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Hypersurface non-rational singularities which look like canonical from its Newton boundary

東京工業大、理　石井　志保子 (Shihoko ISHII)
金沢大学、自然　泊　昌孝 (Masataka TOMARI)

Introduction

Let $(V,p)=\{f=0\}\subset \mathbf{C}^{d+1}$ be a minimal embedding of a hypersurface isolated singularity. For a characterization of rational singularity, Prof. M. Reid had proposed the following conjecture concerning weight filtrations induced from the the weight of each coordinate variables:

Conjecture ([17,§4]). Let (v,p) be a hypersurface isolated singularity as above. If the condition $deg_w(in_w(f))-\sum w_i<0$ hold for any minimal embedding (V,p) over $\mathbf{C}$ and any weighting $(w_1,\ \dots\ ,w_{d+1}), w_i>0$ on the analytic coordinates (we call the condition R here), (V,p) would be a rational singularity , i.e., $p_g(V,p)=0$.

For the 2-dimensional case, it is known that this conjecture is true and that its proof helps the classification of rational double points (cf. (1.4)). Here the 3 weights for simple elliptic singularity had important roles. In [17], M. Reid had hoped that, at least for 3-dimensional case, famous 95 weights for weighted hypersurface normal K3 surface would play a similar important role to give an affirmative answer to the above conjecture. (see §1 for more affirmative information on this conjecture.)

However, unfortunately, we will give a counter example to this conjecture by giving the following 3-dimensional simple K3 singularity which does not belong to usual 95 classes like quasi-homogeneous cases (cf. §3 in detail):

Example. Let (V,p) be a 3-dimensional singularity which is given as a complete intersection in $(\mathbf{C}^5,o)$ as follows:

$$(V,p)=\left(\left\{\begin{array}{ll} q_2(x_1,x_2,x_3)+x_5+h(x) & =0 \\ g_6(x_1,x_2,x_3,,x_4,x_5)+l(x) & =0\end{array}\right\},o\right).$$

Here q_2,g_6 are weighted homogeneous of the type $(1,1,1,2,3)$ of degrees $2,6$ and we assume that they define a weighted complete intersection normal K3 surface $X_{2,6}\subset P(1,1,1,2,3)$. We assume that $h(x),l(x)$ have sufficiently high order and that V has an isolated singularity at the origin o. Then we obtain the following conditions(§2, §3): (1) (V,p) is a hypersurface simple K3 singularity of multiplicity 3 (in particular, this is non-rational) and moreover, (2) the condition R is satisfied. $X_{2,6}$ can be regarded as a degeneration of $X_6\subset P(1,1,1,3)$ (By similar point of views, we have 3 more simple K3 singularity of multiplicity 3. They are also comes from certain degeneration. cf (3.11).)

On the other hand, we know the affirmative answer for Reid's conjecture in the case of simple K3 singularity of multiplicity ([22] and see §3 . Hence our example is a simplest class for which we can seek the counter example.

Here we like to remark on the the condition R from more general theory of filtered blowing up. If we do not restrict our attention about filtration to monomial weight filtrations, then we can show the following about the characterization of rational singularity.

Theorem (1.2) (V, p): normal d-dimensional isolated singularity over $\mathbf{C}$. Then $a_F(O_{V,p}) < 0$ holds for any filtration of ideals F on $O_{V,p}$ which satisfy the conditions of §1 of [23] (cf. (1.1)) if and only if $p_g(V, p) = 0$.

Under the assumption that (V, p) is Cohen-Macaulay, the above theorem gives a characterization of rational singularity. Here a_F for the weight monomial filtration to the minimal embedding of a hypersurface $(V, p) = \{f = 0\} \subset \mathbf{C}^{d+1}$ is written as $a_{F_w}(O_{V,p}) = deg_w(in_w(f)) - \sum w_i$ when the weighting on the coordinates is $w = (w_1, \dots, w_{d+1})$. Hence, we obtain a theorem which generalizes Reid's principle. However, it is difficult to handle all filtration. Our next problem is to find good classes of filtrations which are effective to calculate $a_F(O_{V,p})$ geometrically so that we can find a good criterion about rational singularity.

The contents of the present paper are

1. Introduction to Reid's conjecture.

2. Existence of monomials of. square of quadric polynomials and elementary construction of counter examples.

3. Reid's conjecture and the canonical filtration of purely elliptic singularities.

The first counter example to the Reid conjecture was found by S. Ishii. Her example was 4-dimensional singularity and was studied from the points of views on the Newton boundary. Next M. Tomari had given another proof for her result from the theory of filtered blowing-up and also found a counter example for 3-dimensional case by these arguments. Here we discussed all examples from each points of views and had unified these.

Both authors heartily had received many important suggestions on this subject from Prof. M. Reid. : A beautiful geometric proof for Ishii's first example of 4-dimensional , existence of monogonal K3 surfaces which gives an answer to personal question from Tomari, and the construction of representation of normal K3 surface by the Veroneasian subring (§3). We both heartily thank to Prof. M. Reid. We also express our thanks to our collage of Waseda Tuesday Seminars for their warm encouragements and interests.

In this paper, we also include some new results on 4-dimensional non-rational singularities which look like terminal. These are obtain after the conference.(cf. Corollary (1.9),Theorem (2.10) (2)).

§1. Introduction to Reid's conjecture

(1.1) In this section, we discuss Reid's conjecture from a view point of general theory of filtered blowing up. Let us fix our general setting as follows:

Let (V,p) be a d-dimensional normal singularity over $\mathbf{C}$. As in §1 of [23], a filtration of ideals $F = \{F^k\}_{k\geq 0}$ is a sequence of ideals as follows:

$F^0 = O_{V,p} \supset F^1 = m, F^k \supset F^{k+1}$, and $F^iF^j \subset F^{i+j}$. We set $\mathcal{R} = \oplus_{k\geq 0} F^kT^k \subset O_{V,p}[T]$ and assume that $\mathcal{R}$ is a finitely generated $O_{V,p}$-algebra. We can choose an integer $N > 0$ such that $F^{kN} = F^N \cdot \cdots \cdot F^N$ ($k \geq 0$ times), and we assume $ht(F^N) = d$.

Let $\psi : X = \mathrm{Proj}(\mathcal{R}) \to V$ the filtered blowing up by F.

Then we can show the following.

Theorem (1.2). Let (V,p) be a normal d-dimensional isolated singularity over $\mathbf{C}$. Then $a_F(O_{V,p}) < 0$ holds for any filtration of ideals F on $O_{V,p}$ which satisfy the conditions (1.1), if and only if $p_g(O_{V,p}) = 0$.

Here $a_F(O_{V,p})$ is the a-invariant for the filtered rings for d-cohomology $H^d_m(O_{V,p})$ as follows: Let

$$F^k H^d_m(O_{V,p}) = \mathrm{Im}\{R^{d-1}\psi_* O_X(k) \to H^{d-1}(X - \psi^{-1}(p), O_X) \cong H^d_m(O_{V,p})\}$$

and we set $a_F(O_{V,p}) = \max\{k \mid F^k H^d_m(O_{V,p}) \neq 0\}$ (cf. §1 [23]).

Proof. First, we assume that the condition $a_F(O_{V,p}) < 0$ holds for any filtration of ideals F. There is a resolution of singularities by a composition of blowing-ups as follows (by the big theorem of Hironaka [6])

$$V = V_0 \xleftarrow{\psi_1} V_1 \xleftarrow{\psi_2} V_2 \xleftarrow{\psi_3} \;....\; \xleftarrow{\psi_N} V_N$$

where the center of the blowing up $\psi_i : V_i \to V_{i-1}$ is contained in the singular locus $\mathrm{Sing}(V_{i-1})$, for $i = 1$, ..., N. The composition $\psi_1 \circ \psi_2 \circ \ldots \circ \psi_N$ is obtained as a blowing up with the center ideal $\mathcal{I}$ where the support of $\mathcal{I}$ is contained in the singular locus of $V = V_0$ (cf. EGA [5], or Hironaka-Rossi[7]). Since V is an isolated singularity, I is an m-primary ideal.

We introduce the filtration F as the $\mathcal{I}$-adic filtration, that is $F^n = \mathcal{I}^n$ for $n \geq 0$ and $F^n = O_{V,p}$ for $n \leq -1$. Now $\psi : V_N = \mathrm{Proj}(\oplus_{n\geq 0}\mathcal{I}^n) \to V$ is a resolution of singularity. We can introduce the natural filtration on $H^d_m(O_{V,p})$ by F §1 of [23], and we have

$$F^0 H^d_m(O_{V,p}) = \mathrm{Im}\{R^{d-1}\psi_* O_{V_N} \to H^{d-1}(V_N - \psi^{-1}(p), O_{V_N}) \cong H^d_m(O_{V,p})\}$$

and this is isomorphic to $R^{d-1}\psi_*(O_{V_N})$, since by the Grauert-Riemenschneider vanishing theorem and the relative duality for projective morphism we have

$$H^{d-1}_{\psi^{-1}(p)}(V_N, O_{V_N})^\wedge \cong H^1(V_N, \omega_{V_N}) = 0.$$

Hence the condition $a_F(O_{V,p}) < 0$ implies $R^{d-1}\psi_*(O_{V_N}) = 0$.

Now we will show the converse direction. Assume $p_g(V,p) = 0$. Let $F = \{F^k\}$ be a filtration of ideals of $O_{V,p}$ in the sense of §1 of [TW]. That is $\oplus_{k\geq 0} F^kT^k \subset O_{V,p}[T]$

is a finitely generated $O_{V,p}$-algebra. Let $N \in \mathbf{N}$ satisfy the relation $F^{kN} = F^N \cdot \ldots \cdot F^N, (k \geq 0)$. Now F^N defines the topology of this filtration and Support of $O_{V,p}/F^N$ is the exceptional locus of blowing up

$$X = \mathrm{Proj}(\oplus_{k \geq 0} F^k T^k) \to V \supset V(F^N).$$

We assume $V(F^N) = \{p\}$. Let $\tau : \tilde{X} \to X$ be a resolution of singularities and $\varphi : \tilde{X} \to V$ the composition of τ and ψ. We set $E = \mathrm{Proj}(\oplus_{k \geq 0} F^k/F^{k+1})$ and $A = \tau^{-1}(E)$, so $A = \varphi^{-1}(p)$. We have the isomorphisms $\tilde{X} - A \cong X - E \cong V - \{p\}$ and the following commutative diagram

$$\begin{array}{cccll} 0 = & H^{d-1}(\tilde{X}, O_{\tilde{X}}) & \to & H^{d-1}(\tilde{X} - A, O_{\tilde{X}}) & \\ & \uparrow & & \uparrow\cong & \\ & H^{d-1}(X, O_X) & \to & H^{d-1}(X - E, O_X) & \cong H^d_m(O_{V,p}) \end{array}$$

Hence $H^{d-1}(X, O_X) \to H^{d-1}(X - E, O_X) \cong H^d_m(O_{V,p})$ is a zero-map. For $k \geq 0$, we have $O_X(k) \subset O_X$, therefore

$$F^k H^d_m(O_{V,p}) = \mathrm{Im}\left\{H^{d-1}(X, O_X(k)) \to H^{d-1}(X - E, O_X) \cong H^d_m(O_{V,p})\right\} = 0.$$

Hence we obtain the condition $a_F(O_{V,p}) < 0$.

Q.E.D

(1.3) Under the assumption that $V - \{p\}$ has only rational singularities, we have the relations

$$H^q(\tilde{V}, O_{\tilde{V}}) \cong H^{q+1}_m(O_{V,p}) \quad (0 < q < d - 1)$$

for any resolution of singularities $\tilde{V} \to V$. (This is standard. e.g. see an argument in [28])

Hence if we assume (V, p) is isolated singularity and the Cohen-Macaulay condition on $O_{V,p}$, the theorem above gives the criterion for rational singularity. However, it is difficult "*to study every blowing-up* ". For the computation of a_F, we recall the following. By [23], $a_F(O_{V,p}) = a(gr_F(O_{V,p}))$ when $gr_F(O_{V,p})$ is a Cohen-Macaulay ring (see Proposition(1.17) of [23] for more precise information).

One of most familiar blowing-ups are so called weighted blowing-up induced from the weighting on the coordinates of minimal embedding of the singularity.

Let (V, p) be a hypersurface isolated singularity represented as follows:

$$(V, p) = (\{(x_1, \ldots, x_{d+1}) \in \mathbf{C}^{d+1} \mid f(x_1, \ldots, x_{d+1}) = 0\}, o)$$

where f is a convergent power series and we assume this is a minimal embedding. If we choose some weighting on the variables as $\alpha = (\alpha_1, \ldots, \alpha_{d+1})$, we obtain the following relation for the weight filtration F_α

$$a_{F_\alpha}(O_{V,p}) = a(gr_{F_\alpha}(O_{V,p}) = deg_\alpha(in_{F_\alpha}(f)) - \sum_{i=1}^{d+1} \alpha_i.$$

Please refere [4, Goto-Watanabe, On graded rings I] for the generalities of the a-invarinats of the graded rings.

Let us recall some results for 3-variables (2-dimensional singularity). We don't know the first reference for the following studies in the literture. However, it is Prof. M. Reid who stated this point of view in [17] and discussed again carefully in [19]. We also the following here for the aim to state what the meaning of Reid's conjecture, and finally reduce the arguments to D. Kirby's classical studies.

Theorem (1.4) . Let us assume $\{f=0\}-\{o\}$ is regular and $a_\alpha(f)<0$ for any positive weight α and any analytic coordinate $\mathbf{x}$. Then $\{f=0\}$ defines a rational singularity at the origin o.

n fact the following arguments also provide the complete explicit classification of the defining equation for the rational double points, that is A.D.E. equations. This is one of motivation to consider simillar problem in three dimensional case. Also we will observe that we only need treat the essential *three* weights instead of all weights to check our assertion.

Proof. First we set $\alpha_1=(1,1,1)$. The weighted Taylor expansion is nothing but the usual Taylor expansion. We have $a_{\alpha_1}(f)=\mathrm{ord}(f)-3<0$, and this means that the multiplicity of the singularity is two or one.

We can choose a suitable coordinate system x_1, x_2, x_3 as $f=x_1^2-A(x_2,x_3)$ times certain unit in $\mathbf{C}\{x_1,x_2,x_3\}$. Here we can easily see the unit multiplication can be ignored essentially.

Now we set the second weight $\alpha_2=(2,1,1)$. Here we have
$a_{\alpha_2}(f)=\min\{\alpha_2(x_1^2),\mathrm{ord}(A(x_2,x_3))\}-(2+1+1)<0$, that is $\mathrm{ord}(A(x_2,x_3))\le 3$.

We can choose a suitable coordinate system x_1', x_2', x_3' as $f=(x_1')^2-\{(x_2')^3+B_2(x_3')x_2'+B_3(x_3')\}$ times certain unit in $\mathbf{C}\{x_1,x_2,x_3\}$.

We set the third weight $\alpha_3=(3,2,1)$. Here we have $a_{\alpha_3}(f)=\min\{\alpha_3(x_1^2)=\alpha_3((x_2')^3),\mathrm{ord}(B_2(x_3))+\alpha_3(x_2'),\mathrm{ord}(B_3(x_3))\}-(3+2+1)<0$, that is $\mathrm{ord}(B_2(x_3))\le 3$ or $\mathrm{ord}(B_3(x_3))\le 5$.

Let $C=\{(x_2')^3+B_2(x_3')x_2'+B_3(x_3')=0\}\subset\mathbf{C}^2$ be the branch locus and C_1 be the proper transform of C by the blowing up at o. One can see that the condition $\mathrm{ord}(B_2(x_3))\le 3$ or $\mathrm{ord}(B_3(x_3))\le 5$ equivalent to that the multiplicity of all points of C_1 is less than or equal two. In [12] D. Kirby had classified the equisingular class of these curves and shown that they are equisingular rigid. In fact the resulting classes are coincide with the famous Klein equations. D. Kirby had induced these conditions by the absolute isolatedness of the double point $\{f=0\}$ and had shown that the absolutely isolated double points are Klein-Du Val singularities.

Q.E.D.

The three weights in the proof are closely related to hypersurface simple elliptic singularites. By careful studies of hierarchy of these classes by K. Saito [20] and others, simple elliptic singularities are the next classes to rational singularities and all non-rational singularities can be deformed into one of these.

Now let us take our attention to 3-dimensional cases. In [17], Prof. M. Reid proposed a strategy to classify 3-dimensional hypersurface rational singularity as an analogy of 2-dimensional case.

And he hoped that so called *the famous* 95 *weights* of 3-dimensional analogy of simple elliptic singularities would play important roll to solve the following conjecture:

Conjecture (1.5).((4.2)of [17], M. Reid). Let $f \in \mathbf{C}\{x_1, x_2, x_3, x_4\}$ and put some weights on the variables $x_i's$ as $wt(x_i) = \alpha_i > 0 \in \mathbf{N}$ for $i = 1, 2, 3, 4$. Let us assume $\{f = 0\} - \{o\}$ is regular and $a_\alpha(f) < 0$ for any positive weight α and any analytic coordinate $\mathbf{x}$. Then $\{f = 0\}$ defines a rational singularity at the origin o.

So far, we have several affirmative answer to this. First , if the singularity admits the graded structure , we have:

Theorem (1.6). (H. Flenner [2], K.-i. Watanabe [26,27]) Let $R = \oplus_{k \geq 0} R_k$ be a normal graded ring. Then $\mathrm{Spec}(R)$ has rational singularity if and only if $\mathrm{Spec}(R) - \{V(R_+)\}$ is rational and $a(R) < 0$.

Hence, if R is a hypersurface isolated singularity defined by a quasi-homogeneous polynomial , the conjecture follows. For the singularities which is non-degenerate with respect to their Newton boundary, the answer is affirmative by M. Reid [17] and M. Oka [16].

Theorem (1.7). (M. Reid [17], M. Oka [16]). Let $f \in \mathbf{C}\{x_1, \ldots, x_{d+1}\}$ be non-degenerate with respect to its Newton boundary $\Gamma_+(f)$ in the sense of Kushinirenko. Then the conjecture of M. Reid is true.

And also refer an affirmative results on simple K3 double points (Theorem (3.5)).

However, we found a counter example as in Introduction. In §we will study the counter example from the point of views on "the theory of Newton boundary itself". And in §3, we will study its by a special interest on "simple K3 singularity and theory of filtered rings ".

Following the result and consequences of Theorem (1.2), we will also discuss the behavior of a_F in the case of terminal singularities.

Theorem (1.8). Let (V, p) be Gorenstein rational singularity and F a filtration of ideas as above. Suppose $G = gr_F(O_{V,p})$ is a Gorenstein ring with $a(G) \geq -1$. Then we obtain the following conditions.

(1) $a(G) = -1$. (2) X is normal and has only rational singularities. (3) $\psi : X \to V$ is a crepant morphism and $\psi^{-1}(o) = E \cong \mathrm{Proj}(G)$ is the exceptional Weil divisor.

Proof. By Theorem (1.2), we have $a_F(O_{V,p}) \leq -1$ and have the relation $a_F(O_{V,p}) = a(G)$ by the Cohen-Macaulay property of G ([23]). Hence we obtain $a(G) = -1$. By Theorem (3.5) [23]), we have $\omega_X \cong O_X(a(G) + 1)$ corresponding to an isomorphism $\omega_{V,p} \cong O_{V,p}$. Therefore $\omega_X \cong O_X(0) = O_X$.

Let $\tau : \tilde{X} \to X$ be a resolution of X such that $\varphi = \psi \circ \tau : \tilde{X} \to V$ is a resolution of singularities of V. By the identification $\omega_V = O_V\omega_o$, we have $\omega_{\tilde{X}} \supset O_{\tilde{X}}\varphi^{-1}(\omega_0) \cong O_{\tilde{X}}$, since $\varphi^{-1}(\omega_0)$ is a holomorphic form on $\tilde{X}$. Here there is an injection $\tau_*(\omega_{\tilde{X}}) \subset \omega_X$ by the natural trace morphism. Hence

$$O_X \subset \tau_*(O_{\tilde{X}}) \subset \tau_*\omega_{\tilde{X}} \subset \omega_X = O_X$$

and these are all equal. Since G is a Cohen-Macaulay ring, so is $X = \mathrm{Proj}(\mathcal{R})$ (cf [23]§1). Hence the relation $\tau_*\omega_{\tilde{X}} = \omega_X$ implies that X has only rational singularities.

Q.E.D

Corollary (1.9). Let (V,p) be a Gorenstein terminal singularity and F a filtration such that G is Gorenstein. Then $a(G) \leq -2$.

However, as seen in Theorem (2.10) (2) , (2.11) , in the case of 4-dimensional singularity, the conditions for the terminal singularity do not follow from $a_F \leq -2$ for only monomial weight filtration.

§2. Existence of monomials of square of quadric polynomials and elementary construction of counter examples

(2.1) In this section, we will discuss the definitive existence of certain monomials of square of quadratic forms in terms of the rank. Let k be an infinite field of the characteristic $\mathrm{char}(k) \neq 2$, and $q \in k[x_1, \dots, x_m]$ be a homogeneous polynomial of degree 2.

We can represent q as a bilinear form as: $q(x) = {}^t xAx$ where A is an $m \times m$ symmetric matrix, and we call the rank of A as the rank of q. First we remark the following:

Lemma (2.2). Let the notation be as above. Then the rank of q is the number of essential variable as seen the following relation:

$$\text{rank of } q = \min \left\{ s \in N \;\middle|\; \begin{array}{c} \text{there are linear forms} l_1, \dots, l_s \\ \text{linearly independent over } k \text{and} \\ q \in k[l_1, \dots, l_s] \end{array} \right\}.$$

Proof. Let $H = \left(\frac{\partial^2}{\partial x_i \partial x_j} q\right)$ be the Hessian of q. We have $H = 2A$. If the rank of A is s, we can choose a suitable coordinate $y_1, y_2, \dots, y_m$ such that $q = \alpha_1 y_1^2 + \dots + \alpha_s y_s^2$ and that $\alpha_i \neq 0 \in k$ for $1 \leq i \leq s$. Here we need the conditions on the base fields. Then the assertion is easy.

Conjecture (2.3). Let $\alpha \in (\mathbf{Q}_{\geq 0})^m$ be a rational weighting which satisfies the conditions;

$$\alpha(x^I) \geq 1 \text{ for any monomials } x^I \in q^2.$$

Then $\sum_{i=1}^{m} \alpha_i \geq \frac{\text{rank } q}{4}$. Moreover, if $\alpha \in (\mathbf{Q}_{>0})^m$ then the following assertion would be hold:

(i) If $m = \text{rank } q$, then $\sum_{i=1}^{m} \alpha_i \geq \frac{\text{rank } q}{4}$.

(ii) If $m > \text{rank}\alpha_i \ q$, then $\sum_{i=1}^{m} \alpha_i > \frac{\text{rank } q}{4}$.

We will prove the conjecture for the case rank $q \leq 4$. First we will show the following:

Lemma (2.4). Let $\alpha \in (\mathbf{Q}_{\geq 0})^m$ be a rational weighting which satisfies the conditions;

$$\alpha(x^I) \geq 1 \text{ for any monomials } x^I \in q^2.$$

Then

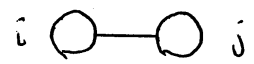
(i) If $x_i^2 \in q$, we have $\alpha_i \geq \frac{1}{4}$.

(ii) If $x_i x_j \in q$ with $i \neq j$, we have $\alpha_i + \alpha_j \geq \frac{1}{2}$.

Proof. If $x_i^2 \in q$, we can easily see that $x_i^4 \in q^2$. Hence, we obtain the relation $\alpha(x_i^4) \geq 1$, that is $\alpha_i \geq \frac{1}{4}$. If $x_i x_j \in q$ with $i \neq j$, we have two cases. In the case that $x_i^2 x_j^2 \in q^2$, we obtain the relation $\alpha(x_i^2 x_j^2) \geq 1$, so $\alpha_i + \alpha_j \geq \frac{1}{2}$. On the contrary assume that there is not $x_i^2 x_j^2$ in q^2. Then to kill the square of $x_i x_j \in q$, there should exits x_i^2 and x_j^2 in q. In this case we obtain the conditions $\alpha_i \geq \frac{1}{4}$ and $\alpha_j \geq \frac{1}{4}$ by the assertion (i). Hence we have proved our assertions.

To prove our Conjecture for special cases, we will introduce the notion of the graph of monomials for the quadratic forms.

Definition (2.5). Let $q \in k[x_1, \ldots, x_m]$ be a quadratic form as above. If $x_i x_j \in q$ with $i \neq j$, then we call "*i and j* "*is a pair* and denote by



if $x_i^2 \in q$, then we denote

Let $\mathcal{P}_q$ be the graph which consists of all wedges and vertices in q defined above. Here any subset of monomials which live in q gives a subgraph of $\mathcal{P}_q$.

Let Γbe a subgraph of $\mathcal{P}_q$ which does not contain x_i^2. We introduce the notion of "a balance subgraph "as: Γ is a balance subgraph when for each vertex of Γ there are same number of pairs in Γ. If Γ contains a x_i^2 , then we define that Γ is a balance subgraph when $\Gamma = \{x_i^2\}$.

Lemma (2.6). Let $\Gamma_1, \ldots, \Gamma_r \subset \mathcal{P}_q$ be some family of balance subgraphs such that there have no common vertex each other. Let $\alpha \in (\mathbf{Q}_{\geq 0})^m$ be a rational weighting which satisfies the conditions;

$$\alpha(x^I) \geq 1 \text{ for any monomials } x^I \in q^2.$$

Then

$$\sum_{i=1}^{m} \alpha_i \geq \frac{1}{4} \sum_{i=1}^{r} |\Gamma_i|.$$

Here $|\Gamma_i|$ denotes the numbers of vertices of Γ_i.

Proof. First we discuss on Γ_1. Let $s = |\Gamma_1|$ and write Γ_1 on the variables $\{x_1, \ldots, x_s\}$. Assume , in Γ_1, each vertex has t pairs. Then Γ_1 has exactly $\frac{st}{2}$ pairs. By Lemma (2.4), we obtain the relation.

$$\sum_{(i,j)\in\Gamma_1} (\alpha_i + \alpha_j) \geq \frac{st}{2} \times \frac{1}{2}.$$

The left hand side is $t(\alpha_1 + \ldots + \alpha_s)$, so we obtain the relation

$$\alpha_1 + \ldots + \alpha_s \geq \frac{s}{4} = \frac{|\Gamma_1|}{4}.$$

Summing up these for all Γ_i, we obtain the assertion of the lemma.

By this lemma, to prove our conjecture (2.3), it is sufficient to show the following:

Conjecture (2.7). Let s be an integer and $q \in k[x_1, \ldots, x_m]$ a quadratic form with rank $q \geq s$. Then there are balance subgraphs $\Gamma_1, \ldots, \Gamma_r \subset \mathcal{P}_q$ such that there have no common vertex each other and that

$$\sum_{i=1}^{r} |\Gamma_i| \geq s.$$

We shall show the following.

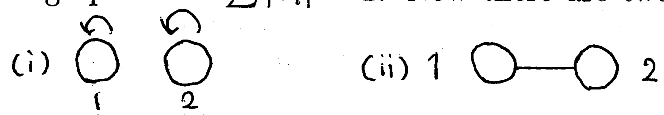
Theorem (2.8). Conjecture (2.7) is true for $s \leq 4$. (Hence (2.3) is also true for this case.)

Proof. We will show the assertion inductively on s step by step.

Let us assume $s = 1$. Then the assumption is $q \neq 0$. Hence there is a non-zero monomial $x^I \in q$. Hence $\Gamma = \{x^I\}$ gives a balance subgraph of $\mathcal{P}_q$.

Let us assume $s = 2$. If there is a balance connected subgraph $\Gamma \subset \mathcal{P}_q$ with $|\Gamma| \geq 2$, then this agrees with our assertion. So assume that there is no such a subgraph. Then q contains only monomials of the forms x_i^2. Then rank $q \geq s$ implies there should at least s x_i^2 in q as desired.

Let us assume $s = 3$. By the assertion for the case rank $q \geq 2$, we already know the existence of a family of independent balance subgraphs Γ_i's in $\mathcal{P}_q$, where the summation of supports is greater than or equal 2. If it reach to 3, these sets give affirmative answer to our assertion. So assume that we only know the existence of independent balance subgraphs with $\sum |\Gamma_i| = 2$. Now there are two cases; (i) $x_1^2, x_2^2 \in q$, (ii) $x_1x_2 \in q$.



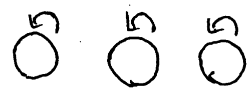
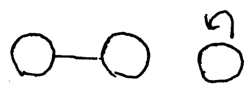
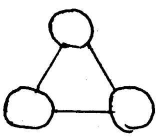
Since the number of essential variables are greater than or equal to 3, there is a monomial of the form x_ix_j with $i \geq 3$in q. Furthermore if $j \geq 3$, then the above x_ix_j gives a balance subgraph which is independent with these of (i) and (ii). So in this case our assertion follows. Now assume x_ix_j with $i \geq 3$ should be x_ix_j with $j \leq 2$.

Now we will discuss the monomials of the form x_3x_j with $j \leq 2$.

In the case (i), if $x_3x_1 \in q$, then $\{x_3x_1\} = \Gamma_1$ and $\{x_2^2\} = \Gamma_2$ give desired subgraphs. In the case $x_3x_2 \in q$ is similar.

In the case (ii). Assume $x_3x_1 \in q$ and $x_3x_2 \in q$. Then $\{x_1x_2, x_2x_3, x_3x_1\} = \Gamma_1$ gives a desired subgraph. Assume $x_3x_1 \in q$ and x_3x_2 is not contained in q. Then there is a monomial $x^I \in q$ which can not be divided by x_1, because we can see rank $q \leq 2$ when $x_1|q$. Hence there is a monomial $M = x_2x_j \in q$ with $j \neq 1$ and $j \neq 3$. So $\{x_1x_3\} = \Gamma_1$ and $\{M\} = \Gamma_2$ give a desired subgraphs.

Let us assume $s = 4$. By the assertion for the case rank $q \geq 3$, we already know the existence of a family of independent balance subgraphs $\Gamma_i's$ in $\mathcal{P}_q$, where the summation of supports is greater than or equal 3. If it reach to 4, these sets give affirmative answer to our assertion. So assume that we only know the existence of independent balance subgraphs with $\sum |\Gamma_i| = 3$. Now there are three cases; (iii) $x_1^2, x_2^2, x_3^2 \in q$, (iv) $x_1x_2, x_3^2 \in q$, (v) $x_1x_2, x_2x_3, x_3x_1 \in q$.

(iii)  (iv)  (v) 

Since the number of essential variables are greater than or equal to 4, there is a monomial of the form x_ix_j in q with $i \geq 4$. Furthermore if $j \geq 4$, then x_ix_j gives a balance subgraph which is independent with these of (iii) ,(iv) and (v). So in this case our assertion follows. Now assume x_ix_j with $i \geq 4$ should be x_ix_j with $j \leq 3$.

Now we will discuss the monomials of the form x_4x_j with $j \leq 3$.

In the case (iii), if $x_4x_1 \in q$, then $\{x_4x_1\} = \Gamma_1$, $\{x_2^2\} = \Gamma_2$, and $\{x_3^2\} = \Gamma_3$ give desired subgraphs. In the cases $x_4x_j \in q$ with $j \geq 2$ are similar.

In the case (v), if $x_4x_1 \in q$, then $\{x_4x_1\} = \Gamma_1$ and $\{x_2x_3\} = \Gamma_2$ give desired subgraphs. In the cases $x_4x_j \in q$ with $j \geq 2$ are similar.

In the case (iv). Assume $x_4x_3 \in q$. Then $\{x_1x_2\} = \Gamma_1$ and $\{x_3x_4\} = \Gamma_2$ give desired subgraphs.

Now we assume there is no x_3x_j with $j \geq 4$ in q.

Assume $x_4x_1 \in q$ and $M = x_jx_2 \in q$ with $j \geq 4$.

Then, in the case $j = 4$, $\{x_1x_2, x_2x_4, x_4x_1\} = \Gamma_1$ and $\{x_3^2\} = \Gamma_2$ give desired subgraph, and, in the case $j \geq 5$, $\{x_1x_4\} = \Gamma_1$, $\{M\} = \Gamma_2$, and $\{x_3^2\} = \Gamma_3$ give desired subgraphs.

Assume $x_4x_1 \in q$ and there is no x_jx_2 with $j \geq 4$ in q. We can write $q = x_3^2 + \tilde{q}$. Since rank $q \geq 4$, we have the condition rank $\tilde{q} \geq 3$. Then there is a monomial $x^I \in \tilde{q}$ which can not be divided by x_1, because we can see rank $\tilde{q} \leq 2$ when $x_1|\tilde{q}$. Hence there is a monomial $M = x_2x_j \in q$ with $j = 2$ or $j = 3$.

If $M = x_2^2$, then $\{x_1x_4\} = \Gamma_1$, $\{M = x_2^2\} = \Gamma_2$, and $\{x_3^2\} = \Gamma_3$ give desired subgraphs.

If $M = x_2x_3$, then $\{x_1x_4\} = \Gamma_1$ and $\{M = x_2x_3\} = \Gamma_2$ give desired subgraphs.

Q.E.D

Hope. There are no counter examples to our conjectures (2.3), (2.7). So we will propose to continue the studies as above.

Theorem (2.9). Let k be an infinite field of the characteristic $\mathrm{char}(k) \neq 2$ and $q \in k[x_1, \dots, x_{d+1}]$. Let $g \in k[[x_1, \dots, x_{d+1}]]$ be a formal power series of ord $g \geq 5$. Let $f = q^2 + g$.

(1) Assume rank $q \geq 4$ and $d \geq 4$. Then, for any analytic coordinates $y_1, \dots, y_{d+1}$ with $k[[x_1, \dots, x_{d+1}]] = k[[y_1, \dots, y_{d+1}]]$ and $\alpha \in (\mathbf{Z}_{>0})^m$ be an integral weighting of the coordinate as $\alpha(y_i) = \alpha_i > 0$, we have the condition

$$a_{F_\alpha}(f) = deg_\alpha(in_{F_\alpha}(f)) - \sum_{i=1}^{d+1} \alpha_i \leq -1.$$

Here $F_\alpha = \{F^l\}$ is the weight filtration on $k[[y_1, \dots, y_{d+1}]]$.

(2) Let k be the complex number field and $\{f = 0\}$ defines an isolated singularity at the origin. We assume ord $= 2m, m \in \mathbf{N}$ and $m \geq d-1 \geq 3$. Then we have the inequality about the geometric genus $p_g(V,p)$ of the singularity $(V,p) = (\{f=0\}, o)$ as follows:

$$p_g(V,p) \geq \binom{m+1}{d} + \binom{m}{d}.$$

(3) Let the situation be as in (2) and further assume rank $q = d+1$ and ord $g = 2(d-1)$. If $\{g = q = 0\}$ is a smooth complete intersection of P^d, then (V,p) is a d-dimensional purely elliptic singularity of the Hodge type $(0, d-1)$.

Proof. (1) Our assumption on the rank q of the initial term q^2 is independent of the choice of coordinate changes. So we will show the assertion for original coordinates as $x_i = y_i$ for all i.

First we state the following claim:

Claim (2.9.1). Let $\alpha \in (\mathbf{Q}_{\geq 0})^m$ be a rational weighting which satisfies the conditions;

$$\alpha(x^I) \geq 1 \text{ for any monomials } x^I \in f.$$

Then $\sum_{i=1}^{d+1} \alpha_i > 1$

Since any monomial x^I of q^2 lives in f, this assertion is a direct consequence of Theorem (2.8).

Now we shall show that (2.9.1) implies the assertions of (1).

Let $\alpha \in (\mathbf{Z}_{>0})^m$ be an integral weighting of the coordinate as $\alpha(x_i) = \alpha_i > 0$. Let $\mu = \min_{x^I \in f} \alpha(x^I)$. Then $\mu > 0$ and we can consider the rational weighting $\alpha' = \frac{1}{\mu}\alpha = \left(\frac{\alpha_1}{\mu}, \dots, \frac{\alpha_{d+1}}{\mu}\right)$. We have $\alpha'(x^I) \geq \frac{1}{\mu} \min_{x^I \in f} \alpha(x^I) = 1$ for any $x^I \in f$. Hence by (2.9.1) , we obtain $\sum_{i=1}^{d+1} \frac{\alpha_i}{\mu} > 1$. This implies the relation $a_{F_\alpha}(f) = \mu - \sum_{i=1}^{d+1} \alpha_i < 0$.

(2) We have the isomorphism of rings

$$O_{V,p} \cong \mathbf{C}\{x_1, \dots, x_{d+1}\}/f \cong \mathbf{C}\{x_1, \dots, x_{d+1}, y\}/\left(q+y, y^2+g(x_1, \dots, x_{d+1})\right).$$

Now let us introduce the weight filtration $\mathcal{F}$ on $\mathbf{C}\{x_1, \dots, x_{d+1}, y\}$ by the weighting $\alpha(x_i) = 1$ for all i and $\alpha(y) = m$. Let F be the induced filtration on the local ring $O_{V,p}$ from $\mathcal{F}$. Then we have the relation

$$G = gr_F(O_{V,p}) \cong \mathbf{C}[x_1, \dots, x_{d+1}, y]/\left(q, y^2 + g_{2m}(x_1, \dots, x_{d+1})\right).$$

Here g_{2m} is the initial part of g. G is a complete intersection. In particular G is a Gorenstein ring with $a(G) = 2 + 2m - (d + 1 + m) = m - d + 1$. By [Theorem (4.2), Tomari-Watanabe], we obtain the following inequality

$$p_g(V,p) \geq \sum_{k=0}^{a(G)} \dim G_k.$$

Since $\deg(y) = m > a(G) = m - d + 1$, we have the relation

$$G_k = \mathbf{C}[x_1, \dots, x_{d+1}]_k / q\mathbf{C}[x_1, \dots, x_{d+1}]_{(k-2)},$$

hence $\dim G_k = \binom{k+d}{d} - \binom{k-2+d}{d}$. So we obtain the assertion.

(3) We can construct the resolution of singularity (V,p) explicitly. Then we can check the criterion for purely ellipticity and the Hodge type.

However, in §3, we will see the assertion in terms of filtered rings (Theorem (3.6)).

We will give a similar statement which also gives an counter example even for 3-dimensional singularities.

Theorem (2.10). Let $f \in \mathbf{C}[x_1, \dots, x_{d+1}]$ be a convergent power series of the form

$$f = x_1^3 + \left(q_2(x_2, \dots, x_{d+1})^2 + D_2(x_1, \dots, x_{d+1})x_1^2\right) + h(x),$$

where $q_2 \in \mathbf{C}[x_2, \dots, x_{d+1}]$ and $D_2 \in \mathbf{C}[x_1, \dots, x_{d+1}]$ are homogeneous polynomial of degree 2. Assume rank $q_2 \geq 3$ and ord $h \geq 5$. Then, for any analytic coordinates $y_1, \dots, y_{d+1}$ with $\mathbf{C}\{x_1, \dots, x_{d+1}\} = \mathbf{C}\{y_1, \dots, y_{d+1}\}$ and $\alpha \in (\mathbf{Z}_{>0})^m$ be an integral weighting of the coordinate as $\alpha(y_i) = \alpha_i > 0$, we have the condition

$$a_{F_\alpha}(f) = deg_\alpha(in_{F_\alpha}(f)) - \sum_{i=1}^{d+1} \alpha_i \leq -1.$$

Here $F_\alpha = \{F^l\}$ is the weight filtration on $\mathbf{C}[y_1, \dots, y_{d+1}]$.

(2) Furthermore, if rank $q_2 \geq 4$ (in particular, $d \geq 4$), we can show the condition $a_{F_\alpha}(f) \leq -2$ for $\alpha \in (\mathbf{Z}_{>0})^m$.

Further, we can also show the the following criterion about non-rationality.

Theorem (2.11). Let $m \in \mathbf{N} \geq 1$. Let (V_m, o) be an isolated singularity defined as the following way:

$$V_m = \left\{ \begin{array}{l} q_2(x_1, x_2, x_3) + y + h(x_1, x_2, x_3, y, z) = 0 \\ g_{6m}(x_1, x_2, x_3, y, z) + l(x_1, x_2, x_3, y, z) = 0 \end{array} \right\} \subset \mathbf{C}^5,$$

where $q_2 \in \mathbf{C}[x_1, x_2, x_3]$ is a homogeneous polynomial of degree 2, and $g_{6m} \in \mathbf{C}[x_1, x_2, x_3, y, z]$ is a weighted homogeneous polynomial of the type $(1, 1, 1, 3m, 2m)$ and of the degree $6m$. Further we assume that $y^2, z^3 \in g_{6m}$, ord $h \geq 3$, and ord $l \geq 6m + 1$. Then we have the following inequality about the geometric genus $p_g(V_m, o)$ of the singularity (V_m, o).

$$p_g(V_m, o) \geq m^2$$

By these theorems, we can see that the example of 3-dimensional singularity in §1 gives a counter example for a conjecture of Professor M. Reid.

By the way, here, we will give the proof of these theorems.

(2.12) *Proof* of (2.10).

First we will see the coordinate transform y to x as follows: $x = Ay + \mathbf{P}(y)$ where $A = (a_{ij}) \in GL(d+1, \mathbf{C})$ and $\mathbf{P}(y) = \begin{pmatrix} P^{[1]}(y) \\ P^{[2]}(y) \\ \vdots \\ P^{[d+1]}(y) \end{pmatrix}$ and $P^{[i]} \in (y_1, \ \dots, y_{d+1})^2 \subset \mathbf{C}\{y_1, \ \dots, y_{d+1}\}$ for $1 \leq i \leq d+1$. We will denote $G(y) = f(Ay + \mathbf{P}(y)) \in \mathbf{C}\{y_1, \ \dots, y_{d+1}\}$. Here we will expand G as follows:

$$G = G_3 + G_4 + \ \dots..$$

where G_i denotes a homogeneous polynomial of degree i, and we obtain the relations: $G_3 = \left(\sum_{j=1}^{d+1} a_{1j} y_j\right)^3$ and

$$G_4 = \left(\sum_{j=1}^{d+1} a_{1j} y_j\right)^2 \left(3P_2^{[1]}(y) + D_2(Ay)\right) + q_2 \left(\sum_{j=1}^{d+1} a_{2j} y_j, \cdots, \sum_{j=1}^{d+1} a_{d+1,j} y_j\right)^2$$

where $P_2^{[1]}$ is the initial term of expansion of $P^{[1]}$ into homogeneous polynomials as $P^{[1]} = P_2^{[1]} + P_3^{[1]} + \ \dots..$

As same as in the proof of (2.8), we will concern the rational weight $\alpha \in (\mathbf{Q}_{\geq 0})^m$ be a rational weighting which satisfies the conditions;

$$\alpha(x^I) \geq 1 \ \text{ for any monomials } x^I \in G.$$

Case 1. There are at least three y_i^3 which is alive in G_3.

Say $y_1^3, y_2^3, y_3^3 \in G_3$. These also live in G. So $\alpha(y_i^3) \geq 1$, and we obtain $\alpha_i \geq \frac{1}{3}$ for $i = 1, 2, 3$. Hence $\sum_{i=1}^{d+1} \alpha_i \geq 1 + \sum_{i=4}^{d+1} > 1$, because $d \geq 3$.

Case 2. Three is exactly one $y_i^3 \in G_3$.

Say $y_1^3 \in G_3$. Then we have the relation $G_3 = a_{11}^3 y_1^3$ and $a_{12} = \cdots = a_{1,d+1} = 0$. Here we obtain the relation $\alpha_1 \geq \frac{1}{3}$ as same as in the case 1. We can represent A as follows:

$$A = \begin{pmatrix} a_{11} & 0 \cdots 0 \\ * & A' \end{pmatrix}$$

with $A' \in GL(d, \mathbf{C})$. Now G_4 is written as

$$G_4 = a_{11}^2 y_1^2 \left(3P_2^{[1]}(y) + D_2(Ay)\right) + q_2 \left(\sum_{j=2}^{d+1} a_{2j} y_j, \cdots, \sum_{j=2}^{d+1} a_{d+1,j} y_j\right)^2 + y_1 R(y)$$

with $R(y)$ is a polynomial. Here $q_2 \left(\sum_{j=2}^{d+1} a_{2j} y_j, \cdots, \sum_{j=2}^{d+1} a_{d+1,j} y_j\right) = \tilde{q_2} \in \mathbf{C}[y_2, \cdots, y_{d+1}]$ has rank $\tilde{q_2} \geq 3$, since $A' \in GL(d, \mathbf{C})$. All monomials of $\tilde{q_2}^2$ lives in G_4 (and hence in G), we obtain the conditions

$$\alpha(x^I) \geq 1 \text{ for any monomials } x^I \in \tilde{q_2}^2.$$

Hence we obtain the relation $\sum_{i=2}^{d+1} \alpha_i \geq \frac{3}{4}$. Therefore we obtain $\sum_{i=1}^{d+1} \alpha_i \geq \frac{1}{3} + \frac{3}{4} > 1$.

Case 3. Three are exactly two $y_i^3 \in G_3$.

Say $y_1^3, y_2^3 \in G_3$. Then we have the relation $G_3 = (a_{11} y_1 + a_{12})^2$ and $a_{13} = \cdots = a_{1,d+1} = 0$. Here we obtain the relation $\alpha_1 \geq \frac{1}{3}$ and $\alpha_2 \geq \frac{1}{3}$ as same as in the case 1. By Laplace expansion of $det(A) = a_{11} det(A_{11}) - a_{12} det(A_{12})$, where A_{ij} is co-multiple factor of the matrix A. Here $det(A_{11}) \neq 0$ or $det(A_{12}) \neq 0$. Now we will assume the condition $det(A_{11}) \neq 0$.

Now G_4 is written as

$$G_4 = (a_{11} y_1 + a_{12} y_2)^2 \left(3P_2^{[1]}(y) + D_2(Ay)\right) + q_2 \left(\sum_{j=2}^{d+1} a_{2j} y_j, \cdots, \sum_{j=2}^{d+1} a_{d+1,j} y_j\right)^2 + y_1 R(y)$$

with $R(y)$ is a polynomial. Here $q_2 \left(\sum_{j=2}^{d+1} a_{2j} y_j, \cdots, \sum_{j=2}^{d+1} a_{d+1,j} y_j\right) = \tilde{q_2} \in \mathbf{C}[y_2, \cdots, y_{d+1}]$ has rank $\tilde{q_2} \geq 3$, since $A_{11} \in GL(d, \mathbf{C})$.

By Theorem (2.8), we may assume , essentially, the one of the followings happen: (a) $y_2 y_3, y_4^2 \in \tilde{q_2}$, (b) $y_2^2, y_3 y_4 \in \tilde{q_2}$, (c) $y_2 y_3, y_3 y_4, y_4 y_2 \in \tilde{q_2}$, and (d) $y_2^2, y_3^2, y_4^2 \in \tilde{q_2}$.

Subcase 1. $y_3 y_4 \in \tilde{q_2}$ or $y_3^2, y_4^2 \in \tilde{q_2}$ happens.

Then the monomials on y_3, y_4 of degree 4 in $\tilde{q_2}^2$ lives in G_4. So we have the condition $\alpha_3 + \alpha_4 \geq \frac{1}{2}$. Hence we obtain the relation $\sum_{i=1}^{d+1} \alpha_i \geq \frac{2}{3} + \frac{1}{2} > 1$.

Subcase 2. Suppose the conditions of the subcase 1 does not happen. For the cases (a),(b),(c) and (d), we can assume that $y_4^2, y_2y_3 \in \tilde{q_2}$ and $\tilde{q_2}$ does not contain y_3y_4 and y_3^2.

Let us study the existence of the monomial $y_4^2y_2y_3$. Here $y_4^2 \cdot y_2y_3 = y_3y_4 \cdot y_2y_4$. But in $\tilde{q_2}^2$ there is only one possibility for getting this monomial as $y_4^2 \cdot y_2y_3$. Therefore this lives in $\tilde{q_2}^2$ and also in G_4. Hence we obtain the relation $\alpha(y_4^2 \cdot y_2y_3) \geq 1$, and $2\alpha_4 + \alpha_2 + \alpha_3 \geq 1$.

Here if $\alpha_1 > \alpha_4$, then $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > 2\alpha_4 + \alpha_2 + \alpha_3 \geq 1$.

Or if $\alpha_4 \geq \alpha_1 (\geq \frac{1}{3})$, then $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \geq \frac{1}{3} + \frac{1}{3} + \alpha_3 + \frac{1}{3} > 1$.

Therefore we obtain the assertion of (1).

Proof of (2) of (2.10).

First we state the following claim:

Claim (2.12.1). Let $\alpha \in (\mathbf{Q}_{>0})^m$ be a rational weighting which satisfies the conditions;

$$\alpha(x^I) \geq 1 \text{ for any monomials } x^I \in G.$$

Let us denote $\alpha_i = \dfrac{p_i}{q_i}$ with $p_i, q_i \in \mathbf{N}$ and $(p_i, q_i) = 1$ for $i = 1, \cdots, d+1$. Let ν be the least common multiple of $q_1, \cdots, q_{d+1}$. Then $\sum_{i=1}^{d+1} \alpha_i > 1 + \dfrac{1}{\nu}$

We will show that this claim implies our assertion of Theorem (2.10)(2).

Let $\alpha \in (\mathbf{Z}_{>0})^m$ be an integral weighting of the coordinate as $\alpha(x_i) = \alpha_i > 0$. Let $\mu = \min_{x^I \in f} \alpha(x^I)$. Then $\mu > 0$ and we can consider the rational weighting $\alpha' = \frac{1}{\mu}\alpha = \left(\frac{\alpha_1}{\mu}, \dots, \frac{\alpha_{d+1}}{\mu}\right)$. We have $\alpha'(x^I) \geq \frac{1}{\mu}\min_{x^I \in f} \alpha(x^I) = 1$ for any $x^I \in f$. Hence by (2.9.1) , we obtain $\sum_{i=1}^{d+1} \frac{\alpha_i}{\mu} > 1 + \dfrac{1}{\nu}$. By definition, μ is divided by ν, so we have $\mu \geq \nu$. This implies the relation

$$a_{F_\alpha}(f) = \mu - \sum_{i=1}^{d+1} \alpha_i < \mu - \mu\left(1 + \frac{1}{\nu}\right) = -\frac{\mu}{\nu} \leq -1.$$

Now we will show the claim.

(2.13) Let us employ the notations of (1) ; e.g. let us assume $x = Ay + \mathbf{P}(y)$ and $G = G_3 + G_4 +$ ..... as in (1).

Case 1. There are at least three y_i^3 which is alive in G_3.

Say $y_1^3, y_2^3, y_3^3 \in G_3$. These also live in G. So $\alpha(y_i^3) \geq 1$, and we obtain $\alpha_i \geq \frac{1}{3}$ for $i = 1, 2, 3$. Hence $\sum_{i=1}^{d+1} \alpha_i \geq 1 + \alpha_4 + \alpha_5 \geq 1 + \dfrac{2}{\nu}$, because $d \geq 4$.

Case 2. Three is exactly one $y_i^3 \in G_3$.

Say $y_1^3 \in G_3$. Then we have the relation $G_3 = a_{11}^3 y_1^3$ and $a_{12} = \cdots = a_{1,d+1} = 0$. Here we obtain the relation $\alpha_1 \geq \frac{1}{3}$ as same as in the case 1. We can represent A

as follows:

$$A = \begin{pmatrix} a_{11} & 0 \cdots 0 \\ * & A' \end{pmatrix}$$

with $A' \in GL(d, \mathbf{C})$. Now G_4 is written as

$$G_4 = a_{11}^2 y_1^2 \left(3P_2^{[1]}(y) + D_2(Ay)\right) + q_2 \left(\sum_{j=2}^{d+1} a_{2j}y_j, \cdots, \sum_{j=2}^{d+1} a_{d+1,j}y_j\right)^2 + y_1 R(y)$$

with $R(y)$ is a polynomial. Here $q_2 \left(\sum_{j=2}^{d+1} a_{2j}y_j, \cdots, \sum_{j=2}^{d+1} a_{d+1,j}y_j\right) = \tilde{q_2} \in \mathbf{C}[y_2, \cdots, y_{d+1}]$ has rank $\tilde{q_2} \geq 4$, since $A' \in GL(d, \mathbf{C})$. All monomials of $\tilde{q_2}^2$ lives in G_4 (and hence in G), we obtain the conditions

$$\alpha(x^I) \geq 1 \text{ for any monomials } x^I \in \tilde{q_2}^2.$$

Hence we obtain the relation $\sum_{i=2}^{d+1} \alpha_i \geq \frac{4}{4}$. We obtain $\sum_{i=1}^{d+1} \alpha_i \geq \frac{1}{3} + 1$. If $\nu > 3$, we have done. If $\nu = 3$, then $\alpha_i \geq \frac{1}{3}$ holds for all i. In this case we have $\sum_{i=1}^{d+1} \alpha_i \geq \frac{5}{3}$. Further if $\nu = 2$, the arguments are more easy.

Case 3. Three are exactly two $y_i^3 \in G_3$.

Say $y_1^3, y_2^3 \in G_3$. Then we have the relation $G_3 = (a_{11}y_1 + a_{12})^2$ and $a_{13} = \cdots = a_{1,d+1} = 0$. Here we obtain the relation $\alpha_1 \geq \frac{1}{3}$ and $\alpha_2 \geq \frac{1}{3}$ as same as in the case 1. Our subclaim is:

Subclaim (2.13.2). There are indecis $i, j \geq 3$ with $i \neq j$ and $\alpha_i + \alpha_j \geq \frac{1}{2}$.

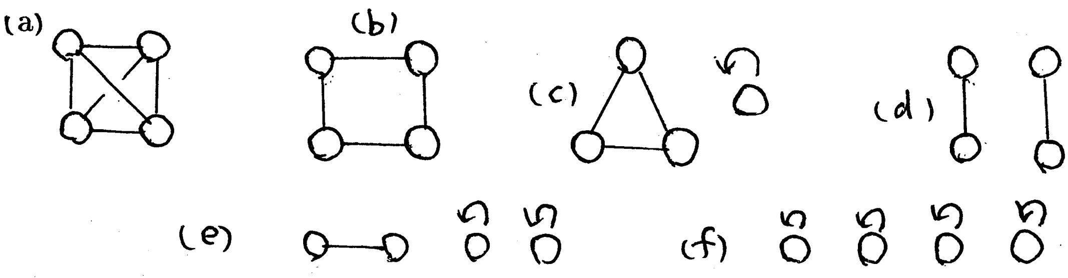
Assume we have shown the subclaim and say $i = 3, j = 4$. Then we obtain the relation $\sum_{i=1}^{d+1} \alpha_i \geq \frac{2}{3} + \frac{1}{2} + \alpha_5 > 1 + \frac{1}{\nu}$. So we have done.

We will show the subclaim. As same as in the proof of (1), we can assume that Now we will assume the condition $det(A_{11}) \neq 0$ and G_4 is written as

$$G_4 = (a_{11}y_1 + a_{12}y_2)^2 \left(3P_2^{[1]}(y) + D_2(Ay)\right) + q_2 \left(\sum_{j=2}^{d+1} a_{2j}y_j, \cdots, \sum_{j=2}^{d+1} a_{d+1,j}y_j\right)^2 + y_1 R(y)$$

with $R(y)$ is a polynomial. Here $q_2 \left(\sum_{j=2}^{d+1} a_{2j}y_j, \cdots, \sum_{j=2}^{d+1} a_{d+1,j}y_j\right) = \tilde{q_2} \in \mathbf{C}[y_2, \cdots, y_{d+1}]$ has rank $\tilde{q_2} \geq 4$, since $A_{11} \in GL(d, \mathbf{C})$.

By Theorem (2.8), there are different 4 indecis $i, j, k, l \geq 2$ and one of following 6 cases occur: (a) $y_iy_j, y_iy_k, y_iy_l, y_jy_k, y_jy_l, y_ky_l \in \tilde{q_2}$, (b) $y_iy_j, y_jy_k, y_ky_l, y_ly_i \in \tilde{q_2}$, (c) $y_iy_j, y_jy_k, y_ky_i, y_l^2 \in \tilde{q_2}$, (d) $y_iy_j, y_ky_l \in \tilde{q_2}$, (e) $y_iy_j, y_k^2, y_l^2 \in \tilde{q_2}$, and (f) $y_i^2, y_j^2, y_k^2, y_l^2 \in \tilde{q_2}$.



Hence there indecis $m, n \geq 3$ with $m \neq n$ where either $y_m y_n \in \tilde{q_2}$ or $y_m^2, y_n^2 \in \tilde{q_2}$ hold. Since the monomials of variables y_i ($i \geq 3$) in $\tilde{q_2}^2$ still live in G_4, we can conclude that $\alpha_m + \alpha_n \geq \frac{1}{2}$ by Lemma (2.4).

Theorem (2.14). Let $m \in \mathbf{N} \geq 1$. Let (V_m, o) be an isolated singularity defined as the following way:

$$V_m = \left\{ \begin{array}{l} q_2(x_1, x_2, x_3, x_4) + y + h(x_1, x_2, x_3, x_4, y, z) = 0 \\ g_{6m}(x_1, x_2, x_3, x_4, y, z) + l(x_1, x_2, x_3, x_4, y, z) = 0 \end{array} \right\} \subset \mathbf{C}^6,$$

where $q_2 \in \mathbf{C}[x_1, x_2, x_3, x_4]$ is a homogeneous polynomial of degree 2, and $g_{6m} \in \mathbf{C}[x_1, x_2, x_3, x_4, y, z]$ is a weighted homogeneous polynomial of the type $(1, 1, 1, 1, 3m, 2m)$ and of the degree $6m$. Further we assume that $y^2, z^3 \in g_{6m}$, ord $h \geq 3$, and ord $l \geq 6m + 1$. Then we have the following inequality about the geometric genus $p_g(V_m, o)$ of the singularity (V_m, o).

$$p_g(V_m, o) \geq \frac{m(m-1)(2m-1)}{6}$$

Proof. We have

$$O_{V,p} \cong \mathbf{C}\{x_1, \ldots, x_4, y, z\} / (q + y + h, g + l).$$

Now let us introduce the weight filtration $\mathcal{F}$ on $\mathbf{C}\{x_1, \ldots, x_4, y, z\}$ by the weighting $\alpha(x_i) = 1$ for all i , $\alpha(y) = 3m$, and $\alpha(z) = 2m$. Let F be the induced filtration on the local ring $O_{V,p}$ from $\mathcal{F}$. Then we have the relation

$$G = gr_F(O_{V,p}) \cong \mathbf{C}[x_1, \ldots, x_4, y, z] / (q, g_{6m}).$$

G is a complete intersection. In particular G is a Gorenstein ring with $a(G) = 2 + 6m - (4 + 2m + 3m) = m - 2$. By [Theorem (4.2), 23], we obtain the following inequality

$$p_g(V, p) \geq \sum_{k=0}^{a(G)} \dim G_k.$$

Since $\deg(y) = 3m > \deg(z) = 2m > a(G) = m - 2$, we have the relation

$$G_k = \mathbf{C}[x_1, \ldots, x_4]_k / q\mathbf{C}[x_1, \ldots, x_4]_{(k-2)},$$

hence $\dim G_k = \binom{k+1}{3} - \binom{k}{3}$. So we obtain the assertion.

§3. Reid's conjecture and the canonical filtration of purely elliptic singularities.

Definition (3.1). A normal isolated singularity (V,p) is purely elliptic if $\delta_m(V,p) = 1$ for every integer $m \geq 1$. Here δ_m ia the L^2-plurigenus which can be computed as

$$\delta_m(V,p) = \dim \omega_V^{[m]} / \varphi_*(\omega_{\tilde{V}}^{[m]}((m-1)A_{red}))$$

for a good resolution $\varphi : (\tilde{V}, A) \to (V,p)$. In the case (V,p) is Gorenstein, this class is not canonical but log canonical. Further we can introduce the notion of maximal rank of Hodge type for these class by means of the Deligne canonical Mixed Hodge Structure on the exceptional locus [Ishii, 8,9], and divided into d cases $(0,0), (0,d-1), \cdots, (0,d-1)$. For 2-dimensional case, the purely elliptic of $(0,1)$ is nothing but a simple elliptic singularity. For 3-dimensional case, [10,Ishii-Watanabe] defines a simple K3 singularity as Gorenstein purely elliptic singularity of type $(0,2)$.

It has a geometric characterization in terms of minimal model of a resolution of singularities.

Theorem (3.2)(Ishii-Watanabe [10]). Let (V,p) be a 3-dimensional normal isolated singularities. Then the following two conditions are equivalent:

(1) (V,p) is a simple K3 singularity.

(2) There is a $\mathbf{Q}$-factorial terminal modification $\psi : (W,E) \to (V,p)$ such that $\psi^{-1}(p) = E$ is a normal K3 surface with rational double points and ω_W is relative ψ-nef.

(3.3) This class is also characterized by the existence of a special filtration.

Theorem (3.3.1) ([Tomari, 21 (4.2)]). Let (V,p) be a normal d-dimensional isolated singularity such that the canonical sheaf ω_W is trivial. Then the following two conditions are equivalent.

(1) (V,p) is a purely elliptic singularity of type $(0,d-1)$ and the relative canonical algebra of (V,p) is finitely generated.

(2) There is a filtration $\{F^k\}_{k\in\mathbb{Z}}$ on the local ring $O_{V,p}$ which satisfies the following conditions:
(2)-0. $\mathcal{R} = \oplus_{k\geq 0} F^k T^k \subset O_{V,p}[T]$ is finitely generated over $O_{V,p}$ and satisfies the conditions of [Tomari-Watanabe, §1].
(2)-1. $G = \oplus_{k\geq 0} F^k/F^{k+1}$ is a normal domain.
In a representation $G = R(E,D)$ by the theorem of Pinkham-Demazure, we have
(2)-2. $E = \mathrm{Proj}(G)$ has only rational singularity,

(2)-3. D is an ample integral Weil divisor, and
(2)-4. $\omega_E \cong O_E$.

Under these two equivalent conditions, we have the relations

$$H^q_m(O_{V,p}) \cong H^q_{G_+}(G) \cong H^{q-1}(E, O_E) \quad \text{for} \quad 2 \leq q \leq d-1.$$

In particular, under the assumption of the Gorenstein condition of (V,p), we have the vanishing $H^{q-1}(E, O_E)$ for $1 \leq q \leq d-2$.

The filtration in the above is unique and induced from the canonical model of a resolution of singularity. We call it the canonical filtration. In this situation we have the condition $a(gr_F(O_{V,p})) = 0$. So we can see the following:

Theorem (3.3.2)(§4,§5 of [Tomari, 21]. Let (V,p) be a normal d-dimensional Gorenstein isolated singularity over $\mathbf{C}$. Let us assume (V,p) is a purely elliptic of the Hodge type $(0, d-1)$. Let $F = \{F^k\}_{k \geq 0}$ be the canonical filtration such that the associated Rees algebra $\mathcal{R} = \oplus_{k \geq 0} F^k T^k \subset O_{V,p}[T]$ is a finitely generated $O_{V,p}$-algebra. Further assume the associated graded ring $G = gr_F(O_{V,p})$ is a hypersurface.

Then we have the following conclusions;

(1) (V,p) is a hypersurface.

(2) There is a minimal embedding $(V,p) \subset (\mathbf{C}^{d+1}, o)$ with an analytic coordinate $(x_1, \cdots, x_{d+1})$ with positive weighting $(q_1, \cdots, q_{d+1})$ which induces the canonical filtration F.

(3) Let us represent $\{f = 0\} = V \subset \mathbf{C}^{d+1}$ with $f \in \mathbf{C}\{x_1, \cdots, x_{d+1}\}$. In the weighted Taylor expansion of f : $f = \sum_{i \geq \rho} f_i$ with respect to the weight $(q_1, \cdots, q_{d+1})$, the initial term f_ρ is a weighted homogeneous polynomial which satisfies the followings:

(3-1) $\rho = \sum_{i=1}^{d+1} q_i$,

(3-2) $\{f_\rho = 0\} - \{o\}$ has only rational singularities, and

(3-3) the Newton boundary $\Gamma(f_\rho) \subset \mathbf{R}^{d+1}$ of f_ρ is d-dimensional and contains $(1, \cdots, 1) \in \mathbf{R}^{d+1}$ in its relative interior.

Remark (3.4). (1) In the case $d = 3$, the weight (q_1, q_2, q_3, q_4) and the degree ρ is one of famous 95 weights. These are classified by M. Reid, A.N. Fletcher and T. Yonemura. [17, 2, 29] (also see [21]).

(2) We have stated Reid's conjecture as a criterion for rational singularity in the introduction of the present paper. Also we can regard the conjecture suggest the existence of certain good minimal embedding with weighting for non-rational singularity where $a_F \geq 0$. The above theorem says that, for purely elliptic hypersurface singularity of a special type, a sufficient condition to check Reid's hope is to show that the associated graded ring of the canonical filtration is a hypersurface.

In view of Remark (3.4) (2), we will review the the following:

Theorem (3.5) ([tomari, 22]). Let (V,p) be a simple K3 singularity of multiplicity two. Let $F = \{F^k\}_{k\geq 0}$ be the canonical filtration. Then $gr_F(O_{V,p})$ is a hypersurface of multiplicity two.

Hence we can not construct counter examples to Reid's conjecture for the simple K3 singularity of multiplicity two. However we have observed in §2, that there are non-rational hypersurface which look like canonical by means of monomial weight filtrations of their minimal embeddings. In what follows, we will give some examples of purely elliptic singularities which give counter example for Reid's conjecture by the theory of filtered blowing-ups (by using Theorems (3.2), (3.3.1) ,(3.3.2)).

Now let us consider the situation as follows:

Theorem (3.6). Let $d \geq 4$. Let

$$(V,p) = \left\{ \begin{array}{c} q(x_1, \cdots, x_{d+1}) + y = 0 \\ g_{2(d-1)}(x_1, \cdots, x_{d+1}) + y^2 + h = 0 \end{array} \right\} \subset \mathbf{C}^{d+2}$$

be an isolated singularity at o where $q, g_{2(d-1)} \in \mathbf{C}[x_1, \cdots, x_{d+1}]$ are homogeneous of degree $2, 2(d-1)$ respectively. We assume $h \in \mathbf{C}\{x_1, \cdots, x_{d+1}, y\}$ has the order $\geq 2(d-1)+1$. If

$$\left\{ \begin{array}{c} q(x_1, \cdots, x_{d+1}) = 0 \\ g_{2(d-1)}(x_1, \cdots, x_{d+1}) = 0 \end{array} \right\} \subset \mathbf{P}^d$$

has only rational singularities, then (V,p) is a d-dimensional hypersurface purely elliptic singularity of type $(0, d-1)$. Moreover (V,p) gives a counter example of Reid's conjecture.

Proof. To use Theorem (3.3.1) : criterion for purely ellipticity of the singularity, we shall introduce a filtration of ideals on the local ring as follows.

Let us represent the local ring $O_{V,p}$ as $O_{V,p} = \mathbf{C}\{x_1, \cdots, x_{d+1}, y\}/(q+y, g_{2(d-1)}+y^2+h)$ and introduce the filtration $F = \{F^k\}_{k\geq 0}$ by the following weighting $wt(x_i) = 1, 1 \leq i \leq d+1, wt(y) = d-1$.

We can see that

$$G = gr_F(O_{V,p}) = \mathbf{C}[x_1, \cdots, x_{d+1}, y]/(q, g_{2(d-1)} + y^2)$$

and this is a Gorenstein ring with $a(G) = 2 + 2(d-1) - (d+1) - (d-1) = 0$. By our assumption, G is a normal graded ring with the Demazure construction $G = R(X_{2,2(d-1)}, D)$ where $X_{2,2(d-1)} = \mathrm{Proj}(G)$ and D is an ample divisor which corresponds to a homogeneous element of G of degree 1. Here we know that $X_{2,2(d-1)}$ is a double covering of P^d branched along $\left\{ \begin{array}{c} q(x_1, \cdots, x_{d+1}) = 0 \\ g_{2(d-1)}(x_1, \cdots, x_{d+1}) = 0 \end{array} \right\} \subset \mathbf{P}^d$.

We can choose D is the pull back of the hyperplane of $\mathbf{P}^d$. So $X_{2,2(d-1)}$ has only Gorenstein rational singularities. By Theorem (3.3), we can conclude that (V,p) is a Gorenstein purely elliptic singularity of the Hodge type $(0,d-1)$ and F is the canonical filtration.

Theorem (3.7)(Tomari [21, (4.10) ,(4.13)]. Let (V,p) be a normal d-dimensional hypersurface purely elliptic singularity of type $(0,d-1)$. Let F_α be a monomial filtration on $O_{V,p}$ induced from certain weight α for a minimal embedding. Suppose $a(gr_{F_\alpha}(O_{V,p})) \geq 0$. Then F_α is the canonical filtration.

We have seen that the associated graded ring of the canonical filtration is not a hypersurface. So any monomial filtration F_α induced from a minimal embedding is not the canonical filtration. Hence $a_{F_\alpha}(O_{V,p}) < 0$ by Theorem (3.7). Now we conclude that (V,p) is a counter example to Reid's conjecture.

Q.E.D

Now the following theorem gives a counter example to Reid's conjecture for 3-dimensional simple K3 singularity of multiplicity three.

Theorem (3.8). Let

$$(V,p) = \left\{ \begin{array}{c} q(x_1,x_2,x_3) + z = 0 \\ g_6(x_1,x_2,x_3,y,z) + h = 0 \end{array} \right\} \subset \mathbf{C}^5$$

be a 3-dimensional isolated singularity at o where $q \in \mathbf{C}[x_1,x_2,x_3]$ is homogeneous of degree 2 , and $g_6 \in \mathbf{C}[x_1,x_2,x_3,y,z]$ a weighted homogeneous polynomial of degree 6 with respect to $wt(x_i) = 1, i = 1,2,3, wt(y) = 2, wt(z) = 3$ respectively. We assume $h \in \mathbf{C}\{x_1, \cdots, x_{d+1}, y, z\}$ has the order ≥ 7. If $\mathbf{C}[x_1, \cdots, x_{d+1}, y]/(q, g_6)$ is normal and

$$X_{2,6} = \left\{q(x_1,x_2,x_3) = 0 g_{2(d-1)}(x_1,x_2,x_3,y,z) = 0\right\} \subset P(1,1,1,2,3)$$

has only rational singularities, then (V,p) is a 3-dimensional hypersurface simple K3 singularity. Moreover (V,p) gives a counter example of Reid's conjecture.

Proof. The arguments are similar as in (3.6).

Let us represent the local ring $O_{V,p}$ as $O_{V,p} = \mathbf{C}\{x_1, \cdots, x_{d+1}, y\}/(q+z, g_6+h)$ and introduce the filtration $F = \{F^k\}_{k \geq 0}$ by the following weighting $wt(x_i) = 1, i = 1,2,3, wt(y) = 2, wt(z) = 3$.

We can see that

$$G = gr_F(O_{V,p}) = \mathbf{C}[x_1, \cdots, x_{d+1}, y]/(q, g_6)$$

and this is a Gorenstein ring with $a(G) = 2+6-(1+1+1+2+3) = 0$. By our assumption, G is a normal graded ring with the Demazure construction $G = R(X_{2,6}, D)$

where $X_{2,2(d-1)} = \mathrm{Proj}(G)$ has only Gorenstein rational singularities. By Theorem (3.3), we can conclude that (V,p) is a Gorenstein purely elliptic singularity of the Hodge type $(0,2)$ and F is the canonical filtration.

We have seen that the associated graded ring of the canonical filtration is not a hypersurface. So any monomial filtration F_α induced from a minimal embedding is not the canonical filtration. Hence $a_{F_\alpha}(O_{V,p}) < 0$ by Theorem (3.7). Now we conclude that (V,p) is a counter example to Reid's conjecture.

Q.E.D

Example (3.9). A normal K3 surface of the conditions of Theorem (3.8) is given by the following explicit equation.

$$X_{2,6} = \left\{ \begin{array}{c} x_2^2 - x_1x_3 = 0 \\ x_1^6 + x_3^6 + y^3 + z^2 = 0 \end{array} \right\} \subset P(1,1,1,2,3).$$

The second author had learned the existence of this K3 surface from Prof. M. Reid as "*monogonal K3 surface* "which is an important phenomena in the theory of family of polarized K3 surfaces of degree two. In fact, this is a limit of nonsingular K3 surface of degree 2 as follows:

$$X_t = \left\{ \begin{array}{c} x_2^2 - x_1x_3 + ty = 0 \\ x_1^6 + x_3^6 + y^3 + z^2 = 0 \end{array} \right\} \subset P(1,1,1,2,3) \quad \text{where} \quad t \in \mathbf{C}.$$

If $t \neq 0$, then X_t is embedded in $P(1,1,1,3)$ as

$$X_t = \left\{ x_1^6 + x_3^6 + \left(x_1x_3 - x_2^2\right)^3 + z^2 = 0 \right\}.$$

Hence we can also observe the degeneration of simple K3 singularities as follows:

$$(V_t, o) = \left(\left\{ \begin{array}{c} x_2^2 - x_1x_3 + ty + z = 0 \\ x_1^6 + x_3^6 + y^3 + z^2 + h = 0 \end{array} \right\}, o \right) \subset (\mathbf{C}^5, o) \quad \text{where} \quad t \in \mathbf{C}.$$

(In [13], S. Mori already had constructed a certain degeneration of K3 surfaces of a different type in certain weighted projective space, where the weighted degree changes by specialization.)

In this family, the filtration F induced from the weight $wt(x_i) = 1, 1 \le i \le 3, wt(y) = 2, wt(z) = 3$ gives the canonical filtration for all t. Hence the simultaneous filtered blowing-up of this family gives a simultaneous canonical model. In particular this family has the constant γ_m. The singularity (V_0, o) stands outside of famous 95 classes from the view of the Newton boundary. But this is a limit of one class of 95classes which has simultaneous canonical models.

A natural question is

Question (3.10). Let (V,p) be a hypersurface simple K3 singularity of multiplicity 3 such that the associated graded ring of the canonical filtration is not a hypersurface. (Hence this gives a conjecture of Reid as in Theorem (3.8).) Then is it a limit of simple K3 singularity of multiplicity 2 which has simultaneous canonical model?

(3.11) We will give three more similar example of simple K3 triple points in the below. Our approach heavily depend on the following observation of Prof. M. Reid. Let us consider $(\{x^{12}+y^{12}+z^3+w^2=0\}, o)$. This is an simple K3 singularity. We can consider the normal graded ring $R = \mathbf{C}[x,y,z,w]/x^{12}+y^{12}+z^3+w^2$. Here we can represent $R = R(X,D)$ as Demazure's construction, where $X = \mathrm{Proj}(R)$ is a normal K3 surface and D an integral ample Weil divisor. Let $R^{[2]}$ be the second Veronesian subring of R. We obtain

$$\begin{aligned} R^{[2]} &= \mathbf{C}[x^2, xy, y^2, z, w]/x^{12}+y^{12}+z^3+w^2 \\ &\cong \mathbf{C}[s,t,u,z,w]/(s^6+t^6+z^3+w^2, t^2-su). \end{aligned}$$

Hence we obtain the defining equation of $X_{2,6}$ by the second Veronesian of $X_{12} \subset P(1,1,4,6)$.

By the similar procedure, we can obtain the following three examples:

(3.11.1) $X_{4,12} \subset P(1,2,3,4,6)$ as the second Veronesian of $X_{24} \subset P(1,3,8,12)$.

(3.11.2) $X_{6,18} \subset P(1,3,5,6,9)$ as the second Veronesian of $X_{36} \subset P(1,5,12,18)$.

(3.11.3) $X_{8,24} \subset P(3,4,5,8,12)$ as the second Veronesian of $X_{48} \subset P(3,5,16,24)$.

For these three cases, we can attach hypersurface simple K3 triple points where the associated graded ring of the canonical filtration is not a hypersurface. Hence these three singularities also give counter example to Reid's conjecture.

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