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A Gibbs Theory for States of Boson Systems

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Abstract

Based on certain conditional expectations leading to conditional local states we provide a Gibbs Theory analogous to [20, 21]. As example, we look at specifications corresponding to coherent states and the ideal Bose gas.

1 Introduction

In classical statistical mechanics one wants to determine equilibrium states for infinite particle systems, suppose the interaction between the particles is given. The first possible definition is proposed by the so called thermodynamical limit. Dobrushin, Lanford and Ruelle [6, 16] proposed, based on an equivalence theorem for lattice systems, another scheme: The equilibrium state should be invariant under the action of certain (sub-) stochastic kernels. Related to the potential Φ these kernels are formally determined as $Q_\Phi(\,\cdot\,|\mathfrak{M}_{\Lambda^c})$, the conditional probability inside a bounded region Λ under the condition of the outside σ-field $\mathfrak{M}_{\Lambda^c}$.

The situation in quantum statistical mechanics is not so simple. One can introduce the same formalism [1], but the main problem is the existence of the conditional expectations [2, 22]. The requirement of the existence of such norm one projections which leave the state invariant would shrink to much the set of possible (equilibrium) states. From [18] we know that for locally normal states of boson system there exist some analogon, the conditional local states. This approach avoids the problems of existence of conditional expectations as it works essentially with projections on the center of certain von Neumann algebras. From this approach we establish a clear-cut connection to classical abstract theory of Gibbs measures as developed in [20, 21].

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2 Preliminaries

Let G be a complete separable metric space. The (bounded) Borel sets are $\mathfrak{B}$ and $\mathfrak{G}$ respectively. We use the description $\Lambda^c = G\setminus\Lambda$ for the complement of $\Lambda \in \mathfrak{G}$ and $\mathbb{1}_\Lambda$ for the indicator function

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of Λ. δ_x is the Dirac measure concentrated in $x \in G$, $\mathfrak{o}$ denotes the zero measure on $\mathfrak{G}$. Further, we fix a measure ν on $(G, \mathfrak{G})$ which is locally finite ($\nu(\Lambda) < \infty$ for all $\Lambda \in \mathfrak{B}$).

Now let M be the set of all locally finite counting measures on G, i.e.

$$M = \{\varphi : \varphi \text{ is measure on } \mathfrak{G},\ \varphi(\Lambda) \in \mathbb{N} \text{ for all } \Lambda \in \mathfrak{B}\}.$$

If $\Lambda \in \mathfrak{G}$ and $\varphi \in M$ we denote by φ_Λ the measure $\varphi_\Lambda(\cdot) = \varphi(\cdot \cap \Lambda)$. Under abuse of notation, we write $x \in \varphi$ if $\varphi(\{x\}) > 0$. $\mathfrak{M}_\Lambda$ is the smallest σ-field on M making the maps $\varphi \mapsto \varphi(\Lambda')$ measurable for all $\Lambda' \in \mathfrak{B}, \Lambda' \subseteq \Lambda$. We set $\mathfrak{M} = \mathfrak{M}_G$ and call a probability measure on $(M, \mathfrak{M})$ *point process.*

Remark 1 It is well-known [19] that any $\varphi \in M$ is a countable sum of Dirac measures. So we can interprete φ as point configuration and a point process as distribution of a random point configuration.

For $\Lambda \in \mathfrak{G}$ we define a σ-finite measure F_Λ on $(M, \mathfrak{M})$ by setting for $Y \in \mathfrak{M}$

$$F_\Lambda(Y) = \mathbb{1}_Y(\mathfrak{o}) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \mathbb{1}_Y(\textstyle\sum_{i=1}^n \delta_{x_i}) \nu^{\otimes n}(\mathrm{d}x_1, \ldots, \mathrm{d}x_n). \tag{1}$$

We set $\mathcal{M}_\Lambda = L^2_{\mathbb{C}}(M, \mathfrak{M}, F_\Lambda)$. Note that $\mathcal{M} = \mathcal{M}_G$ is canonically isomorphic to the symmetric Fock space over $L^2(G, \nu)$ (cf. e.g. [13, Satz 2.5]).

Now we introduce the quasilocal algebra corresponding to locally finite boson systems. If $\Lambda, \Lambda' \in \mathfrak{G}$ are disjoint $\mathcal{M}_\Lambda \otimes \mathcal{M}_{\Lambda'}$ and $\mathcal{M}_{\Lambda \cup \Lambda'}$ are isomorphic under the isomorphism $I_{\Lambda,\Lambda'}$ characterized by

$$I_{\Lambda,\Lambda'}(\psi_1 \otimes \psi_2)(\varphi) = \psi_1(\varphi_\Lambda)\psi_2(\varphi_{\Lambda'}) \qquad F\text{–a.e.}$$

for all $\psi_2 \in \mathcal{M}_\Lambda$, $\psi_2 \in \mathcal{M}_{\Lambda'}$. Under this identification we set for $\Lambda \in \mathfrak{B}$

$$\mathcal{A}_\Lambda = \mathfrak{L}(\mathcal{M}_\Lambda) \otimes \mathbb{1}_{\Lambda^c} \hookrightarrow \mathfrak{L}(\mathcal{M}).$$

Thereby $\mathbb{1}_{\Lambda^c}$ is the identity in $\mathfrak{L}(\mathcal{M}_{\Lambda^c})$. The algebra of quasilocal observables is given by $\mathcal{A} = \overline{\bigcup_{\Lambda \in \mathfrak{B}} \mathcal{A}_\Lambda}$, where the bar denotes the closure in the uniform topology of $\mathfrak{L}(\mathcal{M})$. Then the pair $(\mathcal{A}, (\mathcal{A}_\Lambda)_{\Lambda \in \mathfrak{B}})$ is a (bosonic) quasilocal algebra in the sense of [3, Definition 2.6.3].

As usual a state ω on $\mathcal{A}$ is a positive continuous linear functional on $\mathcal{A}$.

Definition 1 (cf. [3, Definition 2.6.6]) *A state ω on $\mathcal{A}$ is called* locally normal state *if for all $\Lambda \in \mathfrak{B}$ there is a trace class operator ϱ_Λ on $\mathcal{M}_\Lambda$ such that for all $A \in \mathfrak{L}(\mathcal{M}_\Lambda)$*

$$\omega_\Lambda(A \otimes \mathbb{1}_{\Lambda^c}) = \omega(A \otimes \mathbb{1}_{\Lambda^c}) = \operatorname{tr}(\varrho_\Lambda A).$$

Now one can search for good descriptions for locally normal states. A first possibility is to fix $(\omega_\Lambda)_{\Lambda \in \mathfrak{B}}$. But the compatibility condition is easy to handle. Another method developed in [10, 14] is the characterization of locally normal states by their position distribution and the conditional local states. As it was proved in [17] this provides a complete description for all locally normal states.

For $Y \in \mathfrak{M}$ let $\mathrm{O}_Y \in \mathfrak{L}(\mathcal{M})$ be the operator of multiplication by $\mathbb{1}_Y$, i.e. for all $\psi \in \mathcal{M}$

$$(\mathrm{O}_Y \psi)(\varphi) = \mathbb{1}_Y(\varphi)\psi(\varphi) \qquad F\text{–a.e.}$$

Obviously, for $\Lambda \in \mathfrak{B}$ and $Y \in \mathfrak{M}_\Lambda$ we have $\mathrm{O}_Y \in \mathcal{A}_\Lambda$. A special case of the following result was already proved in [9] but it is valid also in the above described general situation.

Proposition 1 (cf. [10, Theorem 2.15]) *For any locally normal state on $\mathcal{A}$ there exists exactly one point process P_ω on G which fulfills for all $\Lambda \in \mathfrak{B}$ and all $Y \in \mathfrak{M}_\Lambda$*

$$\mathrm{P}_\omega(Y) = \omega(\mathrm{O}_Y). \qquad \blacksquare$$

Remark 2 O_Y is interpreted as position measurement. Thus we call P_ω the *position distribution* of ω.

Theorem 2 ([18, Theorem 3]) *Let ω be a locally normal state on $\mathcal{A}$. Then there is a family $\left(\omega_\Lambda^\varphi\right)_{\Lambda\in\mathfrak{B},\varphi\in M}$ fulfilling for all $\Lambda \in \mathfrak{B}$ the following conditions:*

(i) *For all $\varphi \in M$ ω_Λ^φ is a (normal) state on $\mathcal{A}_\Lambda$.*

(ii) *For all $A \in \mathcal{A}_\Lambda$ the mapping $\varphi \mapsto \omega_\Lambda^\varphi(A)$ is $\mathfrak{M}_{\Lambda^c}$ measurable.*

(iii) *For all $A \in \mathcal{A}_\Lambda$ and all $Y \in \mathfrak{M}_{\Lambda'}$ for some $\Lambda' \in \mathfrak{B}$, $\Lambda \cap \Lambda' = \emptyset$ it holds*

$$\omega(A\mathrm{O}_Y) = \int_Y \mathrm{P}_\omega(\mathrm{d}\varphi)\,\omega_\Lambda^\varphi(A). \tag{2}$$

If $\left(\theta_\Lambda^\varphi\right)_{\Lambda\in\mathfrak{B},\varphi\in M}$ is another family fulfilling (i)–(iii) *then for all $\Lambda \in \mathfrak{B}$ for P_ω-a.a. $\varphi \in M$*

$$\omega_\Lambda^\varphi = \theta_\Lambda^\varphi. \blacksquare$$

The states ω_Λ^φ are called *conditional local states.*

Remark 3 The conditional local states are related to certain conditional expectations. Let $(H_\omega, \pi_\omega, \Omega_\omega)$ be the GNS-triple w.r.t. the locally normal state ω. The position distribution is the restriction of $\langle \Omega_\omega, \cdot\, \Omega_\omega\rangle$ to the range of the canonical embedding of $L^\infty(M, \mathfrak{M}, \mathrm{P}_\omega)$ into $\pi_\omega(\mathcal{A})''$. Then

$$E_\Lambda^\omega(A) = \varphi \mapsto \omega_\Lambda^\varphi(A(\varphi)).$$

is the (Umegaki) conditional expectation from the algebra given by the embedding of

$$\mathcal{A}_\Lambda \otimes L^\infty(M, \mathfrak{M}_{\Lambda^c}, \mathrm{P}_\omega) \hookrightarrow \pi_\omega(\mathcal{A})''$$

onto the embedding of $L^\infty(M, \mathfrak{M}_{\Lambda^c}, \mathrm{P}_\omega)$. The latter being the center of $\mathcal{A}_\Lambda \otimes L^\infty(M, \mathfrak{M}_{\Lambda^c}, \mathrm{P}_\omega)$ assures existence of the conditional expectation [22]. In general, we work with the conditional local states, but the GNS picture is useful for reference.

Definition 2 *The* tail–field $\mathfrak{M}^\infty$ *is defined as* $\mathfrak{M}^\infty = \bigcap_{\Lambda\in\mathfrak{B}} \mathfrak{M}_{\Lambda^c}$. *We say that* a point process Q is $\mathfrak{M}^\infty$ trivial *if $Q(Y) \in \{0, 1\}$ for all $Y \in \mathfrak{M}^\infty$.*

Remark 4 Sometimes $\mathfrak{M}^\infty$ measurable bounded functions are called observables at infinity (see [16]). They are interpreted as quantities which can be observed from outside the system.

There exists also a conditional expectation onto the tail field $\mathfrak{M}^\infty$. Namely there are locally normal states $({}^\infty\omega^\varphi)_{\varphi\in M}$ fulfilling P_ω-a.s. for all $A \in \bigcup_{\Lambda\in\mathfrak{B}} \mathcal{A}_\Lambda$

$${}^\infty\omega^\varphi(A) = \lim_{n\to\infty} \omega_{\Lambda_n}^\varphi(A)$$

where $(\Lambda_n)_{n\in\mathbb{N}} \subset \mathfrak{B}$ is a cofinal sequence. Moreover, for all $A \in \mathcal{A}_\Lambda$

$$^{\infty}\omega^{\varphi}(A) = \mathbb{E}_{\mathrm{P}_\omega}\left(\omega_\Lambda^{(\cdot)}(A)\middle|\ \mathfrak{M}^{\infty}\right)(\varphi) \qquad \mathrm{P}_\omega\text{–a.s.}$$

and for all $A \in \mathcal{A}$

$$\omega(A) = \int \mathrm{P}_\omega(\mathrm{d}\varphi)\, ^{\infty}\omega^{\varphi}(A). \tag{3}$$

The choice of position distribution and conditional local states to determine some locally normal state is not completely arbitrary.

Lemma 3 *Let ω be a locally normal state and $\Lambda \in \mathfrak{B}$. The P_ω-a.s. for all $Y \in \mathfrak{M}_\Lambda$*

$$\omega_\Lambda^{\varphi}(\mathrm{O}_Y) = \mathrm{P}_\omega\left(Y \mid \mathfrak{M}_{\Lambda^c}\right)(\varphi).\blacksquare$$

Lemma 4 *For $\Lambda', \Lambda \in \mathfrak{B}$ with $\Lambda \supseteq \Lambda'$ it holds true for P_ω-a.a. $\varphi \in M$, all $A \in \mathcal{A}_\Lambda$ and all $Y \in \mathfrak{M}_{\Lambda\setminus\Lambda'}$ that*

$$\int_Y \mathrm{P}_\omega\left(\mathrm{d}\tilde{\varphi} \mid \mathfrak{M}_{\Lambda^c}\right)(\varphi)\omega_{\Lambda'}^{\tilde{\varphi}_\Lambda+\varphi_{\Lambda^c}}(A) = \omega_\Lambda^{\varphi}(A\mathrm{O}_Y).\blacksquare$$

Remark 5 This relation is just the projectivity of the conditional expectations related to the conditional local states.

3 Classical Gibbs Theory for Point Processes

We just rephrase the so called Gibbs formalism used in [15,20,21] to formalize the conditional expectations $Q\left(\cdot \mid \mathfrak{M}_{\Lambda^c}\right)(\varphi)$ for point processes Q.

Definition 3 (cf. [21, Definition 2.1.1]) *Let be $\mathfrak{C} \subseteq \mathfrak{B}$ a set system. A triple $(\pi, \mathfrak{R}, \mathfrak{C})$ where $\pi = (\pi_\Lambda)_{\Lambda\in\mathfrak{C}}$, $\mathfrak{R} = (R_\Lambda)_{\Lambda\in\mathfrak{C}}$ is called* local specification with sets of regularity $\mathfrak{R}$ *if*

(LS1) *$R_\Lambda \in \mathfrak{M}_{\Lambda^c}$ for all $\Lambda \in \mathfrak{C}$.*

(LS2) *For all $\Lambda \in \mathfrak{C}$ we have $\pi_\Lambda : M \times \mathfrak{M} \longmapsto [0,1]$ and $\pi_\Lambda(\varphi, \cdot)$ is for all $\varphi \in R_\Lambda$ a probability measure on $\mathfrak{M}$.*

(LS3) *If $\Lambda \in \mathfrak{C}$ and $\varphi \notin R_\Lambda$ then $\pi_\Lambda(\varphi, \cdot) = 0$.*

(LS4) *For all $\Lambda \in \mathfrak{C}$ and $Y \in \mathfrak{M}$ the function $\pi_\Lambda(\cdot, Y)$ is $\mathfrak{M}_{\Lambda^c}$ measurable.*

(LS5) *If $\Lambda \in \mathfrak{C}$, $\varphi \in M$ and $Y \in \mathfrak{M}$, $Z \in \mathfrak{M}_{\Lambda^c}$ then*

$$\pi_\Lambda(\varphi, Y \cap Z) = \mathbb{1}_Z(\varphi)\pi_\Lambda(\varphi, Y).$$

(LS6) *For all* $\Lambda, \Lambda' \in \mathfrak{C}$, $\Lambda \subseteq \Lambda'$ *and all* $\varphi \in M$:

$$\pi_{\Lambda'} * \pi_\Lambda(\varphi, \cdot) = \int \pi_{\Lambda'}(\varphi, \mathrm{d}\tilde{\varphi})\pi_\Lambda(\tilde{\varphi}, \cdot) = \pi_{\Lambda'}(\varphi, \cdot).$$

We will assume that $\mathfrak{C}$ contains a cofinal sequence and $G \notin \mathfrak{C}$.

Remark 6 The regularity sets are introduced due to the following reason. In some examples the natural way to define the "conditional distributions" $\pi_\Lambda(\varphi, \cdot)$ may fail for some "irregular" configurations φ. The way out is the assumption that invariant measures Q are concentrated on regular configurations. On the other hand the measures $Q(\,\cdot \mid \mathfrak{M}_{\Lambda^c})(\varphi)$ are defined only on a Q essential set of φ so that we can redefine them on the inessential set of irregular configurations. If we set them 0 on that set then that set has automatically Q measure zero for all invariant measures Q.

Moreover, the standard construction of π is like follows: Let $\Phi : M \longmapsto \mathbb{R}$ be the interaction of some given *finite* point configuration. The energy function H is then given by $H(\varphi) = \sum_{\hat{\varphi} \leq \varphi} \Phi(\hat{\varphi})$, in general this is infinite if φ is itself infinite. For $\Lambda \in \mathfrak{B}$ define the conditional energy function $H_\Lambda^\varphi : M_\Lambda \longmapsto \mathbb{R}$ by

$$H_\Lambda^\varphi(\hat{\varphi}) = \sum_{\substack{\tilde{\varphi} \leq \varphi_{\Lambda^c} + \hat{\varphi} \\ \tilde{\varphi}(\Lambda) > 0}} \Phi(\tilde{\varphi}). \tag{4}$$

Such a way we can interprete $H_\Lambda^\varphi(\hat{\varphi})$ as the energy difference occuring if we add the configuration $\hat{\varphi}$ to the points of the configuration φ. The ansatz for the conditional probability in equilibrium w.r.t. H is

$$Q(Y \mid \mathfrak{M}_{\Lambda^c})(\varphi) := \pi_\Lambda^H(Y, \varphi) = \frac{\int_Y F(\mathrm{d}\varphi_1) \exp\{-H_\Lambda^\varphi(\varphi_1)\}}{\int_{M_\Lambda} F(\mathrm{d}\varphi_1) \exp\{-H_\Lambda^\varphi(\varphi_1)\}}. \tag{5}$$

Unfortunately, H_Λ^φ is an infinite series and we face again a convergence problem. Now we set

$$R_\Lambda = \{\varphi : \sum_{\substack{\tilde{\varphi} \leq \varphi_{\Lambda^c} + \hat{\varphi} \\ \tilde{\varphi}(\Lambda) > 0}} \Phi(\tilde{\varphi}) \text{ converges absolutely}\}$$

and the associated local specification π^H is well defined. Recently [7] it was pointed out that pointwise convergence of the sum in (4) is the right object in this framework, as stronger assumptions shrink to much the set of feasible Gibbs measures.

Definition 4 *A point process Q is called* Gibbs process w.r.t. the local specification $(\pi, \mathfrak{R}, \mathfrak{C})$ *if it fulfils for all $\Lambda \in \mathfrak{C}$ and $Y \in \mathfrak{M}$ the DLR equations*

$$Q(Y) = \int Q(\mathrm{d}\varphi)\, \pi_\Lambda(\varphi, Y). \tag{6}$$

We denote the set of Gibbs processes by $\mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C})$.

The DLR equations determine the conditional distributions.

Lemma 5 *Q fulfils $Q \in \mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C})$ iff for all $\Lambda \in \mathfrak{C}$ it holds Q-a.s. for all $Y \in \mathfrak{M}_\Lambda$*

$$\pi_\Lambda(Y, \varphi) = Q(Y \mid \mathfrak{M}_{\Lambda^c})(\varphi). \blacksquare$$

Due to [21] the set $\mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C})$ is closed w.r.t. mixings, especially convex. The set of extremal Gibbs processes is denoted by $\mathrm{ex}\mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C})$.

The tail-field plays an important rôle in classical Gibbs theory.

Lemma 6 ([20, Theorem 2.1]) (i) *$Q \in \mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C})$ iff Q is trivial on $\mathfrak{M}^\infty$.*

(ii) *If it holds $Q_{1\restriction\mathfrak{M}^\infty} = Q_{2\restriction\mathfrak{M}^\infty}$ for $Q_1, Q_2 \in \mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C})$ then $Q_1 = Q_2$.* ∎

In the case $\mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C}) \neq \emptyset$ one gets an entrance boundary for all Gibbs processes:

Proposition 7 ([20, Theorem 2.2]) *Assume the local specification $(\pi, \mathfrak{R}, \mathfrak{C})$ has at least one Gibbs process. Then there is a stochastic kernel π^∞ from $\mathfrak{M}$ to $\mathfrak{M}^\infty$ with the following properties:*

(π^∞1) *For all $\varphi \in M$ we have $\pi^\infty(\varphi, \cdot\,) \in \mathrm{ex}\mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C})$.*

(π^∞2) *For all $Q \in \mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C})$ and $Y \in \mathfrak{M}$*

$$Q\,(Y \mid \mathfrak{M}^\infty)(\,\cdot\,) = \pi^\infty(\,\cdot\,, Y) \qquad Q\text{–a.s.}$$

(π^∞3) *For all $\varphi \in M$*

$$\pi^\infty(\varphi, \{\hat{\varphi} : \pi^\infty(\hat{\varphi}, \cdot\,) = \pi^\infty(\varphi, \cdot\,)\}) = 1.$$

(π^∞4) *$Q \in \mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C})$ is true iff for all $Y \in \mathfrak{M}$*

$$Q(Y) = \int Q(\mathrm{d}\varphi)\, \pi^\infty(\varphi, Y). \qquad ∎$$

Formally π_Λ is a measure on the whole $\mathfrak{M}$, but for arbitrary $Y \in \mathfrak{M}$ it holds (cf. [21, Satz 2.1.4])

$$\pi_\Lambda(\varphi, Y) = \pi_\Lambda(\varphi, \{\hat{\varphi} : \hat{\varphi}_\Lambda + \varphi_{\Lambda^c} \in Y\}).$$

Thus π_Λ is fixed by its values on $\mathfrak{M}_\Lambda$. Following this we get the following connection to purely locally defined local specifications:

Lemma 8 *Let $\mathfrak{R}$ and $\mathfrak{C}$ be as above. For a family $(\tilde{\pi}_\Lambda)_{\Lambda \in \mathfrak{C}}$ of maps from $M \times \mathfrak{M}_\Lambda$ to $[0,1]$ consider the following conditions:*

(LS′1) *$\tilde{\pi}_\Lambda(\varphi, \cdot\,)$ is for all $\Lambda \in \mathfrak{C}$ and all $\varphi \in R_\Lambda$ a probability measure on $\mathfrak{M}_\Lambda$.*

(LS′2) *If $\Lambda \in \mathfrak{C}$ and $\varphi \notin R_\Lambda$ then $\tilde{\pi}_\Lambda(\varphi, \cdot\,) = 0$.*

(LS′3) *If $\Lambda \in \mathfrak{C}$ and $Y \in \mathfrak{M}_\Lambda$ then $\tilde{\pi}_\Lambda(\,\cdot\,, Y)$ is $\mathfrak{M}_{\Lambda^c}$ measurable.*

(LS′4) *For all $\Lambda, \Lambda' \in \mathfrak{C}$, $\Lambda \subseteq \Lambda'$ all $\varphi \in M$ and all $Y' \in \mathfrak{M}_{\Lambda' \setminus \Lambda}$, $Y \in \mathfrak{M}_\Lambda$ it holds*

$$\tilde{\pi}_{\Lambda'}(\varphi, Y' \cap Y) = \int_{Y'} \tilde{\pi}_{\Lambda'}(\varphi, \mathrm{d}\tilde{\varphi})\tilde{\pi}_\Lambda(\tilde{\varphi}_{\Lambda'} + \varphi_{(\Lambda')^c}, Y). \tag{7}$$

Such a family $\tilde{\pi}$ corresponds uniquely to a local specification π which is determined by

$$\pi_\Lambda(\varphi, Y) = \tilde{\pi}_\Lambda(\varphi, \{\tilde{\varphi} : \tilde{\varphi}_\Lambda + \varphi_{\Lambda^c} \in Y\}).$$

Moreover $Q \in \mathcal{GP}(\pi, \mathfrak{R}, \mathfrak{C})$ iff it holds Q-a.s. for all $Y \in \mathfrak{M}_\Lambda$

$$\tilde{\pi}_\Lambda(Y, \varphi) = Q\,(Y \mid \mathfrak{M}_{\Lambda^c})\,(\varphi).$$

Proof: Assume we are given the local specification $(\pi, \mathfrak{R}, \mathfrak{C})$. For all $\Lambda \in \mathfrak{C}$ and $Y \in \mathfrak{M}_\Lambda$ it follows

$$\tilde{\pi}_\Lambda(\varphi, Y) = \pi_\Lambda(\varphi, Y).$$

It is easy to see that $\tilde{\pi}$ fulfills (LS′1)–(LS′3). The condition (LS′4) we derive like follows: For $\Lambda, \Lambda' \in \mathfrak{C}$, $\Lambda \subseteq \Lambda'$ with $\Lambda'' = \Lambda' \setminus \Lambda$ and $Y_1 \in \mathfrak{M}_\Lambda$, $Y_2 \in \mathfrak{M}_{\Lambda''}$ we conclude from the properties of π:

$$\begin{aligned}
\tilde{\pi}_{\Lambda'}(\varphi, Y_1 \cap Y_2) &= \pi_{\Lambda'}(\varphi, Y_1 \cap Y_2) = \int \pi_{\Lambda'}(\varphi, \mathrm{d}\hat{\varphi})\pi_\Lambda(\hat{\varphi}, Y_1 \cap Y_2) \\
&= \int \pi_{\Lambda'}(\varphi, \mathrm{d}\hat{\varphi})\mathbb{1}_{Y_2}(\hat{\varphi})\pi_\Lambda(\hat{\varphi}, Y_1) = \int_{Y_2} \pi_{\Lambda'}(\varphi, \mathrm{d}\hat{\varphi})\tilde{\pi}_\Lambda(\hat{\varphi}, Y_1) \\
&= \int_{Y_2} \tilde{\pi}_{\Lambda'}(\varphi, \mathrm{d}\hat{\varphi})\tilde{\pi}_\Lambda(\hat{\varphi}_{\Lambda'} + \varphi_{(\Lambda')^c}, Y_1).
\end{aligned}$$

On the other hand, let $\tilde{\pi}$ be given with (LS′1)–(LS′4). Then for the respective π all properties are obvious exept of (LS6). We choose $\Lambda, \Lambda', Y_1, Y_2$ like above and get the following chain:

$$\begin{aligned}
\pi_{\Lambda'} * \pi_\Lambda(\varphi, Y_1 \cap Y_2) &= \int \pi_{\Lambda'}(\varphi, \mathrm{d}\hat{\varphi})\pi_\Lambda(\hat{\varphi}, Y_1 \cap Y_2) \\
&= \int \pi_{\Lambda'}(\varphi, \mathrm{d}\hat{\varphi})\tilde{\pi}_\Lambda(\hat{\varphi}, \{\tilde{\varphi} : \tilde{\varphi}_\Lambda + \hat{\varphi}_{\Lambda^c} \in Y_1 \cap Y_2\}) \\
&= \int \pi_{\Lambda'}(\varphi, \mathrm{d}\hat{\varphi})\mathbb{1}_{Y_2}(\hat{\varphi})\tilde{\pi}_\Lambda(\hat{\varphi}, Y_1) \\
&= \int_{Y_2} \tilde{\pi}_{\Lambda'}(\varphi, \mathrm{d}\hat{\varphi})\tilde{\pi}_\Lambda\left(\hat{\varphi}_{\Lambda'} + \varphi_{(\Lambda')^c}, Y_1\right) \\
&= \tilde{\pi}_{\Lambda'}(\varphi, Y_1 \cap Y_2) = \pi_{\Lambda'}(\varphi, Y_1 \cap Y_2).
\end{aligned}$$

As $\mathfrak{M}_{\Lambda'}$ is generated by the sets of the form $Y_1 \cap Y_2$ with $Y_1 \in \mathfrak{M}_\Lambda$, $Y_2 \in \mathfrak{M}_{\Lambda''}$, $\pi_\Lambda(\varphi, \cdot\,)$ is determined by its values on $\mathfrak{M}_\Lambda$. Thus π is a local specification.

The formula for the Gibbs process is obvious. ∎

4 A Gibbs Formalism for Locally Finite Boson Systems

Now we come to the quantum case. Point processes have to be replaced by (locally normal) states on $\mathcal{A}$. The rôle of the conditional expectation $Q\,(\,\cdot \mid \mathfrak{M}_{\Lambda^c})\,(\varphi)$ is taken by the conditional local states ω_Λ^φ.

Definition 5 *Let $\mathfrak{C} \subseteq \mathfrak{B}$. A triple $(\gamma, \mathfrak{R}, \mathfrak{C})$ is called* generalized local specification with family of regularity sets $\mathfrak{R}$ *if $\gamma = (\gamma_\Lambda)_{\Lambda \in \mathfrak{C}}$ and $\mathfrak{R} = (R_\Lambda)_{\Lambda \in \mathfrak{C}}$ fulfill the following conditions:*

(GLS1) *For all $\Lambda \in \mathfrak{C}$ is $R_\Lambda \in \mathfrak{M}_{\Lambda^c}$.*

(GLS2) *For all $\Lambda \in \mathfrak{C}$ we have $\gamma_\Lambda : M \times \mathcal{A}_\Lambda \longmapsto \mathbb{C}$ and $\gamma_\Lambda(\varphi, \cdot\,)$ is for all $\varphi \in R_\Lambda$ a normal state on $\mathcal{A}_\Lambda$.*

(GLS3) *If $\Lambda \in \mathfrak{C}$ and $\varphi \notin R_\Lambda$ then $\gamma_\Lambda(\varphi, \cdot\,) = 0$.*

(GLS4) *The map $\varphi \mapsto \gamma_\Lambda(\varphi, A)$ is $\mathfrak{M}_{\Lambda^c}$ measurable for all $\Lambda \in \mathfrak{B}$ and $A \in \mathcal{A}_\Lambda$.*

(GLS5) *For all $\Lambda, \Lambda' \in \mathfrak{C}$, $\Lambda \subseteq \Lambda'$, all $\varphi \in M$ and all $A \in \mathcal{A}_\Lambda$ and $Y \in \mathfrak{M}_{\Lambda' \setminus \Lambda}$ it holds true*

$$\gamma_{\Lambda'}(\varphi, A\mathrm{O}_Y) = \int_Y \mathrm{P}_{\gamma_{\Lambda'}(\varphi,\cdot)}(\mathrm{d}\tilde{\varphi})\gamma_\Lambda(\tilde{\varphi}_{\Lambda'} + \varphi_{(\Lambda')^c}, A).$$

Then we define the local specification $(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$ by setting for $Y \in \mathfrak{M}$

$$\pi^\gamma_\Lambda(\varphi, Y) = \gamma_\Lambda\left(\varphi, \mathrm{O}_{\{\hat{\varphi}:\hat{\varphi}_\Lambda + \varphi_{\Lambda^c} \in Y\}}\right).$$

Definition 6 *We call a locally normal state ω on $\mathcal{A}$* Gibbs state w.r.t. the generalized local specification $(\gamma, \mathfrak{R}, \mathfrak{C})$ *if for all $\Lambda \in \mathfrak{C}$ P_ω-a.s.*

$$\gamma_\Lambda(\varphi, \cdot\,) = \omega^\varphi_\Lambda(\,\cdot\,).$$

The set of Gibbs states is denoted $\mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$.

Remark 7 We would like to formulate the Gibbs property with conditional expectations. But there lacks a description of conditional local states as conditional expectations on an algebra universal for all locally normal states ω. Moreover, γ_Λ replaces $\tilde{\pi}_\Lambda$ and not π_Λ in the classical formalism. Roughly speaking, γ_Λ is something like a λ mapping in the sense of [5].

Gibbs states are closely related to Gibbs processes of the associated local specification.

Theorem 9 *Fix a generalized local specification $(\gamma, \mathfrak{R}, \mathfrak{C})$. If $\omega \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ then $\mathrm{P}_\omega \in \mathcal{GP}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$. Conversely, any point process $Q \in \mathcal{GP}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$ determines uniquely a locally normal state $\omega \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ with $\mathrm{P}_\omega = Q$.*

Proof: Let be $\omega \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$. Due to definition it holds for all $\Lambda \in \mathfrak{C}$ and all $Y \in \mathfrak{M}_\Lambda$

$$\int \mathrm{P}_\omega(\mathrm{d}\varphi)\, \mathrm{P}_{\gamma_\Lambda(\varphi,\cdot)}(Y) = \omega(\mathrm{O}_Y) = \mathrm{P}_\omega(Y).$$

Thus $\mathrm{P}_\omega \in \mathcal{GS}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$.

Now assume $Q \in \mathcal{GS}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$. We set for $\Lambda \in \mathfrak{C}$

$$\omega_\Lambda(\,\cdot\,) := \int Q(\mathrm{d}\varphi)\, \gamma_\Lambda(\varphi, \cdot\,).$$

Suppose $\mathfrak{C} \ni \Lambda' \supseteq \Lambda$. Due to $Q \in \mathcal{GP}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$ and Lemma 5 we get for each $A \in \mathcal{A}_\Lambda$

$$\begin{aligned} \omega_{\Lambda'}(A) &= \int Q(\mathrm{d}\varphi)\,\gamma_{\Lambda'}(\varphi, AO_M) = \int Q(\mathrm{d}\varphi) \int \pi^\gamma(\varphi, \mathrm{d}\tilde{\varphi})\gamma_\Lambda(\tilde{\varphi}_{\Lambda'} + \varphi_{(\Lambda')^c}, A) \\ &= \int Q(\mathrm{d}\varphi) \int Q^\varphi_{\Lambda'}(\mathrm{d}\tilde{\varphi})\gamma_\Lambda(\tilde{\varphi} + \varphi_{(\Lambda')^c}, A) = \int Q(\mathrm{d}\varphi)\,\gamma_\Lambda(\varphi, A) = \omega_\Lambda(A). \end{aligned}$$

Thus the local states $(\omega_\Lambda)_{\Lambda \in \mathfrak{C}}$ are compatible. As $\mathfrak{C}$ contains a cofinal sequence they determine uniquely a locally normal state ω on $\mathcal{A}$. Because of $Q \in \mathcal{GP}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$ we have $\mathrm{P}_\omega = Q$ and $\omega \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$. The uniqueness of ω follows immediately from the definition of conditional local states. ∎

Now we look at mixings, an immediate consequence of [21] and the above proposition is

Proposition 10 *The set $\mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ is closed w.r.t. mixings.*∎

Again, $\mathrm{ex}\mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ denotes the set of extremal Gibbs states.

Lemma 11 *Let $(\gamma, \mathfrak{R}, \mathfrak{C})$ be a generalized local specification. Then*

(i) *$\omega \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ is extremal iff $\mathrm{P}_\omega \in \mathrm{ex}\mathcal{GP}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$.*

(ii) *for $\omega_1, \omega_2 \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ the measures P_{ω_1} and P_{ω_2} coincide on $\mathfrak{M}^\infty$ iff $\omega_1 = \omega_2$.*

Proof: 1° For $\omega \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ and $\mathrm{P}_\omega \in \mathrm{ex}\mathcal{GP}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$ we choose $\lambda \in (0,1)$ and locally normal states $\omega_1, \omega_2 \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ with $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$. Thus $\mathrm{P}_\omega = \lambda \mathrm{P}_{\omega_1} + (1-\lambda)\mathrm{P}_{\omega_2}$, according to $\mathrm{P}_{\omega_1}, \mathrm{P}_{\omega_2} \in \mathcal{GP}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$ also $\mathrm{P}_{\omega_1} = \mathrm{P}_{\omega_2}$. With Theorem 9 we derive $\omega_1 = \omega_2 = \omega$.
Now assume that $\omega \in \mathrm{ex}\mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ and there are $Q_1, Q_2 \in \mathcal{GP}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$ with $\mathrm{P}_\omega = \lambda Q_1 + (1-\lambda)Q_2$. Due to Theorem 9 there are $\omega_1, \omega_2 \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ with $\mathrm{P}_{\omega_i} = Q_i$ for $i = 1, 2$. The definitions of Gibbs states and conditional local states yield $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$. This implies $\omega_1 = \omega_2$ and $Q_1 = Q_2$. Thus $\mathrm{P}_\omega \in \mathrm{ex}\mathcal{GP}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$.
2° Take $\omega_1, \omega_2 \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ with $\mathrm{P}_{\omega_1}{\restriction_{\mathfrak{M}^\infty}} = \mathrm{P}_{\omega_2}{\restriction_{\mathfrak{M}^\infty}}$. Theorem 9 implies $\mathrm{P}_{\omega_1}, \mathrm{P}_{\omega_2} \in \mathcal{GP}(\pi^\gamma, \mathfrak{R}, \mathfrak{C})$. From Lemma 6 we conclude $\mathrm{P}_{\omega_1} = \mathrm{P}_{\omega_2}$ and Theorem 9 yields $\omega_1 = \omega_2$. ∎

We can also construct an entrance boundary:

Theorem 12 *Assume $\mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C}) \neq \emptyset$ for a generalized local specification $(\gamma, \mathfrak{R}, \mathfrak{C})$. Then there are two maps $\gamma^\infty : M \times \mathcal{A} \longmapsto \mathbb{C}$ and $\pi^\infty : M \times \mathfrak{M} \longmapsto [0,1]$ with the following properties:*

($\gamma^\infty 1$) *$\gamma^\infty(\varphi, \cdot\,)$ is a locally normal state on $\mathcal{A}$ for each $\varphi \in M$.*

($\gamma^\infty 2$) *$\pi^\infty(\varphi, \cdot\,)$ is for all $\varphi \in M$ the position distribution of $\gamma^\infty(\varphi, \cdot\,)$.*

($\gamma^\infty 3$) *The maps $\varphi \mapsto \gamma^\infty(\varphi, A)$ and $\varphi \mapsto \pi^\infty(\varphi, Y)$ are $\mathfrak{M}^\infty$ measurable for all $A \in \mathcal{A}$ and $Y \in \mathfrak{M}$ respectively.*

($\gamma^\infty 4$) *For all $\omega \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ the following formula is P_ω-a.s. valid:*

$$^\infty\omega^\varphi(\,\cdot\,) = \gamma^\infty(\varphi, \cdot\,).$$

($\gamma^{\infty}5$) *For each* $\varphi \in M$ *it holds* $\gamma^{\infty}(\varphi,\,\cdot\,) \in \mathrm{ex}\mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$.

($\gamma^{\infty}6$) *If* $\varphi \in M$ *then* $\pi^{\infty}(\varphi, \{\hat{\varphi} : \gamma^{\infty}(\hat{\varphi},\,\cdot\,) = \gamma^{\infty}(\varphi,\,\cdot\,)\}) = 1$.

($\gamma^{\infty}7$) *A locally normal state* ω *on* $\mathcal{A}$ *is from* $\mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ *iff*

$$\int \mathrm{P}_{\omega}(\mathrm{d}\varphi)\, \gamma^{\infty}(\varphi,\,\cdot\,) = \omega(\,\cdot\,). \tag{8}$$

Proof: Due to Theorem 9 we have $\mathcal{GP}(\pi^{\gamma}, \mathfrak{R}, \mathfrak{C}) \neq \emptyset$. Now Proposition 7 provides a stochastic kernel π^{∞} with the properties ($\pi^{\infty}1$)–($\pi^{\infty}4$). Due to Theorem 9 and property ($\pi^{\infty}1$) there exists for any $\varphi \in M$ a unique locally normal state $\gamma^{\infty}(\varphi,\,\cdot\,) \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ with $\mathrm{P}_{\gamma^{\infty}(\varphi,\cdot)}(Y) = \pi^{\infty}(\varphi, Y)$ for all $Y \in \mathfrak{M}$. ($\pi^{\infty}1$) together with Theorem 9 and Lemma 11 implies $\gamma^{\infty}(\varphi,\,\cdot\,) \in \mathrm{ex}\mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$. Thus ($\gamma^{\infty}1$)–($\gamma^{\infty}3$) and ($\gamma^{\infty}5$) are fulfilled.

The property ($\gamma^{\infty}4$) follows from the respective property ($\pi^{\infty}2$) of π^{∞} and the properties of ${}^{\infty}\omega^{\varphi}$.

The locally normal states $\gamma^{\infty}(\varphi,\,\cdot\,)$ are determined by $\pi^{\infty}(\varphi,\,\cdot\,)$. Thus ($\gamma^{\infty}6$) follows from ($\pi^{\infty}3$).

Suppose $\omega(\,\cdot\,) = \int \mathrm{P}_{\omega}(\mathrm{d}\varphi)\, \gamma^{\infty}(\varphi,\,\cdot\,)$. Proposition 10 and condition ($\gamma^{\infty}3$) imply $\omega \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$. On the other hand we can derive from $\omega \in \mathcal{GS}(\gamma, \mathfrak{R}, \mathfrak{C})$ with property ($\gamma^{\infty}4$), Equation (3) and the properties of ${}^{\infty}\omega^{\varphi}$

$$\omega(\,\cdot\,) = \int \mathrm{P}_{\omega}(\mathrm{d}\varphi)\, {}^{\infty}\omega^{\varphi}(\,\cdot\,) = \int \mathrm{P}_{\omega}(\mathrm{d}\varphi)\, \gamma^{\infty}(\varphi,\,\cdot\,). \blacksquare$$

5 Examples

Example 1 (Coherent states) *Define for* $h \in L^2(G, \nu)$ *the exponential vector* $\exp_h \in \mathcal{M}$ *by*

$$\exp_h(\varphi) = \begin{cases} \prod\limits_{x \in \varphi} (h(x))^{\varphi(\{x\})} & \text{if } 0 < \varphi(G) < \infty \\ 1 & \text{if } \varphi = o \\ 0 & \text{otherwise} \end{cases}$$

and let $L^2_{\mathrm{loc}}(G, \nu) = \{g : \int_{\Lambda} \nu(\mathrm{d}x) |g(x)|^2 < \infty \forall \Lambda \in \mathfrak{B}\}$ *denote the space of locally square integrable functions. We assign to* $g \in L^2_{\mathrm{loc}}(G, \nu)$ *the generalized local specification* $(\gamma^g, \mathfrak{R}, \mathfrak{B})$ *with* $\mathfrak{R}_{\Lambda} = M$ *and*

$$\gamma^g_{\Lambda}(\varphi, A \otimes \mathbb{1}_{\Lambda^c}) = \frac{\langle \exp_{g_{\Lambda}}, A \exp_{g_{\Lambda}} \rangle}{\exp\{\|g_{\Lambda}\|\}} \tag{9}$$

for all $\Lambda \in \mathfrak{B}$ *and all local observables* $A \in \mathfrak{L}(\mathcal{M}_{\Lambda})$. *Thereby* g_{Λ} *is an abbreviation for* $g \cdot \mathbb{1}_{\Lambda}$. *It is easy to see that* γ^g *is a generalized local specification. As* γ^g_{Λ} *is independent of* φ *it holds* $\omega_{\Lambda} = \gamma^g_{\Lambda}$ *for any Gibbs state* ω. *Thus there is exactly one Gibbs state, the coherent state corresponding to* g *in the sense of [12].*

Example 2 (Ideal Bose gas) *We want to deal with the ideal Bose gas in the above formalism. Assume $G = \mathbb{R}^d$, equipped with Lebesgue measure ℓ^d. Suppose we are given a potential, i.e. a function $U : M \longmapsto \mathbb{R}$. Under certain conditions on the function U the operator*

$$H = -\mathrm{d}\Gamma(\Delta) + U,$$

where U acts by multiplication, $\mathrm{d}\Gamma$ is the differential second quantization and Δ is some (selfadjoint) form of the Laplacian, is selfadjoint and should give the Hamiltonian of the particle system. If H has suitable spectrum, $\mathrm{e}^{-\beta H}$ is of trace class and represents the unnormalized (normal) equilibrium state of the system at inverse temperature β.

Like in [11] we do not want to go into self-adjointness problems but assume that the unnormalized kernel of the trace class operator representing the normal equilibrium state is given by a Feynman-Kac formula

$$k(\textstyle\sum_{i=1}^n \delta_{x_i}, \sum_{j=1}^n \delta_{y_j}) = \dfrac{1}{Z} z^n \displaystyle\sum_{\pi\in S_n} \int \mu^{nd,\beta}_{(x_1,\dots,x_n),(y_{\pi(1)},\dots,y_{\pi(n)})}(\mathrm{d}(w_1,\dots,w_n))\mathrm{e}^{-\int_0^\beta \ell(\mathrm{d}t) U\left(\sum_{i=1}^n \delta_{w_i(t)}\right)}. \quad (10)$$

Thereby $\mu^{nd,\beta}_{(x_1,\dots,x_n),(y_1,\dots,y_n)}$ is the conditional Wiener measure (cf. [4]), S_n is the group of permutations of $\{1,\dots,n\}$ and Z is a normalizing constant. Additionally, $k(o,o) = 1$ and $k(\varphi_1,\varphi_2) = 0$ if $|\varphi_1| \neq |\varphi_2|$. In the sequel we will follow only that part of the kernel with $|\varphi_1| = |\varphi_2| \in \{1,2,\dots\}$.

As we are dealing with the case of an ideal gas we only accept an outer potential, i.e.

$$U(\textstyle\sum_{i=1}^n \delta_{x_i}) = \sum_{i=1}^n h(x_i)$$

where $h : \mathbb{R}^d \longmapsto \mathbb{R}$. So we can rewrite (10) as follows

$$\begin{aligned} k(\textstyle\sum_{i=1}^n \delta_{x_i}, \sum_{j=1}^n \delta_{y_j}) &= \dfrac{1}{Z} z^n \displaystyle\sum_{\pi\in S_n} \int \mu^{nd,\beta}_{(x_1,\dots,x_n),(y_{\pi(1)},\dots,y_{\pi(n)})}(\mathrm{d}(w_1,\dots,w_n)) \prod_{i=1}^n \mathrm{e}^{-\int_0^\beta h(w_i(t))\ell(\mathrm{d}t)} \\ &= \dfrac{1}{Z} z^n \displaystyle\sum_{\pi\in S_n} \prod_{i=1}^n \int \mu^{d,\beta}_{x_i,y_{\pi(i)}}(\mathrm{d}w_i)\mathrm{e}^{-\int_0^\beta h(w_i(t))\ell(\mathrm{d}t)}. \end{aligned} \quad (11)$$

Like in the proof of [13, Theorem 4.8] the respective conditional local states can be given by its unnormalized kernel k_Λ^φ (assume $\varphi(\Lambda) = 0$) through

$$k_\Lambda^{\sum_{l=1}^p \delta_{u_l}}(\textstyle\sum_{i=1}^n \delta_{x_i}, \sum_{j=1}^n \delta_{y_j}) = z^n \displaystyle\sum_{\pi\in S_{n+p}} \prod_{i=1}^{n+p} \int \mu^{d,\beta}_{s_i,t_{\pi(i)}}(\mathrm{d}w_i)\mathrm{e}^{-\int_0^\beta h(w_i(t))\ell(\mathrm{d}t)}. \quad (12)$$

where $(s_1,\dots,s_{n+p}) = (x_1,\dots,x_n,u_1,\dots,u_p)$ and t is defined similarly by y and u.

The first crucial point is that in the case of no interaction the Gibbs state is in general not a normal state (cf. [4]). To cover this situation we want to apply the Gibbs formalism from above:

Formally the conditional local states ω_Λ^φ are given by (12), but as φ has in general infinitely many points, we must deal with some limit procedure to define $\gamma_\Lambda(\varphi,\,\cdot\,)$ for infinite φ. To this goal we rescale k_Λ^φ by dividing it by $k_\Lambda^\varphi(o,o)$, leading to $\tilde{k}_\Lambda^\varphi(\varphi_1,\varphi_2) = \frac{k_\Lambda^\varphi(\varphi_1,\varphi_2)}{k_\Lambda^\varphi(o,o)}$ and bring this kernel into a more suitable form.

Remark 8 It is easy to see, that

$$\tilde{k}_\Lambda^\varphi(\varphi_1,\varphi_2) = k_\omega(\varphi_1,\varphi_2,\varphi)$$

if $\varphi(\Lambda)=0$. k_ω is the so called conditional reduced density matrix (cf. [10]).

With the abbreviation $a(x,y) = \int \mu_{x,y}^{d,\beta}(\mathrm{d}w)\mathrm{e}^{-\int_0^\beta h(w(s))\ell(\mathrm{d}s)}$ we derive

$$\tilde{k}_\Lambda^{\sum_{l=1}^p \delta_{u_l}}\left(\sum_{i=1}^n \delta_{x_i}, \sum_{j=1}^n \delta_{y_j}\right) = z^n \frac{\sum_{\pi\in S_{n+p}} \prod_{i=1}^{n+p} a(r_i, s_{\pi i})}{\sum_{\tau\in S_p} \prod_{i=1}^p a(u_i, u_{\tau i})} \tag{13}$$

Set $\tilde{\varphi} = \sum_{i=1}^n \delta_{x_i}$, $\hat{\varphi} = \sum_{j=1}^n \delta_{y_j}$ and $\varphi = \sum_{l=1}^p \delta_{u_l}$. In the denominator of the RHS we can divide $\tilde{\varphi}$ and $\hat{\varphi}$ in two parts respectively: $\tilde{\varphi} = \vartheta_1 + \vartheta_2$, $\hat{\varphi} = \vartheta_3 + \vartheta_4$, thereby assuming that ϑ_3 is exactly the image of $\tilde{\varphi}$ and ϑ_1 in $\hat{\varphi}$. Thus ϑ_2 is mapped into φ and ϑ_4 is the complete image of points from φ. Denote by $I(\varphi_1,\varphi_2)$ the set of all injections from the support of φ_1 into the support of φ_2. This yields

$$\tilde{k}_\Lambda^\varphi(\tilde{\varphi},\hat{\varphi}) = \frac{z^{|\tilde{\varphi}|}}{\sum\limits_{\tau\in I(\varphi,\varphi)} \prod\limits_{u\in\varphi} a(u,\tau(u))} \times \Bigg(\sum_{\tilde{\varphi}=\vartheta_1+\vartheta_2} \sum_{\hat{\varphi}=\vartheta_3+\vartheta_4} \sum_{q_1\in I(\vartheta_2,\varphi)} \sum_{q_2\in I(\vartheta_4,\varphi)}$$
$$\sum_{\tau\in I(\varphi-q_1(\vartheta_2),\varphi-q_2(\vartheta_4))} \prod_{x\in\vartheta_2} a(x,q_1(x)) \prod_{y\in\vartheta_4} a(q_2(y),y) \prod_{u\in(\varphi-q_1(\vartheta_2))} a(u,\tau(u)) \Bigg)$$

Now

$$\frac{\sum\limits_{\tau\in I(\varphi-q_1(\vartheta_2),\varphi-q_2(\vartheta_4))} \prod\limits_{u\in(\varphi-q_1(\vartheta_2))} a(u,\tau(u))}{\sum\limits_{\tau\in I(\varphi,\varphi)} \prod\limits_{u\in\varphi} a(u,\tau(u))}$$

is some probability for a random permutation of the support of φ with transition rates given by a, cf. [8]. Assume P_a^φ is the associated probability law, i.e.

$$P_a^\varphi(Y) = \frac{\sum\limits_{\tau\in Y} \prod\limits_{u\in\varphi} a(u,\tau(u))}{\sum\limits_{\tau\in I(\varphi,\varphi)} \prod\limits_{u\in\varphi} a(u,\tau(u))}.$$

Then

$$\tilde{k}_\Lambda^\varphi(\tilde{\varphi},\hat{\varphi}) = z^{|\tilde{\varphi}|} \sum_{\tilde{\varphi}=\vartheta_1+\vartheta_2} \sum_{\hat{\varphi}=\vartheta_3+\vartheta_4} \sum_{q_1\in I(\vartheta_2,\varphi)} \sum_{q_2\in I(\vartheta_4,\varphi)}$$
$$\prod_{x\in\vartheta_2} a(x,q_1(x))^{-1} \prod_{y\in\vartheta_4} a(q_2(y),y)^{-1} P_a^\varphi(\{\tau\in I(\varphi,\varphi) : \tau(q_1(\vartheta_2)) = q_2(\vartheta_4)\}) \tag{14}$$

Under some conditions on φ, P_a^φ makes sense also for infinite point configurations φ (cf. [8]). If we have a "good" point configuration, we can define an unnormalized kernel of ω_Λ^φ by (14). The question whether we have a good configuration is related to some classical Gibbs problem on the countable phase space φ, in [8] it was dealt with such problems. That reference provides also some uniqueness condition. But it is also possible that P_a^φ is only a subprobability measure. Then we get

$$\tilde{k}_\Lambda^\varphi(\mathrm{o},\mathrm{o}) = P_a^\varphi(I(\varphi,\varphi)) < 1$$

which does not coincide with our assumption $\tilde{k}_\Lambda^\varphi(\mathrm{o},\mathrm{o}) = 1$. Thus our sets of regularity should contain only configurations for which P_a^φ exists uniquely as probability measure.

From a limit we get by the reduction due to Theorem 9 another (classical) Gibbs problem, which waits for a solution. Roughly speaking, the first Gibbs problem is related to momentum (as the Feynman-Kac formula provides some perturbation of the contraction semigroup associated to the square of the momentum operator), whereas the second one comes from positions.

The above algorithm should remain applicable if boundary conditions are involved and Bose-Einstein condensation occurs. In that case μ and a depend on additional parameters. Thus the limit problem is more complicated and no solution in sight.

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