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ON THE DETERMINATION OF THE HEAT CONDUCTIVITY FROM THE HEAT FLOW

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Introduction

We study the inverse problem to determine $a(t)$ of the parabolic system

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= a(t) \frac{\partial^{2} u}{\partial x^{2}} \quad (0 < x < \infty, 0 < t < T), \\
u(x, 0) &= 0 \quad (0 \leq x < \infty), \\
u(0, t) &= f(t) \quad (0 \leq t < T), \\
- a(t) \frac{\partial u}{\partial x}(0, t) &= g(t) \quad (0 < t < T), 
\end{aligned}
$$

(0.1)

so that this (overspecified) system admits a classical solution $u(x, t)$ satisfying, for each $T' < T$,

$$
\sup_{0 < t < T'} \left\{ |u(x, t)| + \left| \frac{\partial u}{\partial x}(x, t) \right| \right\} = O(e^{\alpha x}) \quad (x \to \infty).
$$

(0.2)

with some constant $\alpha < 2$.

This problem was studied by several authors ([1,2,3,5]), and various existence and uniqueness results were established. However, they have been accomplished under the assumption that $f(t)$ is a monotonically nondecreasing function. The purpose of the present paper is to investigate the problem without this assumption.

Let us assume that

(I) $a(t)$ is positive and continuous for $0 \leq t < T$,

(II) $f(t)$ is continuous for $0 \leq t < T$ and $f(0) = 0$.

Then the system

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= a(t) \frac{\partial^{2} u}{\partial x^{2}} \quad (0 < x < \infty, 0 < t < T), \\
u(x, 0) &= 0 \quad (0 \leq x < \infty), \\
u(0, t) &= f(t) \quad (0 \leq t < T), 
\end{aligned}
$$

is uniquely solvable under the assumption (0.2), and the solution $u(x, t)$ can be expressed as

$$
u(x, t) = -2 \int_{0}^{t} \frac{\partial H}{\partial x} \left( x, \int_{\tau}^{t} a(\tau)d\tau \right) a(\tau)f(\tau)d\tau,$$

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where $H(x, t)$ is the fundamental solution of the heat equation:

$$H(x, t) := \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

Hence, as was shown in [2] (also see [5]), if $f$ is differentiable then the inverse problem mentioned in the beginning is equivalent to finding a positive solution $a(t)$ of the nonlinear integral equation

$$\frac{1}{\sqrt{\pi}} a(t) \int_{0}^{t} \frac{f'(\tau)}{\left(\int_{\tau}^{t} a(r)dr\right)^{1/2}} d\tau = g(t) \quad (0 < t < T).$$

(0.3)

We hereafter focus our attention on the equation (0.3). The main goal here is to show that the equation (0.3) is solvable near $t = 0$ and the continuation of the solution can be made as far as it is bounded above, without the monotonicity of $f(t)$.

Throughout this paper we use the notation

$$C_+(I) := \{a(t) \in C(I) \mid a(t) > 0 \quad (t \in I)\}.$$ 

In Section 1 we shall establish a uniqueness result. In Section 2 we shall establish a local existence result. In Section 3 we shall discuss the continuation of solution. The main result will be given in Section 4.

1. Uniqueness

In this section we shall establish the following uniqueness result:

**Theorem 1.1.** Assume that

(i) $f(t) \in C[0, T] \cap C^1(0, T)$, $\lim_{t \to 0} t^{1-\mu} f'(t) > 0$ with some $\mu > 0$;

(ii) $g(t) \in C_+(0, T)$.

If $a_1(t), a_2(t) \in C_+[0, T]$ are solutions of (0.3) then $a_1(t) \equiv a_2(t)$.

Before the proof we shall give some remarks on the assumptions:

**Remark 1.2.** By the substitution $\tau = t\rho$, (0.3) can be rewritten as

$$t^{\mu-1/2} a(t) \int_{0}^{1} \frac{(t\rho)^{1-\mu} f'(t\rho)}{\left(\int_{\rho}^{1} a(tr)dr\right)^{1/2}} d\rho = \sqrt{\pi} g(t) \quad (0 < t < T).$$

(1.1)

Accordingly the assumption (i) implies that there exists the limit

$$\lim_{t \to 0} t^{1/2-\mu} g(t) > 0$$

(1.2)

In addition to the assumption (ii) we assume that $g(t) \in C[0, T)$. Then it follows from (1.2) that the condition $\mu \geq 1/2$ is necessary. Moreover if (0.3) has a solution $a(t) \in C_+[0, T)$ then (0.3) holds even at $t = 0$. 
We now give the proof of Theorem 1.1. Let $T_1 \in (0, T)$ be fixed. By (1.1) we obtain for $0 < t \leq T_1$,
\[
a_2(t) \int_0^1 \frac{(tp)^{1-\mu} f'(tp)}{\left( \int_0^1 a_2(tr)dr \right)^{1/2} \rho^{1-\mu}} \, d\rho = a_1(t) \int_0^1 \frac{(tp)^{1-\mu} f'(tp)}{\left( \int_0^1 a_1(tr)dr \right)^{1/2} \rho^{1-\mu}} \, d\rho.
\]
By taking the limit as $t \to 0$, this yields
\[
a_2(0) = a_1(0).
\] (1.4)

We put
\[
b(t) := a_2(t) - a_1(t), \quad p(t) := \int_0^1 \frac{(tp)^{1-\mu} f'(tp)}{\left( \int_0^1 a_2(tr)dr \right)^{1/2} \rho^{1-\mu}} \, d\rho.
\]
Then, from (1.3), we have
\[
b(t)p(t) = (a_2(t) - a_1(t))p(t)
\]
\[
= a_1(t) \int_0^1 \left\{ \frac{1}{\left( \int_0^1 a_1(tr)dr \right)^{1/2}} - \frac{1}{\left( \int_0^1 a_2(tr)dr \right)^{1/2}} \right\} (tp)^{1-\mu} f'(tp) \, d\rho \rho^{1-\mu}
\]
\[
=a_1(t) \int_0^1 \frac{\int_0^1 b(tr)dr}{\prod_{j=1}^2 \left( \int_0^1 a_j(tr)dr \right)^{1/2}} (tp)^{1-\mu} f'(tp) \, d\rho \rho^{1-\mu},
\]
where we have used interchange of the order of integration. Therefore, by setting
\[
\Phi(t, \sigma) := \frac{a_1(t)}{p(t)} \int_0^\sigma \frac{(tp)^{1-\mu} f'(tp)}{\prod_{j=1}^2 \left( \int_0^1 a_j(tr)dr \right)^{1/2}} \frac{d\rho}{\rho^{1-\mu}}
\]
we arrive at
\[
b(t) = \int_0^1 \Phi(t, \sigma)b(t\sigma)d\sigma \quad (0 \leq t \leq T').
\] (1.5)

In view of (1.1), $p(t) = \sqrt{\pi} t^{1/2-\mu} a_2(t)^{-1} g(t)$. Hence, by the assumption (ii), $p(t)$ is positive for $0 < t \leq T'$. But, in view of the definition of $p(t)$ and the assumption (i), $p(t)$ is a continuous function on the interval $[0, T_1]$ with $p(0) > 0$. So $\min_{0 \leq t \leq T'} p(t) =: c > 0$. This shows that
\[
|\Phi(t, \sigma)| \leq M_1 \int_0^\sigma \frac{1}{(1-\rho)^{3/2} \rho^{1-\mu}} \, d\rho \leq \frac{M}{(1-\sigma)^{1/2}}.
\] (1.6)
Moreover, from (1.4), we get
\[
\Phi(\sigma) := \lim_{t \to 0} \Phi(t, \sigma) = \frac{1}{2} \int_{0}^{1} \frac{1}{(1-\rho)^{1/2}\rho^{1-\mu}} \int_{0}^{\sigma} \frac{d\rho}{(1-\rho)^{3/2}\rho^{1-\mu}}
\]
\[
= \frac{1}{2} \frac{1}{B(\mu,1/2)} \int_{0}^{\sigma} \frac{d\rho}{(1-\rho)^{3/2}\rho^{1-\mu}} > 0,
\]
(1.7)
where \(B(\cdot, \cdot)\) denote the beta function. Note that this convergence is uniform with respect to \(\sigma\) in the following sense:
\[
\lim_{t \to 0} \sup_{0 \leq \sigma < 1} (1-\sigma)^{1/2} |\Phi(t, \sigma) - \Phi(\sigma)| = 0.
\]
(1.8)

We now define
\[
J_{\Phi} \hat{z}(t) := \int_{0}^{1} \Phi(\sigma) \hat{z}(t\sigma) d\sigma \quad (0 \leq t \leq \Lambda)
\]
for all \(\hat{z}(t)\) in the Banach space \(C[0, \Lambda]\) of all continuous functions on \([0, \Lambda]\) (with norm \(|| \cdot ||_{\Lambda}\) given \(||\hat{z}||_{\Lambda} := \max_{0 \leq t \leq \Lambda} |\hat{z}(t)|\)). Then \(J_{\Phi}\) is a bounded linear operator from \(C[0, \Lambda]\) to itself, and the operator norm \(||J_{\Phi}||_{\Lambda}\) of \(J_{\Phi} : C[0, \Lambda] \to C[0, \Lambda]\) is computed as
\[
||J_{\Phi}||_{\Lambda} = \int_{0}^{1} |\Phi(\sigma)| d\sigma = \frac{1}{2} \frac{1}{B(\mu,1/2)} \int_{0}^{1} \int_{0}^{\sigma} \frac{d\rho}{(1-\rho)^{3/2}\rho^{1-\mu}}
\]
\[
= \frac{1}{2} \frac{1}{B(\mu,1/2)} \int_{0}^{\sigma} \frac{d\rho}{(1-\rho)^{3/2}\rho^{1-\mu}} \int_{\rho}^{1} d\sigma = \frac{1}{2}.
\]

Accordingly, by means of the Neumann series, the operator \(I - J_{\Phi} : C[0, \Lambda] \to C[0, \Lambda]\) has the bounded inverse \((I - J_{\Phi})^{-1}\), where \(I\) denotes the identity operator in \(C[0, \Lambda]\).

Since (1.5) can be written as
\[
(I - J_{\Phi})b(t) = \int_{0}^{1} [\Phi(t, \sigma) - \Phi(\sigma)] b(t\sigma) d\sigma,
\]
we obtain for \(0 < \Lambda \leq T_{1}\),
\[
||b||_{\Lambda} \leq ||(I - J_{\Phi})^{-1}||_{\Lambda} \max_{0 \leq t \leq \Lambda} \int_{0}^{1} |\Phi(t, \sigma) - \Phi(\sigma)| d\sigma ||b||_{\Lambda}
\]
\[
\leq 2 \int_{0}^{1} \max_{0 \leq t \leq \Lambda} (1-\sigma)^{1/2} |\Phi(t, \sigma) - \Phi(\sigma)| d\sigma ||b||_{\Lambda}.
\]
This, together with (1.8), shows that there exists \(\delta > 0\) such that \(||b||_{\delta} = 0\), that is, \(b(t) = 0\) for any \(t \in [0, \delta]\).
For $\delta \leq t \leq T_1$ it follows from (1.5), (1.6) that

$$|b(t)| = \left| \int_0^1 \Phi(t, \sigma)b(t\sigma)d\sigma \right| \leq M \int_0^1 \frac{|b(t\sigma)|}{(1-\sigma)^{1/2}}d\sigma$$

$$= \frac{M}{t^{1/2}} \int_{\delta}^{t} \frac{|b(\tau)|}{(t-\tau)^{1/2}}d\tau.$$

This leads to

$$|b(t)| \leq \frac{M^2}{\delta} \int_{\delta}^{t} \frac{d\tau}{(t-\tau)^{1/2}} \int_{\delta}^{\tau} \frac{|b(s)|}{(\tau-s)^{1/2}}ds = \frac{\pi}{\delta} \int_{\delta}^{t} |b(s)|ds \quad (\delta \leq t \leq T_1).$$

By virtue of Gronwall’s inequality this shows that $b(t) = 0 \quad (\delta \leq t \leq T_1)$. The proof of Theorem 1.1 is complete.

We wish to point out that, even under the assumption that $f(t)$ is monotonically nondecreasing, there appear cases in which Theorem 1.1 is of vital importance. For instance, we consider the case $f(t) \equiv t, g(t) = (2/\sqrt{\pi})t^{1/2}$. Then it is clear that $a(t) \equiv 1$ is a solution of (0.3). Since the assumptions in Theorem 1.1 are satisfied we can apply the theorem to conclude that this trivial solution is a unique solution of (0.3).

2. Local existence

In this section we shall establish the following local existence theorem:

**Theorem 2.1.** Assume that, with some $\mu > 0$,

(i) $f(t) \in C[0, T) \cap C^1(0, T), \lim_{t \to 0} t^{1/2} - f'(t) > 0$;

(ii) $g(t) \in C_+(0, T), \lim_{t \to 0} t^{1/2} - \mu g(t) > 0$.

Then, for sufficiently small $T_0 > 0$, (0.3) has a solution $a(t) \in C_+[0, T_0]$.

Since the assumptions (i) and (ii) imply that $f'(t) > 0, g(t) > 0$ near $t = 0$, in the case $1/2 \leq \mu$, this result is a direct consequence of [5, Theorem 3]; and also, in the case $1/2 \leq \mu < 1$, of [2, Theorem 4]. We give an alternative proof of Theorem 2.1, however, in order to make the present paper readable, and in order to make the spirit in the paper transparent.

**Proof of Theorem 2.1.** Let $f(t), g(t)$ be a function satisfying (i), (ii) and put

$$P := \lim_{t \to 0} t^{1-\mu} f'(t); \quad Q := \lim_{t \to 0} t^{1/2-\mu} g(t),$$

Moreover we define a function $g_0(t)$ by

$$g_0(t) := \frac{Q/P}{B(\mu, 1/2)} \frac{f'((t-\tau))}{(t-\tau)^{1/2}}d\tau = \frac{Q/P}{B(\mu, 1/2)} t^{\mu-1/2} \int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{(1-\rho)^{1/2}\rho^{1-\mu}}d\rho,$$

and consider a mapping defined by

$$F(a(t)) = t^{1/2-\mu} a(t) \int_0^t \frac{f'(\tau)}{\left(\int_0^t a(r)dr\right)^{1/2}}d\tau - \sqrt{\pi} t^{1/2-\mu} g_0(t).$$
It is easy to see that the constant function

\[ a_0(t) := \left( \sqrt{\frac{Q}{P}} \frac{B(\mu, 1/2)}{1} \right)^2 \]

satisfies \( F(a(t)) = 0 \), and that, for each \( T_1 < T \), \( F \) is a \( C^1 \)-mapping of an open neighbourhood of \( a_0 \) in \( C[0, T_1] \) to \( C[0, T_1] \). The Fréchet derivative \( F_a(a_0) \) at \( a_0 \) is computed as,

\[
F_a(a_0) a(t) = A t^{1/2 - \mu} \left\{ \int_0^t \frac{f'(\tau)}{(t - \tau)^{1/2} d\tau} - \frac{1}{2} \int_0^t a(\tau) d\tau \right\} = A \left\{ \omega(t) a(t) - \frac{1}{2} \int_0^1 a(t\sigma) d\sigma \int_0^\sigma \frac{(t\rho)^{1-\mu} f'(t\rho)}{(1-\rho)^{3/2} d\rho} d\rho \right\},
\]

for each \( a(t) \in C[0, T_1] \). Here we set

\[
A := \left( \sqrt{\pi} \frac{Q/P}{B(\mu, 1/2)} \right)^{-1}, \quad \omega(t) := \int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{(1-\rho)^{3/2} \rho^{1-\mu} d\rho}
\]

Let \( h(t) \in C[0, T_1] \) and consider the equation

\[
F_a(a_0) a(t) = h(t), \quad (0 \leq t \leq T_1).
\]  

(2.2)

By assumption, the function \( \omega(t) \) is positive for sufficiently small \( t \). Hence, if \( T_1 \) is sufficiently small then the equation (2.2) is equivalent to

\[
a(t) - \int_0^1 \Omega(t, \sigma) a(t\sigma) d\sigma = \tilde{h}(t), \quad (0 \leq t \leq T_1),
\]

(2.3)

where we put

\[
\Omega(t, \sigma) := \frac{1}{2\omega(t)} \int_0^\sigma \frac{(t\rho)^{1-\mu} f'(t\rho)}{(1-\rho)^{3/2} \rho^{1-\mu} d\rho}, \quad \tilde{h}(t) := (A \omega(t))^{-1} h(t).
\]

By interchange of the order of integration we have

\[
\lim_{t \to 0} \int_0^1 |\Omega(t, \sigma)| d\sigma = \frac{1}{2B(1/2, \mu)} \int_0^1 d\sigma \int_0^\sigma \frac{d\rho}{(1-\rho)^{3/2} \rho^{1-\mu}} = 1/2.
\]

Therefore, by means of the Neumann series, the equation (2.3) is uniquely soluble in the space \( C[0, T_1] \), provided that \( T_1 \) is sufficiently small. This shows that \( F_a(a_0) : C[0, T_1] \to C[0, T_1] \) has a bounded linear inverse. Hence, by the implicit function theorem (see e.g. [4, Theorem 1.20]), we conclude that there exists \( \delta > 0 \) such that the equation \( F(a(t)) = h(t) \) has a solution \( a(t) \) in \( C[0, T_1] \) if \( \max_{0 \leq t \leq T_1} |h(t)| < \delta \).
We now set
\[ k(t) := \sqrt{\pi} t^{1/2 - \mu} g(2t) - \sqrt{\pi} t/2 - \mu g_{0}(1t). \]
By the definition (2.1) it follows that \( \lim_{t \to 0} k(t) = 0 \). Noting that \( \delta \) may depend on \( T_{1} \) we introduce a function \( \tilde{k}(t) \) so that \( \tilde{k}(t) = k(t) \) near 0: in \([0, T_{1}']\), say; and so that \( \max_{0 \leq t \leq T_{1}} |\tilde{k}(t)| < \delta \). Then \( F(a)(t) = \tilde{k}(t) \) has a solution \( a(t) \) in \( C[0, T_{1}] \). Then \( a(t) \) satisfies (0.3) for \( 0 \leq t \leq T_{2}' \). This completes the proof of Theorem 2.1.

3. Continuation

In this section we shall establish the following continuation theorem:

**Theorem 3.1.** Assume that
(i) \( f(t) \in C[0, T) \cap C^{1}(0, T) \);
(ii) \( g(t) \in C_{+}(0, T) \).
Let \( 0 < T_{1} < T \) and there exists a solution \( a(t) \in C_{+}[0, T_{1}] \) of (0.3). Then the solution \( a(t) \) can be continued to the right of \( T_{1} \).

The main idea of the proof of Theorem 3.1 is the use of the implicit function theorem in an appropriate function space setting. Let \( T_{2} \) be fixed so that \( T_{1} < T_{2} < T \) and define a constant function \( a_{0}(t) \) in the interval \([T_{1}, T_{2}]\) by \( a_{0}(t) \equiv a(T_{0}) \) and \( \tilde{a}(t) \) in the interval \([0, T_{2}]\) by
\[
\tilde{a}(t) := \begin{cases} a(t) & (0 \leq t \leq T_{1}), \\ a_{0}(t) & (T_{1} \leq t \leq T_{2}). \end{cases}
\]
Moreover we define a function \( g_{0}(t) \) in \([T_{1}, T_{2}]\) by
\[
g_{0}(t) := \frac{1}{\sqrt{\pi}} a_{0}(t) \int_{0}^{t} \frac{f'(_{\mathcal{T})}}{\left( \int_{\tau}^{t} \tilde{a}(r)dr \right)^{1/2}} d\tau. \tag{3.1}
\]
Let \( X \) be a function space defined by
\[
X := \{ b(t) \in C[T_{1}, T_{2}] | b(T_{1}) = 0 \}
\]
with the maximal norm, and consider the mapping
\[
F(b)(t) := (a_{0}(t) + b(t)) \int_{0}^{t} \frac{f'(_{\mathcal{T})}}{\left( \int_{\tau}^{t} \tilde{a}(r) + \tilde{b}(r)dr \right)^{1/2}} d\tau - \sqrt{\pi} g_{0}(t) \quad (T_{0} \leq t \leq T_{1}),
\]
where
\[
\tilde{b}(t) := \begin{cases} 0 & (0 \leq t \leq T_{1}), \\ b(t) & (T_{1} \leq t \leq T_{2}). \end{cases}
\]
Clearly \( F(0) = 0 \). Moreover we have:
Lemma 3.2. $F$ is a $C^1$-mapping of an open neighbourhood of 0 in $X$ to $X$. The Fréchet derivative $F_b(0)$ at 0 is written as, for $b \in X$,

$$F_b(0)b(t) = \sqrt{\pi \frac{g_0(t)}{a_0(t)}} b(t) - \frac{1}{2} a_0(t) \int_{T_1}^t b(s) ds \int_0^s \frac{f'{}^{(\tau)}}{(\int_{\tau}^t \tilde{a}(r) dr)^{3/2}} d\tau. \quad (3.2)$$

Proof of Lemma 3.2. It is easy to see that $F(b)$ is a continuous mapping of an open neighbourhood of 0 in $X$ to $X$. The Fréchet derivative $F_b(b_0)$ at $b_0$ is computed as,

$$F_b(b_0)b(t) = b(t) \int_0^t \frac{f'{}^{(\tau)}}{(\int_{\tau}^t \tilde{a}(r) dr)^{1/2}} d\tau - \frac{1}{2} (a_0(t) + (b_0(t)) \int_{T_1}^t b(s) ds \int_0^s \frac{f'{}^{(\tau)}}{(\int_{\tau}^t \tilde{a}(r) dr)^{3/2}} d\tau,$$

for $b(t) \in X$. As is easily seen, $F_b(b_0)$ is continuous in $b_0$ in the sense of operator norm. In the case $b_0 = 0$ we have

$$F_b(0)b(t) = b(t) \int_0^t \frac{f'{}^{(\tau)}}{(\int_{\tau}^t \tilde{a}(r) dr)^{1/2}} d\tau - \frac{1}{2} a_0(t) \int_{T_1}^t b(s) ds \int_0^s \frac{f'{}^{(\tau)}}{(\int_{\tau}^t \tilde{a}(r) dr)^{3/2}} d\tau,$$

which, together with (3.1), yields (3.2). The proof of Lemma 3.2 is complete.

We now let $\phi(t) \in X$ and consider the equation

$$F_b(0)b(t) = \phi(t) \quad (T_1 \leq t \leq T_2). \quad (3.3)$$

If $T_2$ is sufficiently near $T_1$ then $g_0(t) > 0$ for $T_1 \leq t \leq T_2$. Therefore (3.3) is equivalent to

$$b(t) - \int_{T_1}^t L(t, s)b(s) ds = \tilde{\phi}(t) \quad (T_1 \leq t \leq T_2), \quad (3.4)$$

where we set

$$L(t, s) := -\frac{1}{2 \sqrt{\pi}} \frac{a_0(t)^2}{g_0(t)} \int_0^s \frac{f'{}^{(\tau)}}{(\int_{\tau}^t \tilde{a}(r) dr)^{3/2}} d\tau \quad (T_1 \leq s \leq t \leq T_2),$$

$$\tilde{\phi}(t) := \frac{a_0(t)}{\sqrt{\pi g_0(t)}} \phi(t) \quad (T_1 \leq t \leq T_2).$$

Since $\tilde{a}(r) > 0$ for $0 \leq t \leq T_2$ there exists a constant $M$ such that $|L(t, s)| \leq M(t - s)^{-1/2}$. So, by a standard solving metod (see e.g [6, §39]) of the Volterra equation of the second kind, it follows that (3.4) has a unique solution $b(t)$ in $X$ for each $\phi(t) \in X$, and that the correspondence $\phi(t) \mapsto b(t)$ is a bounded linear operator in $X$. This shows that $F_b(0) : X \to X$ has a bounded linear inverse.
Hence, by the implicit function theorem (see e.g. [4, Theorem 1.20]), we conclude that there exists $\delta > 0$ such that the equation $F(b)(t) = \sqrt{\pi}(g(t) - g_0(t))$ has a solution $b(t)$ in $X$ if $\max_{T_1 \leq t \leq T_2} \sqrt{\pi}|g(t) - g_0(t)| < \delta$. Noting that $\delta$ may depend on $T_2$ we introduce a function $\tilde{g}(t)$ so that $\tilde{g}(t) = g(t)$ near $T_1$, say; and so that $\max_{T_1 \leq t \leq T_2} \sqrt{\pi}|\tilde{g}(t) - g_0(t)| < \delta$. Then $F(b)(t) = \sqrt{\pi}(\tilde{g}(t) - g_0(t))$ has a solution $b(t)$ in $X$. Noting that $\delta$ may depend on $T_2$ we introduce a function $\tilde{g}(t)$ so that $\tilde{g}(t) = g(t)$ near $T_1$, say; and so that $\max_{T_1 \leq t \leq T_2} \sqrt{\pi}|\tilde{g}(t) - g_0(t)| < \delta$. Then $F(b)(t) = \sqrt{\pi}(\tilde{g}(t) - g_0(t))$ has a solution $b(t)$ in $X$.

Using the solution $b(t)$ we set $a(t) := a_0(t) + b(t)$. Then $a(t)$ satisfies (0.3) for $T_1 \leq t \leq T_2$. This completes the proof of Theorem 3.1.

4. Alternative theorem

In this section we shall establish the following:

**Theorem 4.1.** Assume that, with some $\mu > 0$,

(i) $f(t) \in C[0, T) \cap C^1(0, T)$, $\lim_{t \to 0} t^{1-\mu} f'(t) > 0$;

(ii) $g(t) \in C_+(0, T)$, $\lim_{t \to 0} t^{1/2-\mu} g(t) > 0$.

Then a solution $a(t) \in C_+[0, T_1)$ of (0.3) that does not become infinite as $t \to T_1$ can be continued to the right of $T_1$.

An obvious consequence of Theorem 4.1 is the following:

**Corollary 4.2.** Assume (i) and (ii). If a solution $a(t) \in C_+[0, T_*)$ of (0.3) cannot be continued any further, then $\lim_{t \to T_*} a(t) = +\infty$.

We base the proof of Theorem 4.1 on the following *a priori* property of solutions $a(t)$ of (0.3):

**Lemma 4.3.** Under the same assumption as in Theorem 4.1, a solution $a(t) \in C_+[0, T_1)$ of (0.3) for some $T_1 < T$ satisfies $\inf_{0 \leq t < T_1} a(t) > 0$.

**Proof.** Let $T'_1 < T_1$. From (1.1) we have for $0 \leq t \leq T'_1$,

$$0 < \sqrt{\pi} \min_{0 \leq t \leq T_1} (t^{1/2-\mu} g(t)) \leq \frac{a(t) \int_0^1 (\rho)^{1-\mu} f'(t \rho) \frac{d\rho}{(\int_0^1 \rho a(t \rho) dr)^{1/2}}(\rho^{1-\mu})}{\rho^{1-\mu}} \leq a(t) \max_{0 \leq t \leq T_1} |t^{1-\mu} f'(t)| \left( \min_{0 \leq t \leq T_1} a(t) \right)^{1/2} \int_0^1 \left( 1 - \rho \right)^{1/2} \rho^{1-\mu},$$

which yields

$$\frac{\sqrt{\pi}}{B(1/2, \mu)} \min_{0 \leq t \leq T_1} (t^{1/2-\mu} g(t)) \left( \frac{\max_{0 \leq t \leq T_1} |t^{1-\mu} f'(t)|}{\min_{0 \leq t \leq T_1} a(t)} \right)^{1/2} \leq \left( \min_{0 \leq t \leq T_1} a(t) \right)^{1/2}$$

Noting that the left side is a constant independent of $T'_1$, we complete the proof.

Lemma 4.3 leads to the following *alternative* for a solution of (0.3):
Lemma 4.4. Assume (i) and (ii) in Theorem 4.1, and let $a(t) \in C_+ [0, T_1]$ be a solution of (0.3) for some $T_1 < T$. Then, either $a(t)$ tends to a finite, positive value as $t \to T_1$: $0 < \lim_{t \to T_1} a(t) < \infty$; or $a(t)$ tends to infinity as $t \to T_1$: $\lim_{t \to T_1} a(t) = +\infty$.

Proof. We proceed in two steps.

**Step 1.** We shall show that if $\liminf_{t \to \infty} a(t) < \infty$ then $\sup_{0 \leq t < T_1} a(t) < \infty$. By the assumption there exists a sequence $\{t_k\}_{k=1}^{\infty} \to T_1$ as $k \to \infty$, such that

$$\sup_{k} a(t_k) \leq M_1 < \infty,$$

with some constant $M_1$ independent of $k$. The equation (0.3) can be rewritten as

$$\sqrt{\pi} g(t) = a(t) \int_0^{t_k} \frac{f'(\tau)}{(\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2}} d\tau + a(t) \int_0^{t_k} \frac{1}{(\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2}} - \frac{1}{(\int_{\mathcal{T}}^{t} a(r)dr)^{1/2}} f'(\tau) d\tau + a(t) \int_{t_k}^{t} \frac{f'((\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2}}{d\tau}.$$

Hence we have

$$\sqrt{\pi} g(t) = \sqrt{\pi} \frac{g(t_k)}{a(t_k)} a(t) + I_1(t, t_k) + I_2(t, t_k),$$

where

$$I_1(t, t_k) := -a(t) \int_{t_k}^{t} a(r)dr \times \int_0^{t_k} \frac{f'((\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2}}{d\tau} \left\{ \left(\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2} + (\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2}\right) \right\} d\tau,$$

$$I_2(t, t_k) := a(t) \int_{t_k}^{t} \frac{f'((\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2}}{d\tau}.$$

By subtracting $g(t_k)$ from (4.2) we get

$$\sqrt{\pi} (a(t) - a(t_k)) = \sqrt{\pi} \frac{g(t_k)}{a(t_k)} (g(t) - g(t_k)) - \frac{a(t_k)}{g(t_k)} I_1(t, t_k) - \frac{a(t_k)}{g(t_k)} I_2(t, t_k)$$

for $t \geq t_k$. By setting

$$b_k(t) := a(t) - a(t_k), \ \varphi(t, t_k) :=$$

$$= \int_0^{t_k} \frac{f'((\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2}} {d\tau} \left\{ \left(\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2} + (\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2}\right) \right\} d\tau,$$

$$\psi(t, t_k) := \int_{t_k}^{t} \frac{f'((\int_{\mathcal{T}}^{t_k} a(r)dr)^{1/2}}{d\tau},$$
we obtain

\[
I_1(t, t_k) = -(b_k(t) + a(t_k)) \int_{t_k}^{t} (b_k(r) + a(t_k)) dr \varphi(t, t_k)
\]

\[
= -\varphi(t, t_k) b_k(t) \int_{t_k}^{t} b_k(r) dr - a(t_k) \varphi(t, t_k) \int_{t_k}^{t} b_k(r) dr
\]

\[- b_k(t)(t-t_k)a(t_k)\varphi(t, t_k) - a(t_k)^2(t-t_k)\varphi(t, t_k),
\]

\[
I_2(t, t_k) = b_k(t)\psi(t, t_k) + a(t_k)\psi(t, t_k).
\]

Substituting this in (4.3) shows that

\[
\left[ \sqrt{\pi} - \frac{a(t_k)^2}{g(t_k)}(t-t_k)\varphi(t, t_k) + \frac{a(t_k)}{g(t_k)}\psi(t, t_k) \right] b_k(t)
\]

\[
= A(t) + \frac{a(t_k)^2}{g(t_k)} (t-t_k)\varphi(t, t_k) + \frac{a(t_k)}{g(t_k)} b_k(t) \varphi(t, t_k) \int_{t_k}^{t} b_k(r) dr,
\]

where we put

\[
A(t) := \sqrt{\pi} \frac{a(t_k)}{g(t_k)} (g(t) - g(t_k)) + \frac{a(t_k)^3}{g(t_k)}(t-t_k)\varphi(t, t_k) - \frac{a(t_k)^2}{g(t_k)} \psi(t, t_k).\]

We now set \( m_a := \inf_{0 \leq t < T_1} a(t), M_f := \max_{0 \leq t \leq T_1} |f'(t)|. \) Note that \( m_a > 0 \) by Lemma 4.3. It follows that for \( t_k \leq t < T_1 \)

\[
|\varphi(t, t_k)| \leq \frac{M_f}{m_a^{3/2}} \int_{0}^{t_k} \frac{d\tau}{(t-\tau)(t_k-\tau)^{1/2}} \leq \frac{M_2}{(t-t_k)^{1/2}}
\]

\[
|\psi(t, t_k)| \leq \frac{M_f}{m_a^{1/2}} \int_{t_k}^{t} \frac{d\tau}{(t-\tau)^{1/2}} \leq M_2(t-t_k)^{1/2}
\]

(4.4)

with a constant \( M_2 \) independent of \( k \). This, together with (4.2), shows that

\[
\left| - \frac{a(t_k)^2}{g(t_k)} (t-t_k)\varphi(t, t_k) + \frac{a(t_k)}{g(t_k)} \psi(t, t_k) \right| \leq \sqrt{\pi} - 1 \quad (k \geq N_1),
\]

if we take \( N_1 \) sufficiently large. Accordingly, from (4.1) and (4.4), we have

\[
|b_k(t)| \leq |A(t)| + \frac{M_3 + M_4 |b_k(t)|}{(t-t_k)^{1/2}} \int_{t_k}^{t} |b_k(s)| ds \quad (k \geq N_1).
\]

So, for \( k \geq N_1 \), if \( |b_k(t)| \leq 1 \) then for \( t_k \leq t < T_1 \),

\[
|b_k(t)| \leq |A(t)| + \frac{M_3 + M_4}{(t-t_k)^{1/2}} \int_{t_k}^{t} |b_k(r)| dr
\]

\[
\leq |A(t)| + \frac{M_3 + M_4}{(t-t_k)^{1/2}} \int_{t_k}^{t} \left\{ |A(r)| + \frac{M_3 + M_4}{(r-t_k)^{1/2}} \int_{t_k}^{r} |b_k(s)| ds \right\} dr
\]

\[
\leq B(t) + (M_3 + M_4)^2 \int_{t_k}^{t} |b_k(s)| ds,
\]
where

$$B(t) := |A(t)| + \frac{M_3 + M_4}{(t-t_k)^{1/2}} \int_{t_k}^{t} |A(r)| \, dr.$$  

By the definition of $A(t)$ and (4.4), it follows that $\lim_{t \to t_k} B(t) = 0$ uniformly with respect to $k$. This, together with Gronwall's inequality, shows that $\lim_{t \to t_k} b_k(t) = 0$ uniformly with respect to $k$. Hence if we take $N(= N_1)$ sufficiently large then $|b_k(t)| \leq 1/2$ for $k \geq N$, $t_k \leq t < T_1$, provided that $|b_k(t)| \leq 1$. In other words, for $k \geq N$, $t_k \leq t \leq T_1$, either $|b_k(t)| \geq 1$ or $|b_k(t)| \leq 1/2$. But the former does not occur because $b_k(t)$ is a continuous function in the interval with $b_k(t_k) = 0$. Thereby we conclude that there exists a number $N$ such that, for $k \geq N$, $a(t) \leq a(t_k) + 1/2$ in the interval $t_k \leq t < T_1$. This shows that $\sup_{0 \leq t < T_1} a(t) < \infty$.

**Step 2.** We shall show that if $\sup_{0 \leq t < T_1} a(t) < \infty$ then $a(t)$ tends to a finite, positive value as $t \to T_1$. Let $T_0 \leq s \leq t < T_1$. Using (4.3) we have

$$\sqrt{\pi}(a(t) - a(s)) = \sqrt{\pi} \frac{a(s)}{g(s)} (g(t) - g(s)) - \frac{a(s)}{g(s)} I_1(t, s) \leq \frac{a(s)}{g(s)} I_2(t, s) = \sqrt{\pi} \frac{a(s)}{g(s)} (g(t) - g(s)) + \frac{a(s)a(t)}{g(s)} \int_{s}^{t} a(r) \varphi(t, s) - \frac{a(s)a(t)}{g(s)} \varphi(t, s).$$

It follows from this equality, the assumption $\sup_{0 \leq t < T_1} a(t) < \infty$, (4.4), and the uniform continuity of $g(t)$ that $a(t)$ is uniformly continuous on $[0, T_1]$. Hence $a(t)$ is extended as a continuous function on $[0, T_1]$. The proof of Lemma 4.4 is complete.

We now give the

**Proof of Theorem 4.1.** If a solution $a(t) \in C_+(0, T_1)$ of (0.3) does not become infinite as $t \to T_1$, then, by Lemma 4.4, $a(t)$ is extended as a positive solution on $[0, T_1]$. So, by Theorem 3.1, $a(t)$ can be continued to the right of $T_1$. The proof of Theorem 4.1 is complete.

We treat the case when $f'(t) \geq 0$. The following result is useful.

**Lemma 4.5.** In addition to the assumption in Theorem 4.1 we assume that $f'(t) \geq 0$ for each $t \in (0, T)$. Then any solution $a(t) \in C_+[0, T_1)$ of (0.3) for some $T_1 < T$ satisfies $\sup_{0 \leq t < T_1} a(t) < \infty$.

**Proof.** Let $T_1' < T_1$. It follows from (1.1) that

$$\sqrt{\pi} \max_{0 \leq t \leq T_1} (t^{1/2-\mu} g(t)) \geq a(t) \int_{0}^{1} \frac{(t \rho)^{1-\mu} f'(t \rho)}{(1-\rho)^{1/2} \rho^{1-\mu}} \, d\rho \geq \frac{a(t)}{\max_{0 \leq t \leq T_1} a(t)} \min_{0 \leq t \leq T_1} \int_{0}^{1} \frac{(t \rho)^{1-\mu} f'(t \rho)}{(1-\rho)^{1/2} \rho^{1-\mu}} \, d\rho.$$
for $0 \leq t \leq T_1'$. Since the function
\[ \int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{(1-\rho)^{1/2} \rho^{1-\mu}} d\rho \]
is a positive, continuous function on $[0, T_1]$, we get
\[ \sqrt{\pi} \left( \min_{0 \leq t \leq T_1} \int_0^1 \frac{(t\rho)^{1-\mu} f'(t\rho)}{(1-\rho)^{1/2} \rho^{1-\mu}} d\rho \right)^{-1} \max_{0 \leq t \leq T_1} (t^{1/2-\mu} g(t)) \geq \left( \max_{0 \leq t \leq T_1} a(t) \right)^{1/2}. \]
Noting the left side is a constant independent of $T_1'$ we complete the proof.

By virtue of Lemma 4.5 the following is an immediate consequence of Theorem 4.1:

**Corollary 4.6.** In addition to the assumptions in Theorem 4.1 we assume that $f'(t) \geq 0$ for each $t \in (0, T)$. Then (0.3) has a solution $a(t) \in C_+[0, T)$.

We wish to point out that Corollary 4.5 is also obtained immediately by [5, Chap 1, Theorem 3]. In the case $1/2 \leq \mu < 1$ this follows also from [2].

**References**