$ and $\beta(\varphi) \leq 2$. Thus we have obtained the exact value of Chevalley exponent of $\varphi$: $\beta(\varphi) = 2$.

Hence the completion of $\varphi$ with respect the maximal-ideal-adic topologies is injective by the trivial half of Fact IV. Then, as Osgood has pointed out, $\varphi$ is itself injective and the analytic closure of the image of the holomorphic map associated to $\varphi$ is 3-dimensional, although the source is 2-dimensional.

(This pathology can be removed by a point blowing up of $\mathbb{C}^3$. Namely, take a homomorphism $\pi : \mathbb{C}\{y_1, y_2, y_3\} \longrightarrow \mathbb{C}\{Y_1, Y_2, Y_3\}$ with $\pi(y_1) = Y_1$, $\pi(y_2) = Y_1Y_2$, $\pi(y_3) = Y_1Y_3$. Then the canonically induced map $\psi : \mathbb{C}\{Y_1, Y_2, Y_3\} \longrightarrow \mathbb{C}\{x_1, x_2\}$ with $\psi(Y_1) = x_1$, $\psi(Y_2) = x_2$, $\psi(Y_3) = x_2 \exp x_2$ is surjective. This means that the associated map $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ is an embedding.)

**Whitney's example** ([Wh]). There exists a germ of an analytic set which is not locally isomorphic to algebraic one. A simple example which is normal is the 3-dimensional singularity $X_0$ defined by the equation

$$f := w^6 - xy(y-x)(y-(3+z)x)(y - x\exp z) = 0.$$

Let us consider its local algebra $(R, \mathfrak{m})$ at 0 with

$$R := \mathbb{C}\{x, y, z, w\}/f\mathbb{C}\{x, y, z, w\}.$$

Then $\Phi := \{\bar{x}, \bar{y}, \bar{z}, \bar{w}\}$ form a system of generators of $\mathfrak{m}$ (bar indicates the class $\mod f$). Let us take the 3-dimensional local algebra $(S, \mathfrak{n})$ with

$$S := \mathbb{C}\{s, t, u, v\}/(t + u\exp v)\mathbb{C}\{s, t, u, v\} \cong \mathbb{C}\{s, u, v\}.$$

Then there exists a holomorphic map germ $\theta : S \to R$ such that $\operatorname{grk}\theta = \dim S$ and

$$\theta(\bar{s}) = \bar{x}, \quad \theta(\bar{t}) = \bar{w}^6 - \bar{x}\bar{y}^2(\bar{y}-\bar{x})(\bar{y}-(3+\bar{z})\bar{x}),$$
$$\theta(\bar{u}) = \bar{x}^2\bar{y}(\bar{y}-\bar{x})(\bar{y}-(3+\bar{z})\bar{x}), \quad \theta(\bar{v}) = \bar{z}.$$

Since the elements of $\Phi := \{\bar{x}, \bar{y}, \bar{z}, \bar{w}\} \subset R$ are algebraic over the image of $\Psi := \{\bar{s}, \bar{t}, \bar{u}, \bar{v}, \overline{\exp v}\} \subset S$ and vice versa, it holds that $\alpha(\Phi) = \alpha(\theta(\Psi)) = \alpha(\Psi) = 2$ by Fact III, Fact VI and Fact I. Since every system $\Sigma$ of generators of $\mathfrak{m}$ corresponds to an embedding of the singularity into an affine space, we may call the quantity

$$\alpha(R) \equiv \alpha(X_\xi) := \inf\{\alpha(\Sigma) : \Sigma \text{ is a systems of generators of } \mathfrak{m}\}$$

the *intrinsic growth exponent* of the singularity. In this sense, $\alpha(R) \leq \alpha(\Phi) = 2$. Since $X_\xi$ is locally non algebraic, $\operatorname{trdeg}_{\mathbb{C}}(\Sigma) = 4$ for any system $\Sigma$ of generators of $\mathfrak{m}$. Then we see $4/3 \leq \alpha(R) \leq \alpha(\Psi) = 2$ by Fact II. What is the precise value of $\alpha(R)$?

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