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Chern number formula for ramified coverings

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Abstract

For a ramified covering $f: Y \longrightarrow X$ between compact complex manifolds, we establish a formula relating the Chern numbers of Y and X. We obtain the formula by localizing characteristic classes via the Čech-de Rham cohomology theory. As corollaries,we deduce generalizations of the Riemann-Hurwitz formula and a formula of Hirzebruch for the signature, as well as formulas for other invariants such as the Todd genus.

1 Introduction

Let $f: Y \longrightarrow X$ be a ramified covering between n-dimensional compact complex manifolds with covering multiplicity μ. Let $R_f = \sum_i r_i R_i$ be the ramification divisor of f, and $B_f = \sum_i b_i B_i$ the branch locus of f. We assume that the ramification divisor and the irreducible component of the branch locus are all non-singular. Our main result is

$$
\begin{aligned}
&c_1^{N_1} \cdots c_n^{N_n}(Y) - \mu \cdot c_1^{N_1} \cdots c_n^{N_n}(X) \\
&= \sum_i \left(H_{TR_i}^{(N_1 \cdots N_n)}(c_1(L_{R_i})) \frown [R_i] - \frac{b_i(r_i+1)}{r_i} H_{TB_i}^{(N_1 \cdots N_n)}(c_1(L_{B_i})) \frown [B_i] \right) \\
&= \sum_i \sum_{\alpha=0}^{n-1} \frac{b_i(1-(r_i+1)^{\alpha+1})}{r_i(r_i+1)^\alpha} P_\alpha(c_1(B_i) \cdots c_{n-1}(B_i)) \cdot c_1(L_B)^\alpha \frown [B_i].
\end{aligned}
$$

In the above, $\sum_{i=1}^n iN_i = n$ and we set formally

$$
H_\xi^{(N_1 \cdots N_n)}(l) = l^{-1} \cdot \left(\left(\prod_{i=1}^n (c_i(\xi) + c_{i-1}(\xi) \cdot l)^{N_i} \right) - c_1^{N_1} \cdots c_n^{N_n}(\xi) \right) = \sum_{\alpha=0}^{n-1} P_\alpha(c_1 \cdots c_{n-1}) l^\alpha,
$$

where P_α is the coefficient of l^α of $H(l)$ as a polynomial in l.

We prove the formula for Chern numbers by applying the framework of the localization of characteristic classes based on the Čech de-Rham cohomology theory. ([L1], [L2],

[LS].) Our methods of proof are very elementary and computational. Classically, all sorts of topological invariants can be calculated as the integral value of differential forms through the de Rham theorem, which gives the representation of cohomology classes and describe the explicit corespondence in the Poincaré duality. The Čech-de Rham cohomology theorey plays the same role for relative cohomology groups as the Alexander duality. So applying this analogy, we can localize Chern classes at the ramification set, which gives us more specific geometric information about what is caused by degenerancy of holomorphic maps.

2 Preliminaries

2.1 Čech-de Rham cohomology theory

First we will give a brief sketch of the Čech-de Rham cohomology theory. (see [BT], [L1], [L2], [S].)

1 Definition.

Let X be an n-dimensional C^∞-manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ an open covering of X, whose index set I is a countable ordered set such that $(\alpha_0, \dots, \alpha_p) \in I^{p+1}$ is totally ordered if $U_{\alpha_0} \cap \dots \cap U_{\alpha_p} \neq \phi$. Let us consider the de Rham complex of sheaves of germs of smooth forms on X

$$0 \longrightarrow \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \xrightarrow{d} \mathcal{A}^3 \longrightarrow \dots .$$

Now let $C^p(\mathcal{U}, \mathcal{A}^q)$ be the group of Čech cochains of degree p with values in $\mathcal{A}^q$. The commutativity of the two operators, the Čech coboundary operator δ and the exterior derivative d, gives rise to a double complex $\{C^{p\ q} = C^p(\mathcal{U}, \mathcal{A}^q); \delta, d\}$. The associated single complex $(A^\bullet(\mathcal{U}), D)$ is defined by

$$A^r(\mathcal{U}) = \bigoplus_{p+q=r} C^p(\mathcal{U}, \mathcal{A}^q)$$

$$D = \delta + (-1)^p d.$$

We call the cohomology groups $H^r(A^\bullet(\mathcal{U}))$ of this associate single complex, the Čech-de Rham cohomology groups of X. This cohomology is canonically isomorphic to the classical de Rham cohomology ([BT]).

2 Product structure.

We also define a product structure $A^r(\mathcal{U}) \times A^s(\mathcal{U}) \longrightarrow A^{r+s}(\mathcal{U})$ as

$$(\sigma \smile \tau)_{\alpha_0\cdots\alpha_p} = \sum_{\nu=0}^{p} (-1)^{(r-\nu)(p-\nu)} \sigma_{\alpha_0\cdots\alpha_\nu} \wedge \tau_{\alpha_\nu\cdots\alpha_p}.$$

Then it induces the cup product structure for the cohomology of the Čech de-Rham complex, which is, via the above isomorphism, compatible with the usuall product in de Rham cohomology.

3 Integration.

Next we define the integration on the Čech-de Rham cohomology group which is compatible with the usual integration on the de Rham cohomology group ([L1]). Suppose now that the manifold X is oriented. Before making our definition, we introduce the following concept.

Definition. Let $\mathcal{U}$ and X be as above. A family $\{R_\alpha\}_{\alpha\in I}$ of n-dimensional manifolds R_α with piecewise smooth boundary in X is called *a system of honey-comb cells* adapted to $\mathcal{U}$ if:

(1) $R_\alpha \subset U_\alpha$, $X = \bigcup_\alpha R_\alpha$.

(2) $Int\,(R_\alpha) \cap Int\,(R_\beta) = \phi$ if $\alpha \neq \beta$.

(3) $R_{\alpha_0\cdots\alpha_p} = \bigcap_{\nu=0}^{p} R_{\alpha_\nu}$ is an $(n-p)$-dimensional manifold with piecewise smooth boundary for any $(\alpha_0\cdots\alpha_p) \in I^{p+1}$.

(4) If $(\alpha_0\cdots\alpha_p)$ is maximal, $R_{\alpha_0\cdots\alpha_p}$ has no boundary.

We also give $R_{\alpha_0\cdots\alpha_p}$ an orientation by the following rules.

(1) Each R_α has the same orientation as X.

(2) $R_{\alpha_0(0)\cdots\alpha_p(p)} = sgn\,(\rho)\cdot R_{\alpha_0\cdots\alpha_p}$ for a permutation ρ.

(3) $\partial R_{\alpha_0\cdots\alpha_p} = \sum_\alpha R_{\alpha_0\cdots\alpha_p\alpha}$.

Now suppose that X is compact, and $\{R_\alpha\}_{\alpha\in I}$ a system of honey-comb cells adapted to $\mathcal{U}$. We define the integration on $A^n(\mathcal{U})$ as:

$$\int_X : A^n\,(A^\bullet\,(\mathcal{U})) \longrightarrow \mathbf{C},$$

$$\int_X \sigma = \sum_{p=0}^{n}\left(\sum_{\alpha_0\cdots\alpha_p\in I^{p+1}} \int_{R_{\alpha_0\cdots\alpha_p}} \sigma_{\alpha_0\cdots\alpha_p}\right), \qquad \sigma \in A^n\,(\mathcal{U})\,.$$

Then we see, from the fact that this integration is independent of the choice of the system of honey-comb cells for D-cocycles and it vanishes for D-coboundaries, that it induces the integration on the cohomology group

$$\int_X : H^n\,(A^\bullet\,(\mathcal{U})) \longrightarrow \mathbf{C},$$

which is compatible with the usual integration on the de Rham cohomology.

4 Alexander-Lefschetz duality.

Finally, we describe the Alexander duality in terms of the Čech-de Rham cohomology ([L1] [L2] [S]). We suppose that X is the same as above, and let $S \subset X$ be a compact subset of X which admits a regular neighborhood, $U_0 = X - S$, and U_1 a reguler neighborhood of S. Now we set $\mathcal{U} = \{U_0, U_1\}$ and consider the Čech-de Rham cohomology of X associated with the covering $\mathcal{U}$. We set $A^r\,(\mathcal{U}, U_0) = \ker\,(A^r\,(\mathcal{U}) \to A^r\,(U_0)) = \{\ (\sigma_0, \sigma_1, \sigma_{01})\,|\ \sigma_0 = 0\ \}$ so that we have the exact sequence

$$0 \longrightarrow A^r\,(\mathcal{U}, U_0) \longrightarrow A^r\,(\mathcal{U}) \longrightarrow A^r\,(U_0) \longrightarrow 0.$$

Then we conclude $H^r(A^\bullet(\mathcal{U},U_0)) \cong H^r(X,X-S;\mathbf{C})$ from the de Rham theorem and the five lemma.

Let $\{R_0,R_1\}$ be a system of honey-comb celles adapted to $\mathcal{U}$. Then we still have the integration

$$\int_X : A^n(\mathcal{U},U_0) \longrightarrow \mathbf{C},$$

given by

$$\int_X \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01} ,$$

for $\sigma = (0,\sigma_1,\sigma_{01}) \in A^n(\mathcal{U},U_0)$. This again induces the integration on the relative cohomology

$$\int_X : H^n(A^\bullet(\mathcal{U},U_0)) \longrightarrow \mathbf{C}.$$

The cup product induces the pairing $A^r(\mathcal{U},U_0)\times A^{n-r}(U_1)\to A^n(\mathcal{U},U_0)$, which followed by the integration, gives a bilinier pairing

$$A^r(\mathcal{U},U_0)\times A^{n-r}(U_1) \longrightarrow \mathbf{C} ,$$

which induces the Alexander duality

$$H^r(X,X-S;\mathbf{C}) \cong H^k(A^\bullet(\mathcal{U},U_0)) \cong H^{n-r}(U_1;\mathbf{C})^* \cong H_{n-k}(S;\mathbf{C}).$$

5 Chern-Weil theory for Čech-de Rham cohomology.

We recall some fundamental results of the Chern-Weil theory, the differential geometric treatment of characteristic classes (see [GH]).

Let X be an n-dimensional C^∞-manifold and $\pi: E \longrightarrow X$ a C^∞-complex vector bundle of rank r over X. Then the i-th Chern class $c_i(E)$ in $H^{2i}_{DR}(X:\mathbf{C})$ is represented by

$$c_i(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^i P^i(\Theta),$$

where we denote by P^i the i-th elementary symmetric polynomial, and Θ the curvature matrix of a connection ∇ on E in terms of some frame for E. Then there is the following well-known result for invariant polynomials determined by connection forms ([B]).

Suppose that $\pi: E \longrightarrow X$ is a C^∞-complex vector bundle of rank r over X, and $\nabla_0,\cdots,\nabla_p$, connections on E. Then there exists $P^i(\nabla_0\cdots\nabla_p) \in A^{2(n-i)-p}(X)$ such that

$$dP^i(\nabla_0\cdots\nabla_p) = \sum_{j=1}^{p}(-1)^{j-p-1}P^i\left(\nabla_0\cdots\check{\nabla}_j\cdots\nabla_p\right).$$

The immediate construction of the above secondary term is given as follows. Let us consider the trivial extension $E\times\mathbf{R}^p \longrightarrow X\times\mathbf{R}^p$ of the vector bundle E over $X\times\mathbf{R}^p$, and $\tilde{\pi}: X\times\mathbf{R}^p \longrightarrow X$ the canonical projection. We take $\tilde{\nabla} = (1-t_1-\cdots-t_p)\nabla_0 + t_1\nabla_1+\cdots+t_p\nabla_p$ as a connection on $E\times\mathbf{R}^p$ and we set

$$P^i(\nabla_0\cdots\nabla_p) = \tilde{\pi}_*\left(P^i\left(\tilde{\nabla}\right)\right),$$

then it has the desired property. Here "$\tilde{\pi}_*$" means the integration along the fibers. By applying the above result for invariant polynomials determind by connection forms, we can express the i-th Chern class $c_i(E)$ in $H^{2i}(A^\bullet(\mathcal{U}))$ as follows. ([L1], [L2], [LS].) Let ∇_α be a connection on $E|_{U_\alpha}$ over U_α,

$$H^{2i}_{DR}(X;\mathbf{R}) \cong H^{2i}(A^\bullet(\mathcal{U}))$$
$$c_i(E) \longleftrightarrow [((c_i(\nabla_\alpha)_\alpha, ((c_i(\nabla_{\bar\alpha})_{\bar\alpha\in I^p}))_p)] \,.$$

In particular, for the case where the covering is given by $\mathcal{U} = \{U_0,\ U_1\}$, the Čech-de Rham cocycle $(c_i(\nabla_0),\ c_i(\nabla_1),\ c_i(\nabla_0,\nabla_1))$ represents the i-th Chern class of E.

6 Correspondence between fundamental classes and cohomology classes of divisors.

Let X be an n-dimensional compact complex manifold, and D a divisor on X, with local defining functions $\{f_\alpha\}$ over some open covering $\{U_\alpha\}$ of X. Then, $D = \{f_\alpha, U_\alpha\}$ defines naturally a complex line bundle L_D which has the system of transition functions $\{g_{\alpha\beta} = f_\alpha/f_\beta\}$. We know that, in the Poincaré duality, the Chern class $c_1(L_D)$ represents the dual of the fundamental class of the divisor D,

$$H^2_{DR}(X;\mathbf{C}) \cong H_{2n-2}(X;\mathbf{C})\,,$$
$$c_1(L_D) \longleftrightarrow [D]\,,$$
$$\left(\int_X c_1(L_D)\wedge\varphi = \int_D \varphi, \qquad {}^\forall\varphi \in Z^{2n-2}(X)\right).$$

Here, we find a more specific correspondence between the fundamental homology class and the Chern class of D in the Alexander duality, by localizing the Chern class in terms of the Čech de Rham cohomology theory. For simplification, here we assume that the divisor D is non-singular. (Indeed the following discussion can be applied to the general case. (Originally due to [S].)

Let X be an n-dimensional complex manifold, D a compact non-singular divisor on X, and $L_D \longrightarrow X$ the associated line bundle of D. If D is given by local defining functions $\{f_\alpha\}$, then those functions clearly give a section $f_D = (f_\alpha, U_\alpha)$ of L_D, whose zero locus coincides with D itself. We set $U_0 = X - D$, $\pi\colon U_1 \to D$ a sufficiently small tubular neighborhood, R_1 a closed disk bundle over D which is contained in U_1, and R_0 the complement of the interior of R_1.

We consider the covering $\mathcal{U} = \{U_0, U_1\}$ with a system of honey-comb cells $\{R_0, R_1\}$ adapted to $\mathcal{U}$. Then as is discussed in the previous sections, the class

$$c_1(L_D) = (\ c_1(\nabla_0), c_1(\nabla_1), c_1(\nabla_0,\nabla_1)\ )$$

in the Čech-de Rham cohomology can be localized at D, by taking an f_D-trivial connection ∇_{f_D} as the connection ∇_0 on U_0 so that $c_1(\nabla_{f_D}) = 0$.

Now let us consider the pairng

$$A^2(\mathcal{U}, U_0) \times A^{2n-2}(U_1) \longrightarrow \mathbf{C},$$

and compute

$$\int_X c_1(L_D) \smile \tau_1 = \int_{R_1} c_1(\nabla_1) \wedge \tau_1 + \int_{R_{01}} c_1(\nabla_0, \nabla_1) \wedge \tau_1$$

for $\tau_1 \in A^{2n-2}(U_1)$. We note that the elements of $A^2(\mathcal{U}, U_0)$ are expressed as cocycles whose component on U_0 vanishes.

Since $\pi: U_1 \to D$ is a deformation retract, U_1 and D have the same homotopy type. So we have $H^{2n-2}(U_1) \cong H^{2n-2}(D)$, which implies

$$\tau_1 = \pi^*\theta + d\rho,$$

for some $\theta \in A^{2n-2}(D)$, and $\rho \in A^{2n-3}(U_1)$. Using the Stokes' theorem and $\partial R_1 = -R_{01}$, we compute

$$\int_{R_1} c_1(\nabla_1) \wedge \tau_1 = \int_{R_1} c_1(\nabla_1) \wedge \pi^*\theta + \int_{R_1} c_1(\nabla_1) \wedge d\rho = \int_{R_1} c_1(\nabla_1) \wedge \pi^*\theta - \int_{R_{01}} c_1(\nabla_1) \wedge \rho,$$

$$\begin{aligned} \int_{R_{01}} c_1(\nabla_0, \nabla_1) \wedge \tau_1 &= \int_{R_{01}} c_1(\nabla_0, \nabla_1) \wedge \pi^*\theta + \int_{R_{01}} c_1(\nabla_0, \nabla_1) \wedge d\rho \\ &= \int_{R_{01}} c_1(\nabla_0, \nabla_1) \wedge \pi^*\theta + \int_{R_{01}} c_1(\nabla_1) \wedge \rho + \int_{\partial R_{01}} c_1(\nabla_0, \nabla_1) \wedge \rho. \end{aligned}$$

Hence we have

$$\int_{R_1} c_1(\nabla_1) \wedge \tau_1 + \int_{R_{01}} c_1(\nabla_0, \nabla_1) \wedge \tau_1 = \int_{R_1} c_1(\nabla_1) \wedge \pi^*\theta + \int_{R_{01}} c_1(\nabla_0, \nabla_1) \wedge \pi^*\theta.$$

Let ∇_{N_D} be a connection on the normal bundle N_D of D. Since $L_D|_D \cong N_D$, and also $L_D|_{U_1} \cong \pi^* N_D$, we can take $\pi^*\nabla_{N_D}$ as the connection ∇_1 on $L_D|_{U_1}$ so that we have

$$\int_{R_1} c_1(\nabla_1) \wedge \pi^*\theta = \int_{R_1} c_1(\pi^*\nabla_{N_D}) \wedge \pi^*\theta = \int_{R_1} \pi^*(c_1(\nabla_{N_D}) \wedge \theta) = \int_D c_1(\nabla_{N_D}) \wedge \theta = 0,$$

because the last term is the integration of a $2n$-form on a $(2n-2)$-dimensional submanifold.

Next, we compute the boundary integral $\int_{R_{01}} c_1(\nabla_0, \nabla_1) \wedge \pi^*\theta$. Since the question is purely local, for any fixed point $p \in D$, and $V_p \subset D$ a neighborhood of p, we set $U_p = \pi^{-1}(V_p)$, and take a local coordinate system (U_p, z) around p sufficiently small so that we may assume that $D = \{z_1 = 0\}$ on U_p, $U_p \subset U_1$, and $N_D|_{V_p}$ has a non-vanishing section s_N. Then $\pi^* s_N$ gives a section on U for L_D. If we give a trivialization of L_D by $\pi^* s_N$, then on $U_p - D$

$$f_D = z_1 = z_1 \cdot \pi^* s_N,$$

and therefore the connection form θ_{f_D} of ∇_{f_D} with respect to the frame $\pi^* s_N$ has the form $df_D/f_D = dz_1/z_1$ of the Cauchy kernel on U. To compute the secondary term $c_1(\nabla_0, \nabla_1)$, let $\tilde{\theta} = (1-t)\theta_{f_D} + t\theta_1$

$$c_1(\nabla_0, \nabla_1) = \tilde{\pi}_*(d\tilde{\theta} - \tilde{\theta} \wedge \tilde{\theta}) = \theta_f - \theta_1.$$

Now by the Cauchy integral formula, we have

$$\int_{R_{01} \cap U_p} c_1(\nabla_0, \nabla_1) \wedge \pi^* \theta = \int_{D \cap U_p} \theta = \int_{D \cap U} \tau_1,$$

which implies

$$\int_{R_{01}} c_1(\nabla_0, \nabla_1) \wedge \pi^* \theta = \int_D \theta = \int_D \tau_1.$$

To organize the results of the above calculation, we obtain the correspondence

$$H^2(X, X-D; \mathbf{C}) \cong H^2(A^\bullet(\mathcal{U}, U_0)) \cong H_{2n-2}(D; \mathbf{C}).$$

$$(\ 0, c_1(\nabla_1), c_1(\nabla_0, \nabla_1)\ ) \longleftrightarrow [D]$$

We remark that the above correspondence is more precise than that of the Poincaré duality. (1): We do not need the compactness of the ambient space X. (2): The dual of the Chern class is found in $H_\bullet(D)$, which indicates explicitly the location of singularities.

3 Proof of the main theorem

In this section, we give the proof of the main theorem. Let X and Y be n-dimensional compact complex manifolds, and $f: Y \longrightarrow X$ a ramified covering with covering multiplicity μ. If f gives a simple (unramified) μ-sheeted covering, then we see that $c_*(Y) - \mu c_*(X) = 0$, which suggests us that the gap is brought about the ramification. So we expect that the difference of the Chern classes can be localized at the ramification set.

We recall some basic facts about ramified coverings.

The *ramification divisor* R_f of f is defined as the analytic hypersurface defined by $\{\det(df) = 0\}$. Let $R_f = \sum r_i R_i$ be the irreducible decomposition of R_f. Then we have

$$r_i + 1 = [\mathcal{O}_{Y,y} : f^* \mathcal{O}_{X,f(y)}] \ ,$$

the degree of integral extention $\mathcal{O}_{Y,y}$ over $f^*\mathcal{O}_{X,f(y)}$ for a generic point y on R_i. In other words, r_i indicates the number of deceasing of sheets at R_i.

The *branch locus* B_f of f is defined by the direct image $f^* R_f$ of R_f under f. Let $B_f = \sum b_i B_i$ be the irreducible decomposition of B_f. Then we have

$$b_i = \mu - \# f^{-1}(x)$$

for a generic point x on B_i.

Now we assume that the ramification divisor of f and the irreducible components of the branch locus of f are all non-singular. Here we remark that the branch locus possibly has some self-intersection between other components. It followes from the assamption that the ramificcation divisor of f is non-singular, that $f|_{R_i}: R_i \to B_i$ is non-degenerate so that it gives the un-ramified covering over B_i, with covering multiplicity r_i/b_i.

First let us consider the case where the ramification divisor R_f has only one component, hence the branch locus B_f also does. We set $R_f = r \cdot R$, and $B_f = b \cdot B$.

Let $\omega: V_1 \longrightarrow B$ be a tubular neighborhood of B, and we take a covering $\mathcal{U} = \{U_0, U_1\}$ of Y with, $U_0 = Y - R$, and $\pi: U_1 \longrightarrow R$, a tubular neighborhood of R such that $U_1 \subset f^{-1}(V_1)$. We consider the Čech-de Rham cohomology of Y associate with the covering of $\mathcal{U}$, and set, in $H^{2i}_{DR}(Y) \cong H^{2i}(A^\bullet(\mathcal{U}))$, that

$$c_i(TY) \longleftrightarrow (\ c_i(\nabla_0), c_i(\nabla_1), c_i(\nabla_0, \nabla_1)\ ),$$
$$c_i(f^*TX) \longleftrightarrow (\ c_i(\tilde{\nabla}_0), c_i(\tilde{\nabla}_1), c_i(\tilde{\nabla}_0, \tilde{\nabla}_1)\ ).$$

Since $df: TY \longrightarrow TX$ gives a bundle homomorphism outside the ramification, and since U_1 and V_1 are tubular neighborhoods of R and B respectively, we have

$$TY|_{Y-R} \cong f^*TX|_{Y-R},$$
$$TY|_{U_1} \cong \pi^*N_R \oplus \pi^*TR \cong L_R|_{U_1} \oplus \pi^*TR,$$
$$f^*TX|_{U_1} \cong f^*(\omega^*N_B \oplus \omega^*TB) \cong f^*(L_B \oplus \omega^*TB).$$

In particular on $U_1 - R$, $L_R \cong f^*L_B$ are isomorphic as trivial bundles. Thus we can take connections on each neighborhood as follows:

$$\nabla_0 = \tilde{\nabla}_0$$

such that

$$\tilde{\nabla}_0|_{V_1 - B} = \nabla_{f^*f_B} \oplus f^*\omega^*\nabla_{TB},$$

$$\begin{aligned}\nabla_0|_{U_1 - R} &= \nabla_{f^*f_B} \oplus \pi^*\nabla_{TR} \\ &= \nabla_{f_R} \oplus \pi^*\nabla_{TR},\end{aligned}$$

and

$$\tilde{\nabla}_1 = f^*\nabla_{L_B} \oplus f^*\omega^*\nabla_{TB},$$
$$\nabla_1 = \nabla_{L_R} \oplus \pi^*\nabla_{TR}.$$

In the above, for a non-singular divisor D we denote by f_D, ∇_{f_D} and ∇_{TD}, the section of D, the f_D-trivial connection, and a connection of the tangent bundle of D respectively.

Next we do local computation for secondary terms. Notation and choice of local neighborhood and frames are the same as in 3.1.

$$\begin{aligned}\tilde{A} &= (1-t)\begin{pmatrix} \pi^*\theta_{TR} & 0 \\ 0 & \theta_{f_R} \end{pmatrix} + t\begin{pmatrix} \pi^*\theta_{TR} & 0 \\ 0 & \theta_1 \end{pmatrix} \\ &= \begin{pmatrix} \pi^*\theta_{TR} & 0 \\ 0 & (1-t)\theta_{f_R} + t\theta_1 \end{pmatrix} \\ &= \begin{pmatrix} \pi^*\theta_{TR} & 0 \\ 0 & \tilde{\theta} \end{pmatrix}.\end{aligned}$$

Thus

$$\begin{aligned} P^i(d\tilde{A} - \tilde{A} \wedge \tilde{A}) &= P^i \begin{pmatrix} d\pi^*\theta_{TR} - \pi^*\theta_{TR} \wedge \pi^*\theta_{TR} & 0 \\ 0 & d\tilde{\theta} - \tilde{\theta} \wedge \tilde{\theta} \end{pmatrix} \\ &= (d\tilde{\theta} - \tilde{\theta} \wedge \tilde{\theta}) \wedge P^{i-1}(d\pi^*\theta_{TR} - \pi^*\theta_{TR} \wedge \pi^*\theta_{TR}) \\ &\qquad + P^i(d\pi^*\theta_{TR} - \pi^*\theta_{TR} \wedge \pi^*\theta_{TR}). \end{aligned}$$

Since only $\tilde{\theta}$ involves the fiber coordinate t, it follows from the projection formula that

$$\begin{aligned} c_i(\nabla_0, \nabla_1) &= \tilde{\pi}_* P^i(d\tilde{A} - \tilde{A} \wedge \tilde{A}) \\ &= \tilde{\pi}_*\{(d\tilde{\theta} - \tilde{\theta} \wedge \tilde{\theta}) \wedge P^{i-1}(\pi^*(d\theta_R - \theta_R \wedge \theta_R))\} \\ &= c_1(\nabla_{f_R}, \nabla_{L_R}) \wedge \pi^* c_{i-1}(R). \end{aligned}$$

To express the secondary term of $c_1^{N_1} \cdots c_n^{N_n}(Y) \in H^{2n}(A^\bullet(\mathcal{U}))$, we set

$$H_\xi^{(N_1 \cdots N_N)}(l) = l^{-1}\left(\prod_{i=1}^n (c_i(\xi) + c_{i-1}(\xi) \cdot l)^{N_i} - c_1^{N_1} \cdots c_n^{N_n}(\xi)\right) = \sum_{\alpha=0}^{n-1} P_\alpha(c_1 \cdots c_{n-1}) l^\alpha.$$

Then the Čech-de Rham class of $c_1^{N_1} \cdots c_n^{N_n}(Y)$ is represented by,

$$\left(\pi^*(c_1^{N_1} \cdots c_n^{N_n})(R),\ \prod_{i=1}^n (\pi^* c_i(R) + \pi^* c_{i-1}(R) c_1(L_R))^{N_i},\ c_1(\nabla_{f_R}, \nabla_{L_D}) \wedge H_{TR}^{(N_1 \cdots N_n)}(c_1(L_R))\right).$$

This can be proved by induction for the number of indeterminate c_i as follows. Here we remark that the degree of the class is not necessarily equal to n, the dimension of the ambient spaces. It follows from the inductive hypothesis that

$$c_1^{N_1} \cdots c_k^{N_k}(Y) =$$

$$\left(\pi^*(c_1^{N_1} \cdots c_k^{N_k})(R), \prod_{i=1}^k (\pi^* c_i(R) + \pi^* c_{i-1}(R) c_1(L_R))^{N_i}, c_1(\nabla_{f_R}, \nabla_{L_R}) \wedge H_{TR}^{(N_1 \cdots N_k)}(c_1(L_R))\right).$$

$$c_{k+1}^{N_{k+1}}(Y) =$$

$$\left(c_{k+1}^{N_{k+1}}(R),\ (\pi^* c_{k+1}(R) + \pi^* c_k(R) c_1(L_R))^{N_{k+1}},\ c_1(\nabla_{f_R}, \nabla_{L_R}) \wedge H_{TR}^{(N_{k+1})}(c_1(L_R))\right).$$

Thus, the secondary term of $c_1^{N_1} \cdots c_{k+1}^{N_{k+1}}(Y)$ is

$$c_1^{N_1} \cdots c_{k+1}^{N_{k+1}}(\nabla_0, \nabla_1)$$

$$= c_1^{N_1} \cdots c_k^{N_k}(\nabla_0) \wedge c_{k+1}^{N_{k+1}}(\nabla_0, \nabla_1) + c_1^{N_1} \cdots c_k^{N_k}(\nabla_0, \nabla_1) \wedge c_{k+1}^{N_{k+1}}(\nabla_1) \quad .$$

$$= c_1(\nabla_{f_r}, \nabla_{L_R}) \wedge c_1(L_R)^{-1} \left(\prod_{i=1}^{k+1} (\pi^* c_i(R) + \pi^* c_{i-1}(R) c_1(L_R))^{N_i} - c_1^{N_1} \cdots c_{k+1}^{N_{k+1}}(R) \right)$$

$$= c_1(\nabla_{f_R}, \nabla_{L_D}) \wedge H_{TX}^{(N_1 \cdots N_{k+1})}(c_1(L_R)),$$

which completes the induction.

In particular for the case when $n = \sum_{i=1}^n i \cdot N_i$, from our assumption that the ramification divisor has degree r we have $f^* L_B = (L_R)^{\otimes r+1}$, thus $f^* c_1(L_B) = (r+1) \cdot c_1(L_R)$. Since $f|_R : R \longrightarrow B$ is non-degenerate, it follows from $TR \cong f^* TB$ that $c_i(R) = f^* c_i(B)$. Therefore we have

$$\begin{aligned} H_{TR}^{(N_1 \cdots N_n)}(c_1(L_R)) \frown [R] &= H_{f^*TB}^{(N_1 \cdots N_n)}((r+1)^{-1} \cdot c_1(f^*(L_B))) \frown [(b/r) \cdot B] \\ &= \sum_{\alpha=0}^{n-1} \frac{b}{r(r+1)^\alpha} P_\alpha(c_1 \cdots c_{n-1}) \cdot c_1(L_B)^\alpha \frown [B]. \end{aligned}$$

By calculating the Čech-dė Rham class of $c_1^{N_1} \cdots c_n^{N_n}(f^*TX)$ all the same, we obtain

$$c_1^{N_1} \cdots c_n^{N_n}(TY) - c_1^{N_1} \cdots c_n^{N_n}(f^*TX) =$$
$$\Big(0,\ (***),\ c_1(\nabla_{f_R}, \nabla_{L_R}) \wedge (H_{TR}^{(N_1 \cdots N_n)}(c_1(L_R)) - (r+1) H_{f^*TB}^{(N_1 \cdots N_n)}(c_1(f^* L_B)))\Big).$$

(We omit the component on U_1 since it vanishes by evaluating on R because of overdegree, which gives integration of $2n$-forms on hypersurface, as observed in 3.1.)
Now, as discused in 3.1, it follows from the correspondence of the Alexander duality that

$$c_1^{N_1} \cdots c_n^{N_n}(TY) \frown [Y] - \mu \cdot c_1^{N_1} \cdots c_n^{N_n}(TX) \frown [X]$$

$$\begin{aligned} &= \int_R c_1(\nabla_{f_R}, \nabla_{L_R}) \wedge (H_{TR}^{(N_1 \cdots N_n)}(c_1(L_R)) - (r+1) H_{f^*TB}^{(N_1 \cdots N_n)}(c_1(f^* L_B))) \\ &= H_{TR}^{(N_1 \cdots N_{k+1})}(c_1(L_R)) \frown [R] - (r+1) H_{TB}^{(N_1 \cdots N_{k+1})}(c_1(L_B)) \frown [(b/r) \cdot B] \\ &= \sum_{\alpha=0}^{n-1} \frac{b(1 - (r+1)^{\alpha+1})}{r(r+1)^\alpha} P_\alpha(c_1(B) \cdots c_{n-1}(B)) \cdot c_1(L_B)^\alpha \frown [B]. \end{aligned}$$

We assumed that the ramification divisor of f is non-singular, so we can assume that the tubular neighborhoods of irreducible components of the divisor do not intersect each other. Hence taking independent sum we conclude:

Theorem [Chern number formula for ramified coverings]
Let $f: Y \longrightarrow X$ be a ramified covering with covering multiplicity μ between compact complex manifolds of dimension n, $R_f = \sum_i r_i R_i$ the ramification divisor of f, and $B_f = \sum_i b_i B_i$ the branch locus of f. We assume that the ramification divisor and the irreducible

components B_i of the branch locus B_f are all non-singular, and suppose that $n = \sum_{i+1}^{n} i \cdot N_i$. Then:

$$c_1^{N_1} \cdots c_n^{N_n}(TY) \frown [Y] - \mu \cdot c_1^{N_1} \cdots c_n^{N_n}(TX) \frown [X]$$

$$= \sum_i \left(H_{TR_i}^{(N_1 \cdots N_n)}(c_1(L_{R_i})) \frown [R_i] - (r_i + 1) \cdot H_{TB_i}^{(N_1 \cdots N_n)}(c_1(L_{B_i})) \frown [B_i] \right)$$

$$= \sum_i \sum_{\alpha=0}^{n-1} \frac{b_i(1 - (r_i+1)^{\alpha+1})}{r_i(r_i+1)^{\alpha}} P_\alpha(c_1(B_i) \cdots c_{n-1}(B_i)) \cdot c_1(L_{B_i})^{\alpha} \frown [B_i],$$

where we set

$$H_\xi^{(N_1 \cdots N_n)}(l) = l^{-1} \left(\prod_{i=1}^{n} (c_i(\xi) + c_{i-1}(\xi) \cdot l)^{N_i} - c_1^{N_1} \cdots c_n^{N_n}(\xi) \right) = \sum_{\alpha=0}^{n-1} P_\alpha(c_1 \cdots c_{n-1}) l^{\alpha}.$$

3.1 Applications

In this section, we give some applications of our formula.

The result for the top Chern class implies the generalized Riemann-Hurwitz formula

$$\chi(Y) - \mu \cdot \chi(X) = -\sum_i b_i \cdot \chi(B_i),$$

which is a special case of the formula proved by Y.Yomdin, [Y].

In case that ($\mathbf{n = 2}$):

The result for the second Chern class inplies

$$c_2(TY) \frown [Y] - \mu \cdot c_2(TX) \frown [X] = -\sum_i b_i \cdot \chi(B_i).$$

We remark that the more general formula is proved for algebraic cases. (see [I].)

We can also deduce the formula for the square of the first Chern classes as follows:

$$c_1(TY)^2 \frown [Y] - \mu \cdot c_1(TX)^2 \frown [X] = -\sum_i \left(2b_i \cdot \chi(B_i) + \frac{b_i(r_i+2)}{r_i+1} B_i \cdot B_i \right).$$

Now from the fact that the signature of the surface is expressed by $L_1 = (1/3)p_1 = \frac{1}{3}(-2c_2 + c_1^2)$, (The calculation for T and L-genus is found in [H1]), we obtain:

Theorem [The formula for signature for ramified coverings]
Let $f: Y \longrightarrow X$ be a ramified covering between compact complex analytic surfaces with covering multiplicity μ, $R_f = \sum_i r_i R_i$ the ramification divisor of f, and $B_f = \sum_i b_i B_i$ the

branch locus of f. We assume that ramification divisor and irreducible components B_i of the branch locus B_f are all non-singular. Then

$$\begin{aligned} Sign(Y) - \mu \cdot Sign(X) &= \frac{1}{3}(p_1(Y) - \mu \cdot p_1(X)) \\ &= \frac{1}{3}\{\left(c_1(Y)^2 - \mu \cdot c_1(X)^2\right) - 2\left(c_2(Y) - \mu \cdot c_2(X)\right)\} \\ &= -\sum_i \frac{b_i(r_i+2)}{3(r_i+1)} B_i \cdot B_i \,. \end{aligned}$$

Originally, the formula for signature for cyclic coverings is formulated for 4-manifold as follows.

Theorem [Hirzebruch [H1]] *Let X be a compact oriented differentiable manifold of dimension 4 without boundary on which the cyclic groups G_n of order n acts by orientation preserving diffeomorphisms. Suppose that Y is differential submanifold of X, not necessarily connected, and has codimension 2. And G_n operates freely on $X - Y$. Then*

$$Sign(X) - n \cdot Sign(X/G_n) = -\frac{n^2-1}{3n} Y' \cdot Y'.$$

where Y' is the branch locus in X/G_n.

We can also deduce the formula for the Todd genus, which is $\frac{1}{12}(c_2 + c_1^2)$:

Theorem *Under the same assumption of the above theorem,*

$$\begin{aligned} T(Y) - \mu \cdot T(X) &= \frac{1}{12}\{(c_2(Y) - \mu \cdot c_2(X)) + (c_1(Y)^2 - \mu \cdot c_1(X)^2)\} \\ &= -\sum_i \left(\frac{b_i}{2} T_1(B_i) + \frac{b_i((r_i+1)^2 - 1)}{12 r_i(r_i+1)} B_i \cdot B_i\right). \end{aligned}$$

In general, however, the calculation for the T-genus or the L-genus is more complicated, as examples we introduce formulas for the cases $n = 3, 4, 5$, and 6. (Also see [H1].)
($\mathbf{n = 3}$):

$T_3 = \frac{1}{24} c_1 c_2$,
$H(l) = \frac{1}{24}((c_2 + c_1^2) + c_1 l)$.

$$T(Y) - \mu \cdot T(X) = -\sum_i \left(b_i \frac{T_2(B_i)}{2} + \frac{b_i(1-(r_i+1)^2)}{r_i(r_i+1)} \int_{B_i} \frac{T_1(B_i)}{12} \smile c_1(N_{B_i})\right).$$

($\mathbf{n = 4}$):

$T_4 = \frac{1}{720} \cdot (-c_4 + c_3 c_1 + 3c_2^2 + 4c_2 c_1^2 - c_1^4)$,

$H(l) = \frac{1}{720}(15c_2c_1 + 5(c_2 + c_1^2)l - l^3).$

$$\begin{aligned} T(Y) - \mu \cdot T(X) &= -\sum_i \frac{T_3(B_i)}{2} \\ &\quad + \sum_i \frac{b_i(1-(r_i+1)^2)}{r_i(r_i+1)} \int_{B_i} \frac{T_2(B_i)}{12} \smile c_1(N_{B_i}) \\ &\quad + \sum_i \frac{b_i(1-(r_i+1)^3)}{r_i(r_i+1)^2} \int_{B_i} \frac{c_1^3(N_{B_i})}{720}. \end{aligned}$$

We can also define the signature for $n = 4$, as

$L_2 = \frac{1}{45}(7p_2 - p_1^2) = \frac{1}{45}(14c_4 - 14c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4).$
$H(l) = \frac{1}{45}((-10c_2 + 5c_1^2)l - l^3)$.

$$\begin{aligned} Sign(Y) - \mu \cdot Sign(X) = &-\sum_i \frac{b_i(1-(r_i+1)^2)}{r_i(r_i+1)} \int_{B_i} \frac{L_1(B_i)}{3} \smile c_1(L_{B_i}) \\ &- \sum_i \frac{b_i(1-(r_i+1)^4)}{r_i(r_i+1)^3} \int_{B_i} \frac{c_1(L_{B_i})^3}{45}. \end{aligned}$$

$(\mathbf{n} = \mathbf{5})$:

$T_5 = \frac{1}{1440}(-c_4c_1 + c_3c_1^2 + 3c_2^2c_1 - c_2c_1^3),$
$H(l) = \frac{1}{1440}\{(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4) + 5c_2c_1l - c_1l^3\}.$

$$\begin{aligned} T(Y) - \mu \cdot T(X) &= -\sum_i \frac{T_4(B_i)}{2} \\ &\quad + \sum_i \frac{b_i(1-(r_i+1)^2)}{r_i(r_i+1)} \int_{B_i} \frac{T_3(B_i)}{12} \smile c_1(N_{B_i}) \\ &\quad - \sum_i \frac{b_i(1-r_i+1^4)}{r_i(r_i+1)^3} \int_{B_i} \frac{T_1(B_i)}{720} \smile c_1^3(N_{B_i}). \end{aligned}$$

$(\mathbf{n} = \mathbf{6})$:

$L_3 = \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_2p_1 + 2p_1^3)$

$$= \frac{1}{3^3 \cdot 5 \cdot 7}(-124c_6 + 124c_5c_1 - 72c_4c_2 - 26c_4c_1^2 + 62c_3^2 - 52c_3c_2c_1 + 26c_3c_1^3 + 10c_2^3 + 11c_2^2c_1^2 - 12c_2c_1^4 + 2c_1^6).$$

$H(l) = \frac{1}{3^3 \cdot 5 \cdot 7}\{(98c_4 - 98c_3c_1 + 21c_2^2 + 28c_2c_1^2 - 7c_1^4) \cdot l + (14c_2 - 7c_1^2) \cdot l^3 + 2l^5\}.$

$$Sign(Y) - \mu \cdot Sign(X) = -\sum_i \frac{b_i(1-(r_i+1)^2)}{r_i(r_i+1)} \int_{B_i} \frac{L_2(B_i)}{3} \smile c_1(L_{B_i})$$

$$- \sum_i \frac{b_i(1-(r_i+1)^4)}{r_i(r_i+1)^3} \int_{B_i} \frac{L_1(B_i)}{45} \smile c_1(L_{B_i})^3$$
$$- \sum_i \frac{b_i(1-(r_i+1)^6)}{r_i(r_i+1)^5} \int_{B_i} \frac{2c_1(L_{B_i})^5}{945} .$$

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