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Fundamental groups of complements to hypersurfaces

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1. Statement of results

In this talk, we will present a generalization of Zariski's hyperplane section theorem.

Let S be a hypersurface in a complex projective space $\mathbb{P}^n$ of dimension ≥ 2. We take a linear plane $\mathbb{P}^2$ in $\mathbb{P}^n$ in a general position with respect to S. Zariski's hyperplane section theorem asserts the following isomorphism:

$$\pi_1(\mathbb{P}^2 \setminus (\mathbb{P}^2 \cap S)) \cong \pi_1(\mathbb{P}^n \setminus S).$$

This enables us to calculate the fundamental group of the complement to a hypersurface by van-Kampen Zariski method. This theorem was stated by Zariski in [Z], but the proof had a gap. The first rigorous proof was given by Hamm and Lê in [H-L]. They used the Morse theory.

Now we are going to consider the following situation. Let U be a complex homogeneous variety on which a connected affine algebraic group G acts transitively. The stabilizer group H_p of a point p of U is assumed to be connected. Let $f : X \to U$ be a morphism from a non-singular connected algebraic variety X. We do not assume that f is proper. For an element $\gamma \in G$, let ${}^{\gamma}f : X \to U$ be the composite of f with the action $\gamma : U \to U$ of γ on U. Suppose that we are given a non-zero reduced effective divisor D of U.

Now we consider the following three conditions of f.

(C1) The image of f is of dimension at least 2.

(C2) The locus of all points of X at which the tangential map of f is of rank zero is of codimension at least 2;

$$\dim \{\ x \in X\ ;\ \dim f_{*,x}(T_xX) = 0\} \ \leq\ \dim X - 2.$$

(C3) A morphism $\bar{f} : \bar{X} \to U$ is said to be a nonsingular projective completion of f if $\bar{X}$ is a nonsingular algebraic variety which contains X as its Zariski open dense subset and $\bar{f}$ is a projective morphism which coincides with f on $X \subset \bar{X}$. Now the third condition is that, there is a non-singular projective completion $\bar{f} : \bar{X} \to U$ of f such that, if W_k is an irreducible component of the boundary $W := \bar{X} \setminus X$ with codimension 1 in $\bar{X}$, then $\dim \bar{f}(W_k)$ is at least one.

Our purpose is to calculate the fundamental group $\pi_1({}^{\gamma}f^{-1}(U \setminus D))$ in terms of $\pi_1(X)$ and $\pi_1(U \setminus D)$ *when γ is chosen generally from G.* We can give a clear answer to this problem in the following situations.

I. *Projective spaces*

Let U be a projective space $\mathbb{P}^n$ with $n \geq 2$, and G the group $\mathrm{GL}(n+1)$ of general linear transformations.

Theorem (P). *Suppose that f satisfies the three conditions. Then, for a general $\gamma \in G$, the morphism*

$$^{\gamma}f^{-1}(\mathbb{P}^n \setminus D) \longrightarrow (\mathbb{P}^n \setminus D) \times X$$

given by $x \mapsto ({}^{\gamma}f(x), x)$ induces a surjective homomorphism on the fundamental groups

$$\pi_1({}^{\gamma}f^{-1}(\mathbb{P}^n \setminus D)) \twoheadrightarrow \pi_1(\mathbb{P}^n \setminus D) \times \pi_1(X),$$

and its kernel is isomorphic to the cokernel of the homomorphism $\pi_2(X) \to \pi_2(\mathbb{P}^n)$ induced by f.

Since $\pi_2(\mathbb{P}^n)$ is an infinite cyclic group, the kernel is always a cyclic group.

When X is a projective plane and f is a linear embedding, this theorem is nothing but Zariski's hyperplane section theorem.

II. *Affine spaces*

Let U be an affine space $\mathbb{A}^n$ with $n \geq 2$, and let G be the group of all affine automorphisms of $\mathbb{A}^n$, which is a subgroup of $\mathrm{GL}(n+1)$.

Theorem (A). *Suppose that f satisfies the conditions* (1), (2) *and* (3) *above . Then, for a general $\gamma \in G$, the natural morphism ${}^{\gamma}f^{-1}(\mathbb{A}^n \setminus D) \to (\mathbb{A}^n \setminus D) \times X$ induces an isomorphism*

$$\pi_1({}^{\gamma}f^{-1}(\mathbb{A}^n \setminus D)) \cong \pi_1(\mathbb{A}^n \setminus D) \times \pi_1(X).$$

III. *Grassmannian varieties*

It is natural to expect that theorem of this type holds for other homogeneous varieties. However, even when we consider simple examples like Grassmannian varieties, we have to put some additional conditions on the morphism f.

Let U be the Grassmannian variety $\mathrm{Grass}(r, m)$ of all r-dimensional linear subspaces of an m-dimensional linear space V, where $2 \leq r \leq m-2$. On this variety, the general linear group $G = \mathrm{GL}(V)$ acts transitively with connected stabilizer subgroups. As before, let D be a non-zero reduced effective divisor of U.

Theorem (G). *Suppose that $f : X \to U$ satisfies the conditions* (2) *and* (3) *and moreover* $\dim f(X) \geq \max(r, m-r) + 1$. *Then, for a general $\gamma \in G$, we have an exact sequence*

$$1 \to \mathrm{Coker}(\pi_2(X) \to \pi_2(U)) \to \pi_1({}^{\gamma}f^{-1}(U \setminus D)) \to \pi_1(X) \times \pi_1(U \setminus D) \to 1.$$

There is an example such that $\dim f(X) = 2$ and the exact sequence does not holds.

Example. Let U be the Grassmannian variety Grass $(\mathbb{P}^1, \mathbb{P}^3)$ of all lines in a projective space $\mathbb{P}^3$. We choose a point $P \in \mathbb{P}^3$ and three lines l_1, l_2 and l_3 passing through P in such a way that there are no planes containing three of them. We take as D the reduced divisor of U whose support is given by

$$\{\ p \in U\ ;\ L(p) \cap (l_1 \cup l_2 \cup l_3) \neq \emptyset\},$$

where $L(p) \subset \mathbb{P}^3$ is the line corresponding to $p \in U$. Let $Q \in \mathbb{P}^3$ be another point, and $f : X \to U$ the inclusion of the nonsingular subvariety

$$X \ := \ \{\ p \in U\ ;\ Q \in L(p)\}$$

of U, which is isomorphic to a projective plane. The fundamental group $\pi_1(U \setminus D)$ is isomorphic to $\mathbb{Z}^2$. Indeed, let $H \subset \mathbb{P}^3$ be a plane such that $P \notin H$, and let P_i be the intersection point of l_i with H. Let $H^\vee$ be the dual projective plane of H, and $L_i \subset H^\vee$ the locus of all lines on H passing through P_i. The projection

$$\mathbb{P}^3 \setminus (l_1 \cup l_2 \cup l_3) \ \longrightarrow \ H \setminus \{P_1, P_2, P_3\}$$

with the center P induces a locally trivial morphism

$$U \setminus D \ \longrightarrow \ H^\vee \setminus (L_1 \cup L_2 \cup L_3),$$

every fiber of which is isomorphic to $\mathbb{A}^2$. Since l_1, l_2 and l_3 are not on any plane, the three lines L_1, L_2 and L_3 do not pass through a common point. Hence we have

$$\pi_1(U \setminus D) \ \cong \ \pi_1(H^\vee \setminus (L_1 \cup L_2 \cup L_3)) \ \cong \ \mathbb{Z}^2.$$

On the other hand, for a general $\gamma \in G = \mathrm{GL}\,(4)$, $\pi_1({}^\gamma f^{-1}(U \setminus D))$ is isomorphic to the free group F_2 generated by two elements. Indeed, let $H' \subset \mathbb{P}^3$ be a general plane. Then the projection

$$p_\gamma \ : \ \mathbb{P}^3 \setminus \{\gamma(Q)\} \ \longrightarrow \ H'$$

with the center $\gamma(Q) \in \mathbb{P}^3$ induces an isomorphism

$${}^\gamma f^{-1}(U \setminus D) \ \cong \ H' \setminus (p_\gamma(l_1) \cup p_\gamma(l_2) \cup p_\gamma(l_3)).$$

Since $p_\gamma(l_1)$, $p_\gamma(l_2)$ and $p_\gamma(l_3)$ are three lines on H' passing through the point $p_\gamma(P)$, we obtain

$$\pi_1(H' \setminus (p_\gamma(l_1) \cup p_\gamma(l_2) \cup p_\gamma(l_3))) \ \cong \ F_2.$$

It is obvious that F_2 cannot be an extension of $\mathbb{Z}^2$ by a cyclic group.

2. Corollaries

Theorem (A) has the following corollary. Let S be a non-singular connected surface equipped with a finite morphism $\bar{f} : S \to \mathbb{A}^2$ onto the affine plane. Let $B \subset \mathbb{A}^2$ be the branch locus of $\bar{f}$. Let D be a reduced curve on $\mathbb{A}^2$ and W a curve on S.

Corollary. *We denote by $E \subset \mathbb{A}^2$ the reduced divisor whose support is the union of the branched curve B and the image $\bar{f}(W)$ of W. Suppose that E intersects D at distinct* $\deg D \cdot \deg E$ *points. Then the fundamental group $\pi_1(S \setminus (W \cup \bar{f}^{-1}(D)))$ is isomorphic to $\pi_1(S \setminus W) \times \pi_1(\mathbb{A}^2 \setminus D)$.*

Indeed, the condition

$$\mathrm{Card}(E \cap D) = \deg D \cdot \deg E$$

means that E and D intersect transversely at their non-singular points, and that they do not have any intersection points at infinity. Hence, under this condition, the homeomorphism type of the space $S \setminus (W \cup \bar{f}^{-1}(D))$ does not change even when the morphism $\bar{f}$ is perturbed to ${}^{\gamma}\bar{f}$ by a general affine automorphism γ of the affine plane. Hence, applying Theorem (A) to the restriction $f : S \setminus W \to \mathbb{A}^2$ of $\bar{f}$ to $S \setminus W$, we obtain the corollary.

In particular, when $S = \mathbb{A}^2$ and $W = \emptyset$, we obtain the invariance theorem of the fundamental group of the complement to affine curve. Let $\bar{f} : \mathbb{A}^2 \to \mathbb{A}^2$ be a finite morphism.

Corollary. *Suppose that the branch locus B of $\bar{f}$ intersects D at distinct* $\deg D \cdot \deg B$ *points. Then $\pi_1(\mathbb{A}^2 \setminus \bar{f}^{-1}(D))$ is isomorphic to $\pi_1(\mathbb{A}^2 \setminus D)$.*

On the other hand, when $\bar{f}$ is the identity, this corollary gives Oka-Sakamoto's product theorem ([O-S]).

3. Sketch of the proof

The method of the proof is rather elementary. Most part of the proof consists of simple dimension counts. And we hope that the same method can be applied to other homogeneous varieties. The main ingredient of the proof is the following:

Theorem ([S1]). *Let F be a nonsingular connected projective variety. Let Z be a reduced effective divisor of the product space $\mathbb{A}^N \times F$ of an affine space with F. For a point $a \in \mathbb{A}^N$, let Z_a denote the scheme-theoretic intersection of Z with $\{a\} \times F$, which is regarded as a subscheme of F. Suppose that the locus Ξ of all $a \in \mathbb{A}^N$ such that Z_a is not a reduced divisor of F is of codimension ≥ 2 in $\mathbb{A}^N$. Then, for a general $a \in \mathbb{A}^N$, the inclusion $\{a\} \times F \hookrightarrow \mathbb{A}^N \times F$ induces an isomorphism $\pi_1(F \setminus Z_a) \cong \pi_1((\mathbb{A}^N \times F) \setminus Z)$.*

The proof of this theorem has been already published in [S1]. Roughly speaking, this theorem is shown by regarding the first projection $(\mathbb{A}^N \times F) \setminus Z \to \mathbb{A}^N$ from the complement to Z to the affine space $\mathbb{A}^N$ as something like a local trivial fiber space. Of course, there are some points $a' \in \mathbb{A}^N$ such that $Z_{a'}$ has worse singularity than that of the divisor Z_a over a general point a. Hence the projection $(\mathbb{A}^N \times F) \setminus Z \to \mathbb{A}^N$ is not locally trivial over $\mathbb{A}^N$. Nevertheless, the fundamental group of algebraic variety is not affected by changing a locus of codimension at least 2. Hence, under the condition that Ξ is of codimension ≥ 2 in $\mathbb{A}^N$, the first piece of the homotopy exact sequence

$$\pi_2(\mathbb{A}^N) = 0 \to \pi_1(F \setminus Z_a) \to \pi_1((\mathbb{A}^N \times F) \setminus Z) \to \pi_1(\mathbb{A}^N) = 0$$

still holds.

Now we will show how to derive the generalized Zariski's hyperplane section theorems from Theorem ([S1]) in the case of projective spaces and Grassmannian varieties

First note that we may assume that $W := \bar{X} \backslash W$ is purely of codimension one. Because removing the locus of codimension larger than one from the non-singular algebraic variety does not affect the topological fundamental group. Therefore we can ignore the irreducible component of W with codimension larger than one.

Let U be the Grassmannian variety Grass (r, V) of all r-dimensional linear subspaces in a linear space V, where $1 \leq r \leq \dim V - 2$. This setting covers both of the projective spaces and the Grassmannian varieties. The point of the proofs of Theorems (P) and (G) is to apply Theorem ([S1]) to the case $\mathbb{A}^N = \mathrm{End}\,(V)$. There is a rational map

$$\mathrm{End}\,(V) \times U \quad \cdots \rightarrow \quad U$$

extending the action of the general linear group $G = \mathrm{GL}(V)$ on U. The indeterminate locus of this rational map is of codimension at least 2, so that, as far as the fundamental groups are concerned, we can neglect it. Let

$$G \times \bar{X} \longrightarrow U \times \bar{X}$$

be the morphism given by $(\gamma, x) \mapsto ({}^{\gamma}\bar{f}(x), x)$. This morphism can also be extended to the rational map

$$\mathrm{End}\,(V) \times \bar{X} \quad \cdots \rightarrow \quad U \times \bar{X}.$$

The indeterminate locus of this rational map is also of codimension at least 2. Let $\bar{\mathcal{X}}$ be the Zariski open dense subset of $\mathrm{End}\,(V) \times \bar{X}$ on which the rational map is defined. The point is that the morphism

$$\psi \; : \; \bar{\mathcal{X}} \longrightarrow U \times \bar{X}$$

is locally trivial. The fiber is isomorphic to the space

$$\{\ \gamma \in \mathrm{End}\,(V) \ ; \ \gamma(L) = L\}$$

of all endomorphisms of V which maps a fixed r-dimensional linear subspace $L \in U$ onto L isomorphically. Let us denote this space by Γ_0. Then Γ_0 is isomorphic to $\mathrm{GL}\,(r) \times \mathbb{A}^{m(m-r)}$.

Now we consider the non-zero reduced divisor

$$E := D \times \bar{X} + U \times W$$

on $U \times \bar{X}$. We regard the boundary W as a reduced divisor of $\bar{X}$. Then we have a homotopy exact sequence

$$1 \longrightarrow \pi_2((U \times \bar{X}) \backslash E) \longrightarrow \pi_1(\Gamma_0) \longrightarrow \pi_1(\bar{\mathcal{X}} \backslash \psi^{-1}(E)) \longrightarrow \pi_1((U \times \bar{X}) \backslash E) \longrightarrow 1$$

associated with ψ. Since the complement of the divisor E is nothing but the product of $U \setminus D$ and $X = \bar{X} \setminus W$, we have

$$\pi_2((U \times D) \setminus E) \cong \pi_2(U \setminus D) \times \pi_2(X), \qquad \pi_1((U \times D) \setminus E) \cong \pi_1(U \setminus D) \times \pi_1(X).$$

On the other hand, let Z be the closure of the divisor $\psi^*(E)$ of $\overline{\mathcal{X}}$ in $\mathrm{End}\,(V) \times \bar{X}$; that is, Z is the divisor on $\mathrm{End}\,(V) \times \bar{X}$ whose support is the closure of the support of $\psi^*(E)$ and whose restriction to $\overline{\mathcal{X}}$ coincides with $\psi^*(E)$. Since the complement of $\overline{\mathcal{X}}$ in $\mathrm{End}\,(V) \times \bar{X}$ is of codimension ≥ 2, we have

$$\pi_1(\overline{\mathcal{X}} \setminus \psi^{-1}(E)) \cong \pi_1((\mathrm{End}\,(V) \times \bar{X}) \setminus Z).$$

Now we can prove the following:

Claim. There is a natural natural isomorphism between $\pi_1(\Gamma_0)$ and $\pi_2(U)$ such that the cokernel of the boundary homomorphism $\partial : \pi_2(U \setminus D) \times \pi_2(X) \to \pi_1(\Gamma_0)$ is identified with the cokernel of $f_* : \pi_2(X) \to \pi_2(U)$.

In the proof of this claim, we use the assumption that D is non-zero, so that the homomorphism $\pi_2(U \setminus D) \to \pi_2(U)$ induced by the inclusion is a zero map.

Now we have an exact sequence

$$1 \to \mathrm{Coker}\,(\pi_2(X) \to \pi_2(U)) \to \pi_1((\mathrm{End}\,(V) \times \bar{X}) \setminus Z) \to \pi_1(U \setminus D) \times \pi_1(X) \to 1.$$

As before, for an element γ of $\mathrm{End}\,(V)$, let Z_γ denote the scheme-theoretic intersection of Z with $\{\gamma\} \times \bar{X}$, and we consider it as a sub-scheme of $\bar{X}$. Then, by the definition of Z, if $\gamma \in \mathrm{GL}\,(V)$, the subscheme Z_γ coincides with $W + {}^\gamma\bar{f}^*(D)$, and hence its complement coincides with ${}^\gamma f^{-1}(U \setminus D)$;

$$X \setminus Z_\gamma = {}^\gamma f^{-1}(U \setminus D).$$

Hence, by Theorem([S1]), we have

Claim. If the locus

$$\Xi := \{\ \gamma \in \mathrm{End}\,(V)\ ;\ Z_\gamma \text{ is not a reduced divisor of } \bar{X}\}$$

is of codimension ≥ 2 in the affine space $\mathrm{End}\,(V)$, then $\pi_1({}^\gamma f^{-1}(U \setminus D))$ is isomorphic to $\pi_1((\mathrm{End}\,(V) \times \bar{X}) \setminus Z)$ for a general $\gamma \in \mathrm{End}\,(V)$.

Therefore the proof of theorems has been reduced to the estimation of the dimension of Ξ. It is rather technical, but we can prove the following:

Claim. Suppose that γ is a general element of the irreducible hypersurface $\Delta := \mathrm{End}\,(V) \setminus \mathrm{GL}\,(V)$. Then Z_γ is a reduced divisor of $\bar{X}$; that is, $\Xi \cap \Delta$ is a proper Zariski closed subset.

Claim. Suppose that f satisfies the conditions in the theorems. Then $\Xi \cap \mathrm{GL}\,(V)$ is of codimension ≥ 2 in $\mathrm{GL}\,(V)$.

These two claims show that Ξ is of codimension ≥ 2 in $\mathrm{End}\,(V)$. Thus Theorems (P) and (G) are proved.

For more details, please refer to the preprint [S2].

References

[H-L] H. A. Hamm and D. T. Lê, Un théorème de Zariski du type de Lefschetz, Ann. Sci. École Norm. Sup. **6** (1973), 317 - 366.

[O-S] M. Oka and K. Sakamoto, Product theorem of the fundamental group of a reducible curve, J. Math. Soc. Japan **30** (1978), 599 - 602.

[S1] I. Shimada, Fundamental groups of open algebraic varieties, Topology **34** (1995), 509 - 532.

[S2] I. Shimada, Zariski's hyperplane section theorem for morphisms to homogeneous varieties, preprint.

[Z] O. Zariski, A theorem on the Poincaré group of an algebraic hypersurface, Ann. Math. **38** (1937), 131 - 141.

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