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Existence of Periodic Solutions for Periodic Linear Functional Differential Equations in Banach Spaces (II)

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1 Introduction

Let R be a real line and E a Banach space with a norm $|\cdot|$. If $x : (-\infty, a) \to E$, then a function $x_t : (-\infty, 0] \to E, t \in (-\infty, a)$, is defined by $x_t(\theta) = x(t+\theta), \theta \in (-\infty, 0]$. We deal with the linear functional differential equation with infinite delay in the Banach space E:

$$\text{(L)} \qquad \frac{dx(t)}{dt} = Ax(t) + B(t, x_t) + F(t).$$

Let $\mathcal{B}$ be a Banach space, consisting of functions $\psi : (-\infty, 0] \to E$, which satisfies some axioms demonstrated in Section 2. We assume that Eq.(L) always satisfies the following hypothesis(H):

(i) $A : \mathcal{D}(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $T(t), t \geq 0$, on E;
(ii) $B : R \times \mathcal{B} \to E$ is continuous and $B(t, \cdot) : \mathcal{B} \to E$ is linear;
(iii) $F : R \to E$ is continuous.

If $B(t, \psi)$ and $F(t)$ in Eq.(L) are periodic functions with a period $\omega > 0$, we denote Eq.(L) by Eq.(P_ωL). If $F \equiv 0$, we denote Eq.(L) and Eq.(P_ωL) by Eq.(L_0) and Eq.($P_\omega L_0$), respectively.

Chow and Hale [1] obtained the following two fixed point theorems for a linear affine map on a Banach space. Let X be a Banach space, and $T : X \to X$ a linear affine map $Tx = Lx + z, x \in X$, where $z \in X$ is fixed.

Theorem A. If the range $R(I - L)$ is closed and if there is an $x_0 \in X$ such that $\{x_0, Tx_0, T^2x_0, \cdots\}$ is bounded in X, then T has a fixed point in X.

Theorem B. If there is an $x_0 \in X$ such that $\{x_0, Tx_0, T^2x_0, \cdots\}$ is relatively compact in X, then T has a fixed point in X.

Using Theorem A, we showed a result [8, Corollary 4.9] on the existence of periodic solutions of Eq.($P_\omega L$). Its proof is based on the fact that, if the point 1 is a normal point of L, then the range $R(I - L)$ is closed. More recently, using Theorem B, Hino and Murakami extended our result. The property that C_0-semigroup $T(t)$ is compact for $t > 0$ on E plays an essential role in their proof given in [4]. In such a direction, Li, Lim and Li [5] have also considered the existence of periodic solutions of Eq.($P_\omega L$) with advanced and delay for the case where $A = 0$ and $E = R^n$. However, Theorem B cannot apply even to the case where $B(t, \cdot)$ is a compact operator for each $t \in R$, but either $A = 0$ in Eq.($P_\omega L$), or C_0-semigroup $T(t)$ is compact only for $t \geq t_0$, where t_0 is a positive constant.

The aim of this paper is to show the existence of periodic solutions for Eq.($P_\omega L$) in succession to [8]. In particular, we will discuss directly the closedness of the range $R(I - L)$ in Theorem A in the manner applicable for the case where the point 1 belongs to the essential spectrum of L. To do so, indeed, we make use of the theory of semi-Fredholm operators. As a result, we have general statements, Theorem 3.7 and Corollary 3.9, for the case that the phase space $\mathcal{B} = UC_g$(see Section 2) is a fading memory space; that is, a uniform fading memory space.

2 Preliminaries

First, we will explain the phase space $\mathcal{B}$. Let $\mathcal{B}$ be a normed linear space consisting of some functions mapping $(-\infty, 0]$ into E; the norm in $\mathcal{B}$ is denoted by $|\cdot|_{\mathcal{B}}$. Throughout this paper we assume that $\mathcal{B}$ satisfies the following axioms.

(B-1) If a function $x : (-\infty, \sigma + a) \to E$ is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then

(i) $x_t \in \mathcal{B}$ for all $t \in [\sigma, \sigma + a)$ and x_t is continuous in $t \in [\sigma, \sigma + a)$;

(ii) $H^{-1}|x(t)| \leq |x_t|_{\mathcal{B}} \leq K(t - \sigma)\sup\{|x(s)| : \sigma \leq s \leq t\} + M(t - \sigma)|x_\sigma|_{\mathcal{B}}$ for all $t \in [\sigma, \sigma + a)$, where $H > 0$ is constant, $K : [0, \infty) \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and they are independent of x.

(B-2) The space $\mathcal{B}$ is complete.

Let BC be the set of bounded, continuous functions mapping $(-\infty, 0]$ into E, and C_{00} its subset consisting of functions with compact support. The space C_{00} is automatically contained in the space $\mathcal{B}$ due to (B-1)-(i). The space BC is contained in $\mathcal{B}$ under the additional axiom (C).

(C) If a uniformly bounded sequence $\{\phi^n(\theta)\}$ in C_{00} converges to a function $\phi(\theta)$ uniformly on every compact set of $(-\infty, 0]$, then $\phi \in \mathcal{B}$ and $\lim_{n\to\infty} |\phi^n - \phi|_{\mathcal{B}} = 0$.

In fact, BC is continuouslly imbedded into $\mathcal{B}$; put

$$\|\phi\|_\infty = \sup\{|\phi(\theta)| : \theta \leq 0\} \quad \text{for} \quad \phi \in BC.$$

Lemma 2.1 ([3]) *If the phase space $\mathcal{B}$ satisfies the axiom* (C), *then there is a constant $J > 0$ such that $|\phi|_{\mathcal{B}} \leq J\|\phi\|_\infty$ for all $\phi \in BC$.*

Define operators $S(t) : \mathcal{B} \to \mathcal{B}, t \geq 0$, as

$$[S(t)\phi](\theta) = \begin{cases} \phi(0) & -t \leq \theta \leq 0, \\ \phi(t+\theta) & \theta \leq -t, \end{cases}$$

and denote by $S_0(t)$ be the restriction of $S(t)$ to $\mathcal{B}_0 := \{\phi \in \mathcal{B} : \phi(0) = 0\}$. The phase space $\mathcal{B}$ is called a fading memory space [3] if the axiom (C) holds and $S_0(t)\phi \to 0$ as $t \to \infty$ for each $\phi \in \mathcal{B}_0$. If $\mathcal{B}$ is such a space, then $\|S_0(t)\|$ is bounded for $t \geq 0$. In addition, if $\|S_0(t)\| \to 0$ as $t \to \infty$, then $\mathcal{B}$ is called a uniform fading memory space. If the phase space $\mathcal{B}$ is a fading memory space, then

$$|x_t|_{\mathcal{B}} \leq J \sup\{|x(s)| : \sigma \leq s \leq t\} + M|x_\sigma|_{\mathcal{B}}, \tag{1}$$

where $M = (1 + HJ)\sup_{t\geq 0} \|S_0(t)\|$.

Example. Take the phase space as $\mathcal{B} = UC_g$, the set of continuous functions, $\phi(\theta)$ such that $\phi(\theta)/g(\theta)$ is bounded and uniformly continuous on $(-\infty, 0]$ with the norm

$$\|\phi\| = \sup\{|\phi(\theta)|/g(\theta) : \theta \leq 0\},$$

where $g(\theta)$ is a positive continuous function such that $g(\theta) \to \infty$ as $\theta \to -\infty$. Then $\|S_0(t)\| = \sup_{s\leq 0} g(s)/g(s-t)$, and it is a uniform fading memory space if and only if it is a fading memory space (cf.[3, p.191]).

Next, we recall the definition of the semi-Fredholm operator on Banach space X. A bounded linear operator L on Banach space X is said to be semi-Fredholm if the range $R(L)$ is closed and at least one of nul $L := \dim N(L), N(L) = \{x \in X | Lx = 0\}$, and def $L := \dim X/R(L)$ is finite. The set of all semi-Fred holm operators with nul $L < \infty$ will be denoted by $\mathcal{F}_+(X)$.

Denote by F_T the set of fixed points of a linear affine map T given in Introduction. The following fixed point theorem is derived from Theorem A and properties of semi-Fredholm operators.

Proposition 2.2 *Assume that* $I - L \in \mathcal{F}_+(X)$. *If there is an* $x_0 \in X$ *such that* $\{x_0, Tx_0, T^2x_0, \cdots\}$ *is bounded in* X, *then* $F_T \neq \emptyset$, F_T *is an affine set and* $\dim F_T = \dim N(I - L) < \infty$.

3 The phase space UC_g and the existence of periodic solutions

A solution operator $U(t,0)$ of Eq.($\mathrm{P_\omega L_0}$) endowed with the initial condition $x_0 = \phi \in \mathcal{B}$ is decomposed as $U(t,0)\phi = \widehat{T}(t)\phi + K(t,0)\phi$, where

$$[\widehat{T}(t)\phi](\theta) = \begin{cases} T(t+\theta)\phi(0) & t+\theta \geq 0, \\ \phi(t+\theta) & t+\theta \leq 0. \end{cases}$$

$$[K(t,0)\phi](\theta) = \begin{cases} \displaystyle\int_0^{t+\theta} T(t+\theta-s)B(s, x_s(\sigma,\phi))\, ds & t+\theta \geq 0, \\ 0 & t+\theta \leq 0. \end{cases}$$

In this section, we will show the closedness of the range $R(I - U(\omega,0))$ by using the theory of semi-Fredholm operators, where $U(\omega,0)$ is the solution operator for Eq.($\mathrm{P_\omega L_0}$). Throughout this section we assume, in addition to the axioms (B-1) and (B-2), the following axiom :

(B-3) $|\phi^1 - \phi^2|_{\mathcal{B}} = 0$ for ϕ^1, ϕ^2 in $\mathcal{B}$ if and only if $\phi^1(\theta) = \phi^2(\theta)$ for $\theta \in (-\infty, 0]$.

Lemma 3.1 *If the phase space* $\mathcal{B}$ *satisfies the axiom* (C) *and* $T(t)$ *is a* C_0-*semigroup on* E, *then a function* ϕ *of* $N(I-\widehat{T}(\omega))$ *is an* ω-*periodic continuous function given by* $\phi(\theta) = T(\theta + n\omega)\phi(0), \theta \in [-n\omega, 0], n = 1, 2, \cdots,$ *where* $\phi(0) \in N(I - T(\omega))$, *and*

$$\dim N(I - \widehat{T}(\omega)) = \dim N(I - T(\omega)).$$

Proof. Suppose that $\widehat{T}(\omega)\phi = \phi$. Since $[\widehat{T}(\omega)\phi](\theta) = \phi(\omega+\theta)$ for $\omega+\theta \leq 0$, it follows that $\phi(\omega+\theta) = \phi(\theta)$ for $\theta \leq -\omega$; that is, $\phi(\theta)$ is ω-periodic on $(-\infty, 0]$. Since $\widehat{T}(n\omega) = \widehat{T}(\omega)^n, n = 0, 1, 2, \cdots$, we have that $\widehat{T}(n\omega)\phi = \phi$. On the other hand, if $-n\omega \leq \theta \leq 0$, then $[\widehat{T}(n\omega)\phi](\theta) = T(n\omega+\theta)\phi(0)$; hence, $T(n\omega+\theta)\phi(0) = \phi(\theta)$ for $-n\omega \leq \theta \leq 0$ and $\phi(\theta)$ is continuous on $[-n\omega, 0]$. Set $a = \phi(0)$ and $x(t) = T(t)a, t \geq 0$. Then $x(t) = \phi(t - n\omega)$ as long as $0 \leq t \leq n\omega$. Since n may be arbitrary, we can regarde that $x(t)$ is ω-periodic and continuous in $(-\infty, \infty)$, and $\phi = x_0$. Since $x(\omega) = x(0)$, it follows that $T(\omega)a = a$; that is, $a \in N(I - T(\omega))$.

Conversely, if $a \in N(I-T(\omega))$, then $T(t+\omega)a = T(t)T(\omega)a = T(t)a, t \geq 0$; that is, $T(t)a$ is ω-periodic in $[0, \infty)$. Suppose that $x(t)$ is the ω-periodic extension of $T(t)a$ to $(-\infty, \infty)$, and set $\phi = x_0$. From the axiom (C) we see that ϕ belongs to $\mathcal{B}$. Then it is obvious that $\widehat{T}(\omega)\phi = \phi$. Moreover, the space $N(I - T(\omega))$ is mapped bijectively onto the space $N(I - \widehat{T}(\omega))$. Therefore, the proof is complete.

Let the null space $N(I - T(\omega))$ be of finite dimension. Then there exists a closed subspace M of E such that $E = M \oplus N$, where $N = N(I - T(\omega))$, and let S_M be the restriction of $I - T(\omega)$ to M. Then $S_M : M \to R(I - T(\omega))$ is a continuous, bijective, linear operator. Thus there is the inverse operator S_M^{-1} of S_M. Of course, if $R(I - T(\omega))$ is closed, then S_M^{-1} is continuous.

To prove that the range $R(I - \widehat{T}(\omega))$ is closed, we will solve the equation $(I - \widehat{T}(\omega))\phi = \psi$ and use the above notations.

Proposition 3.2 *Suppose that the phase space $\mathcal{B}$ satisfies the axiom* (C), *and that* $\dim N(I - T(\omega)) < \infty$. *Then $\psi \in R(I - \widehat{T}(\omega))$ if and only if $\psi(0) \in R(I - T(\omega))$ and $U\psi \in \mathcal{B}, \psi \in \mathcal{B}$, where U is defined as*

$$[U\psi](\theta) = \sum_{j=0}^{k-1} \psi(\theta + j\omega) + T(\theta + k\omega)S_M^{-1}\psi(0), \quad \theta \in [-k\omega, -(k-1)\omega], \quad (2)$$

for $k = 1, 2, \cdots$.

Proof. First, we formally solve the equation $(I - \widehat{T}(\omega))\phi = \psi$. The definition of $\widehat{T}(\omega)$ implies that

$$\phi(\theta) - T(\theta+\omega)\phi(0) = \psi(\theta), \ -\omega \leq \theta \ \text{ and } \ \phi(\theta) - \phi(\theta+\omega) = \psi(\theta), \ \theta \leq -\omega.$$

From the first equation it follows that $(I - T(\omega))\phi(0) = \psi(0)$, and $\phi(\theta) = \psi(\theta) + T(\theta+\omega)\phi(0)$ for $-\omega \leq \theta \leq 0$. From the second equation, it follows that, for $k = 2, 3, \cdots$, $\phi(\theta) = \psi(\theta) + \phi(\theta+\omega)$ for $\theta \in [-k\omega, -(k-1)\omega]$. Hence the solution ϕ is determined as

$$\phi(\theta) = \sum_{j=0}^{k-1} \psi(\theta + j\omega) + T(\theta + k\omega)\phi(0), \quad \theta \in [-k\omega, -(k-1)\omega], \qquad (3)$$

$k = 1, 2, \cdots$, uniquely for $\phi(0)$.

Assume that $\psi \in R(I - \widehat{T}(\omega))$. Then $\psi(0) \in R(I - T(\omega))$, and there exists a function $\hat{\phi} \in \mathcal{B}$ satisfying the equation $(I - \widehat{T}(\omega))\hat{\phi} = \psi$. Ovbiously, $\hat{\phi}(0)$ satisfies the equation $(I - T(\omega))\hat{\phi}(0) = \psi(0)$. Furthermore $\hat{\phi}(0)$ is decomposed as $\hat{\phi}(0) = S_M^{-1}\psi(0) + \phi_N(0), \phi_N(0) \in N, S_M^{-1}\psi(0) \in M$. Set $\phi_N(\theta) = T(\theta + k\omega)\phi_N(0), \theta \in [-k\omega, 0], k = 0, 1, 2, \cdots$. Using Lemma 3.1 we see that ϕ_N belongs to $N(I - \widehat{T}(\omega))$. Hence $\hat{\phi} = U\psi + \phi_N$. Needless to say, $U\psi$ belongs to $\mathcal{B}$.

Conversely, assume that $\psi(0) \in R(I - T(\omega))$ and $U\psi \in \mathcal{B}, \psi \in \mathcal{B}$. Then $[(I - \widehat{T}(\omega))U\psi](\theta) = \psi(\theta)$ for every $\theta \in (-\infty, 0]$; that is, $(I - \widehat{T}(\omega))U\psi = \psi$. The proof is complete.

Theorem 3.3 *Suppose that the phase space $\mathcal{B}$ satisfies the axiom* (C), *and that if $|\phi^n - \phi|_{\mathcal{B}} \to 0$ as $n \to \infty$, then $\phi^n(\theta)$ converges to $\phi(\theta)$ uniformly for θ in any compact interval of* $(-\infty, 0]$. *Furthermore, Suppose that $I - T(\omega) \in \mathcal{F}_+(E)$. Then $R(I - \widehat{T}(\omega))$ is closed if and only if there exists a positive constant c such that $|U\psi|_{\mathcal{B}} \leq c|\psi|_{\mathcal{B}}, \psi \in \mathcal{B}$, as long as $\psi(0) \in R(I - T(\omega))$ and $U\psi \in \mathcal{B}$, where U is given by (2).*

Proof. Set $D = \{U\psi : \psi \in R(I - \widehat{T}(\omega))\}$. Let F be the restriction of $I - \widehat{T}(\omega)$ to D. Then the operator $F : D \to R(I - \widehat{T}(\omega))$ have the following properties : $N(F) = \{0\}, FU\psi = \psi$ for $\psi \in R(I - \widehat{T}(\omega)), R(F) = R(I - \widehat{T}(\omega))$, and F is a bounded linear operator. If F is a closed linear operator, then Theorem 3.3 follows from the well known theorem [8, Theorem 5.1, p.70] about the closed range property. If D is a closed subspace, then F is a closed operator. But it is difficult to see that D is closed. So, we show directly that F is a closed operator. To do so, suppose that a sequence $\phi^n := U\psi^n, n = 1, 2, \cdots$ in D converges to a function ϕ in $\mathcal{B}$ and the sequence $F\phi^n = \psi^n$ conveges to a function ψ in $\mathcal{B}$. From the assumption in the theorem it follows that $\phi^n(\theta) \to \phi(\theta), \psi^n(\theta) \to \psi(\theta)$ as $n \to \infty$ uniformly for θ in any compact interval of $(-\infty, 0]$. Since $R(I - T(\omega))$ is closed, we have that $\psi^n(0) \to \psi(0)$ as $n \to \infty$ and $\psi(0) \in R(I - T(\omega))$. Then from the definition of the operator U it follows that $U\psi^n(\theta) \to U\psi(\theta)$ as $n \to \infty$ uniformly for θ in any compact interval $(-\infty, 0]$. This implies th at $U\psi(\theta) = \phi(\theta)$ for all

$\theta \in (-\infty, 0]$. Since $\phi \in \mathcal{B}$, it follows that $\psi \in R(I - \widehat{T}(\omega)), \phi = U\psi \in D$ and $F\phi = \psi$.

From Theorem 5.1 in [7], Chapter III, it follows that $R(I - \widehat{T}(\omega))$ is closed if and only if there is a positive constant c such that $|\phi|_{\mathcal{B}} \leq c|F\phi|_{\mathcal{B}}$ for all $\phi \in D$, which means that $R(I - \widehat{T}(\omega))$ is closed if and only if $|U\psi|_{\mathcal{B}} \leq c|\psi|_{\mathcal{B}}$ for all $\psi \in R(I - \widehat{T}(\omega))$. From Proposition 3.2 we have the conclution of the theorem.

Let BUC be the set of all bounded and uniformly continuous functions from $(-\infty, 0]$ into E with the supremum norm.

Proposition 3.4 *Take the space BUC as the phase space of $\widehat{T}(\omega)$. Then $R(I - \widehat{T}(\omega))$ is not closed in general.*

Proof. It suffices to show that there exists a sequence $\{\phi^n\}$ in BUC such that $|\phi^n|_{\mathcal{B}} \equiv 1$, and $\lim_{n\to\infty} |(I - \widehat{T}(\omega))\phi^n|_{\mathcal{B}} = 0$. Let e be a unit vector of E ; that is, $|e|_{\mathcal{B}} = 1$, and define $x^n(t)$, $n = 1, 2, \cdots$, as

$$x^n(t) = \begin{cases} e & t \leq -n\omega \\ (-t/n\omega)e & -n\omega \leq t \leq 0 \\ 0 & t \geq 0. \end{cases}$$

Set $\phi^n = x_0^n, n = 1, 2, \cdots$. Since $\phi^n(0) = 0$, we have $[\widehat{T}(\omega)(\phi^n)](\theta) = 0$ for $\theta \in [-\omega, 0]$; in other words, $\widehat{T}(\omega)\phi^n = S_0(\omega)\phi^n$. Thus it follows that $(I - \widehat{T}(\omega))\phi^n = \phi^n - S_0(\omega)\phi^n$; hence, $|(I - \widehat{T}(\omega))\phi^n|_{\mathcal{B}} = 1/n \to 0$ as $n \to 0$. Clearly, $|\phi^n|_{\mathcal{B}} \equiv 1$. Thus this is a desired sequence.

Theorem 3.5 *If $\mathcal{B} = UC_g$ is a uniform fading memory space and if $I - T(\omega) \in \mathcal{F}_+(E)$, then the range $R(I - \widehat{T}(\omega))$ is closed ; hence, $I - \widehat{T}(\omega) \in \mathcal{F}_+(\mathcal{B})$.*

Proof. Since $\mathcal{B} = UC_g$ is a uniform fading memory space, there are $M_0 \geq 1$ and $\epsilon_0 > 0$ such that $\|S_0(t)\| \leq M_0 e^{-\epsilon_0 t}$ for $t \geq 0$. Namely,

$$\|S_0(t)\| = \sup_{s\leq 0} \frac{g(s)}{g(s-t)} = \sup_{s\leq -t} \frac{g(s+t)}{g(s)} \leq M_0 e^{-\epsilon_0 t}.$$

Suppose that $\psi(0) \in R(I - T(\omega)), U\psi \in \mathcal{B}, \psi \in \mathcal{B}$. Then we have that, for $\theta \in [-k\omega, -(k-1)\omega], k \geq 1$,

$$\frac{1}{g(\theta)} \left| \sum_{j=0}^{k-1} \psi(\theta + j\omega) \right| \leq \sum_{j=0}^{k-1} \frac{g(\theta + j\omega)}{g(\theta)} \frac{|\psi(\theta + j\omega)|}{g(\theta + j\omega)}$$

$$\leq \sum_{j=0}^{k-1} \|S_0(j\omega)\| \|\psi\|$$
$$\leq \sum_{j=0}^{k-1} M_0 e^{-\epsilon_0 j\omega} \|\psi\| \leq \frac{M_0\|\psi\|}{1-e^{-\epsilon_0\omega}}.$$

On the other hand, since S_M^{-1} is continuous, we have that

$$\frac{1}{g(\theta)}|T(\theta+k\omega)S_M^{-1}\psi(0)| \leq \sup\{\|T(t)\| : 0 \leq t \leq \omega\}\|S_M^{-1}\|\|\psi\|.$$

Summarizing these inequalities, (2) is estimated as

$$\|U\psi\| \leq \left(\frac{M_0}{1-e^{-\epsilon_0\omega}} + \sup\{\|T(t)\| : 0 \leq t \leq \omega\}\|S_M^{-1}\|\right)\|\psi\|, \tag{4}$$

which implies that the range $R(I-\widehat{T}(\omega))$ is closed, because of Theorem 3.3. Since $I - T(\omega) \in \mathcal{F}_+(E)$. From this fact and Lemma 3.1 it follows that $I - \widehat{T}(\omega) \in \mathcal{F}_+(\mathcal{B})$, which proves the theorem.

The following result is well known in the theory of semi-Fredholm operators (refer to [2, Theorems 3.21, 3.22, pp.35-37], or [7, Theorems 6.3, 6.4, p.128]).

Lemma 3.6 *Let $L \in \mathcal{F}_+(X)$.*

1) *If S is a compact operator on X, then $L \pm S \in \mathcal{F}_+(X)$.*

2) *There is a positive number η such that if S is a bounded linear operator on X satisfying $\|S\| < \eta$, then $L \pm S \in \mathcal{F}_+(X)$ and* $\mathrm{nul}(L \pm S) \leq \mathrm{nul}L$.

Summarizing these results we can obtain one of main theorems of this paper.

Theorem 3.7 *Assume that $\mathcal{B} = UC_g$ is a uniform fading memory space and at least one of the following conditions is satisfied :*

(i) *$T(t)$ is a C_0-compact semigroup on E.*

(ii) *For each $t \in R, B(t,\cdot)$ is a compact operator and $I - T(\omega) \in \mathcal{F}_+(E)$.*

If Eq.(P$_\omega$L) *has an E-bounded solution, then it has an ω-periodic solution.*

Proof. The proof easily follows from Theorem 3.5, the assertion 1) in Lemma 3.6 and Proposition 2.2.

Finally, we consider the case where the both of $T(t)$ and $B(t,\cdot)$ are not compact in general. Set $\|B\|_\infty := \sup\{\|B(t)\| \,|\, 0 \leq t < \infty\}$, where

$\|B(t)\|$ is the operator norm of $B(t,\cdot)$. If $\mathcal{B}$ is a fading memory space, if $\|T(t)\| \le M_w e^{wt}, t \ge 0$, and if $\|B\|_\infty < \infty$, then the Gronwall inequality implies that the solution $x(t,\phi)$ of Eq.($\mathrm{P_\omega L_0}$) such that $x_0 = \phi$ satisfies $|x_t(\phi)|_{\mathcal{B}} \le |\phi|_{\mathcal{B}} N(t; \|B\|_\infty)$ for $t > 0$, where

$$N(t; \|B\|_\infty) = (HJM_w + M)\exp\{t(M_w\|B\|_\infty J + \max\{w, 0\})\},$$

and M is the constant in the inequality (1). We denote by $\mathcal{S}(\omega)$ the set of ω-periodic solutions for Eq.($\mathrm{P_\omega L}$).

Theorem 3.8 *Let $T(t)$ be a C_0-semigroup on E such that $\|T(t)\| \le M_w e^{wt}$, and assume that $I - \hat{T}(\omega) \in \mathcal{F}_+(\mathcal{B})$. Let η be given as in Lemma 3.6 (2) for $I - \hat{T}(\omega) \in \mathcal{F}_+(\mathcal{B})$, and assume that $\|B\|_\infty$ is so small as to satisfy the condition*

$$JM_w\|B\|_\infty N(\omega; \|B\|_\infty)\int_0^\omega e^{ws}ds < \eta.$$

If Eq.($\mathrm{P_\omega L}$) has an E-bounded solution, then $\mathcal{S}(\omega)$ is nonempty and

$$\dim \mathcal{S}(\omega) \le \dim N(I - \hat{T}(\omega)) < \infty.$$

Proof. Recall that the solution operator $U(t,0) : \mathcal{B} \to \mathcal{B}$ for Eq.($\mathrm{P_\omega L_0}$) is decomposed as $U(t,0) = \hat{T}(t) + K(t,0)$. Since

$$\begin{aligned} |K(\omega,0)\phi|_{\mathcal{B}} &\le J \sup_{0\le\tau\le\omega}\int_0^\tau \|T(\tau - s)\|\|B(s)\|_\infty |x_s(\phi))|_{\mathcal{B}}\, ds \\ &\le JM_w\|B\|_\infty N(\omega; \|B\|_\infty)|\phi|_{\mathcal{B}} \sup_{0\le\tau\le\omega}\int_0^\tau e^{w(\tau-s)}\, ds, \end{aligned}$$

we have that

$$\|K(\omega,0)\| \le JM_w\|B\|_\infty N(\omega; \|B\|_\infty)|\phi|_{\mathcal{B}}\int_0^\omega e^{ws}ds.$$

Thus, if the right side of this inequality is less than η, then $I - (\hat{T}(\omega) + K(\omega,0)) \in \mathcal{F}_+(\mathcal{B})$; that is, $I - U(\omega,0) \in \mathcal{F}_+(\mathcal{B})$. From Lemma 3.6 we have $\dim N(I - (\hat{T}(\omega) + K(\omega,0)) \le \dim N(I - \hat{T}(\omega))$. This proves the theorem.

Corollary 3.9 *Assume that $\mathcal{B} = UC_g$ is a uniform fading memory space, $I - T(\omega) \in \mathcal{F}_+(E)$, and that $\|B\|_\infty$ satisfies the same condition as in Theorem 3.8. If Eq.($\mathrm{P_\omega L}$) has an E-bounded solution, then $\mathcal{S}(\omega)$ is nonempty and*

$$\dim \mathcal{S}(\omega) \le \dim N(I - T(\omega)) < \infty.$$

References

[1] S.-N. Chow and J.K. Hale, Strongly limit-compact maps, Funkcial. Ekvac., 17(1974), 31-38.

[2] D.E. Edmunds and W.D. Evans, "*Spectral Theory and Differential Operators* ", Oxford Univ. Press, New York, 1987.

[3] Y. Hino, S. Murakami and T. Naito, "*Functional Differential Equations with Infinite Delay*", Lect. Notes Math. 1473, Springer-Verlag, 1991.

[4] Y. Hino, S. Murakami and T. Yoshizawa, Existence of almost periodic solutions of some functional differential equations in a Banach space, Tohoku Math. J., 49(1997), 133-147.

[5] Y. Li, Z. Lim and Z. Li, A Massera type criterion for linear functional differential equations with advanced and delay, J. Math. Appl., 200(1996), 715-725.

[6] T. Naito, J.S. Shin and S. Murakami, On solution semigroups of general functional differential equations, Nonlinear Analysis, Proc. of the Second World Congress of Nonlinear Analysis, 30:7(1977), 4565-4576.

[7] M. Schechter, "*Principles of functional Analysis*", Academic Press, New York and London, 1971.

[8] J.S. Shin and T. Naito, Existence of periodic solutions for periodic linear functional differential equations in Banach spaces(in Japanese). Kokyuroku No, 900 (1995), 148-158.

[9] J.S. Shin and T. Naito, Closed range properties and periodic solutions for linear functional differential equations, submitted.