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The Hot Free Algebra

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Abstract

We consider the stochastic limit of the standard non relativistic QED (but our results also hold for the polaron interaction of a particle with a Boson field). Extending the Fock case results of [AcLu92], we take the initial state of the field to be a Gibbs state at a given temperature. We show that a new algebra, with commutation relations depending on the temperature and acting on a Hilbert module, emerges. This algebra, that we call the Hot Free Algebra, generalizes the QED Hilbert module algebra in the sense of [AcLuVo97c] and [Ske96] and therefore also the Free (or Boltzmannian) algebra. It is interesting to notice that, when the module structure is neglected, the algebra we find is precisely the algebra that was found in the singleton independence central limit theorem of [Aho98a], [Aho98b]. So the present result also gives a natural physical interpretation for that algebra.

(1) Introduction

In the present work we will consider the standard non relativistic quantum electrodynamics (QED) Hamiltonian (neglecting polarization) (but our results also hold for the polaron model, a model describing the interaction of a non-relativistic particle with a Boson field). We investigate this model as an application of the stochastic limit technique which consists in considering the time rescaling $t \to t/\lambda^2$ and then in investigating the asymptotics of the correlation functions for $\lambda \to 0$. This asymptotics captures the dominating terms in the limit of large times and small coupling constant. After this limit the dynamics became in some sense integrable, and one gets explicit formulae for the correlation functions. The name *stochastic limit* is due to the fact that the initial quantum fields are shown to converge to some new fields which are δ-correlated in time, so they exhibit a typical white noise behaviour in the sense of Hida [HiKuPoStr93]. The main result of the present work is that the Boson creators and annihilators converge, in the temperature stochastic limit for the model considered, to some new operators (master fields), defining a new interesting mathematical structure that we call the *Hot Free Algebra.* This is a deformation of the free algebra in two senses:

i) a deformation parameter appears, depending on the temperature.

ii) the commutation relations are Hilbert module rather than Hilbert space relations in the sense they cannot be realized in a usual Hilbert space, but require the introduction of a Hilbert module, in fact of the so called *interacting Hilbert module* (cf. the remark at the end of section (6)).

These new features are due to the strong nonlinearity. In particular, after the stochastic limit, the Bose statistics becomes a Hilbert module deformation of the Boltzmannian (or Free) statistics.

(2) The stochastic limit: general idea

The stochastic limit is a scaling limit of quantum theory. In the weak coupling case this rescaling can be described as follows. Let us consider a system described by the Hamiltonian

$$H = H_o + \lambda H_I$$

and define the evolution on operator $U_t^{(\lambda)} = e^{-itH}e^{itH_o}$, solution of the following Schrödinger equation in interaction picture:

$$\frac{\partial}{\partial t}U_t^{(\lambda)} = -i\lambda H_I(t)\, U_t^{(\lambda)} \quad , \; U_0^{(\lambda)} = 1 \tag{1}$$

$H_I(t) = e^{itH_o}H_I e^{-itH_o}$ is the *evolved interaction Hamiltonian.* Here λ is a small constant and we will investigate the cumulative effect of small perturbations on a large time scale. For this aim we make the time rescaling, in the evolution equation, $t \to t/\lambda^2$ and then take the limit $\lambda \to 0$. This is equivalent to consider the simultaneous limit $\lambda \to 0$, $t \to \infty$ under the condition that $\lambda^2 t$ tends to a constant (interpreted as a new *slow scale* time). This leads to the rescaled equation

$$\frac{\partial}{\partial t}U_{t/\lambda^2}^{(\lambda)} = -\frac{i}{\lambda}H_I(t/\lambda^2)\, U_{t/\lambda^2}^{(\lambda)} \tag{1a}$$

It is natural to conjecture that, if the limits

$$\lim_{\lambda\to 0} U_{t/\lambda^2}^{(\lambda)} = U_t \tag{2}$$

$$\lim_{\lambda\to 0} \frac{1}{\lambda} H_I\left(\frac{t}{\lambda^2}\right) = H_t \tag{3}$$

exist in some topology to be specified, then U_t is the solution of the equation

$$\partial_t U_t = -iH_t U_t \quad ; \qquad U_0 = 1 \tag{4}$$

We use this limit because after the limit many problems become integrable. In this sense, the stochastic limit allows us to calculate the main contributions to the behavior of a quantum system in a regime, of *long times and small coupling.*

We will consider a quantum mechanical system as a triple (algebra of observables $\mathcal{A}$, state space, evolution operator). Moreover, we will take the state space to be the Hilbert space of the GNS-representation generated by the equilibrium state $\langle\cdot\rangle$ for the free evolution on the algebra of observables, at a given inverse temperature β. This is a mean zero (Boson) Gaussian state. We will study the evolution operator in the interaction picture $U_t^{(\lambda)}$. The stochastic limit of the algebra of observables (master algebra) is constructed in the following way. We associate to an observable A its free evolution $A(t) = e^{itH_o}Ae^{-itH_o}$, and we look for observables A_i such that the limit of the correlators

$$\lim_{\lambda\to 0}\langle\frac{1}{\lambda}A_1(t_1/\lambda^2)\ldots\frac{1}{\lambda}A_k(t_k/\lambda^2)\rangle.$$

exist and is non–trivial. By the general reconstruction theorem of [AcFriLe82] , there exists an algebra $\mathcal{B}$, whose elements we denote B_i, and a state $\langle\cdot\rangle$ on $\mathcal{B}$ such that

$$\lim_{\lambda\to 0}\langle\frac{1}{\lambda}A_1(t_1/\lambda^2)\dots\frac{1}{\lambda}A_k(t_k/\lambda^2)\rangle = \langle B_1(t_1)\dots B_k(t_k)\rangle$$

The pair $\{\mathcal{B}, \langle\cdot\rangle\}$ will be called *the stochastic limit of the algebra of observables* or simply the *master algebra.* In fact, for the investigation of the evolution defined by equation (1), it is sufficient to find the stochastic limit for observables that are implicitly defined by the interaction Hamiltonian H_I. The analysis is done by considering matrix elements of the perturbative series expansion of equation (1a) and using Gaussianity to represent this series as a sum of diagrams. Then one separates the negligible diagrams from the relevant ones and finally one resums the series of the relevant diagrams and proves that the result is a unitary operator satisfying an appropriate stochastic equation driven by a given white noise (master field). The first and most important step of this procedure is to determine the structure of the master field and the space where it lives. This is what we do in the present paper for the model considered.

(3) Statement of the problem and main result

We consider the simplest case in which matter is represented by a single particle, say an electron, whose position and momentum we denote respectively by $q = (q_1, \dots, q_d)$ and $p = (p_1, \dots, p_d)$ and satisfy the commutation relations

$$[q_h, p_k] = i\delta_{hk}$$

The EM field is described by Boson operators (in fact operator valued distributions)

$$a(k) = (a_1(k), \dots, a_d(k)) \quad ; \quad a^+(k) = (a_1^+(k), \dots, a_d^+(k))$$

satisfying the *canonical commutation relations*

$$[a_j(k), a_h^+(k')] = \delta_{jh}\delta(k-k')$$

The Hamiltonian of the system under consideration has the form

$$H = H_o + \lambda H_I = \int \omega(k)a^\dagger(k)a(k)\,dk + \frac{1}{2}p^2 + \lambda H_I$$

where ω is a positive function on ^{d}, a typical example is $\omega(k) = |k|$. H_I describes the interaction of a free particle with an EM field neglecting polarization. The interaction between the particle and the EM field is expressed in terms of a potential $A(x)$, describing the field strength at the space point $x \in R^d$, more precisely $A(x)$ is the potential felt by the particle in position x as a consequence of its interaction with the field The explicit form of the interaction Hamiltonian is

$$H_I = p \cdot A(q) + A(q) \cdot p \tag{5}$$

where p, q are as above and

$$A(q) = \int dk \quad \{g(k)e^{ik\cdot q} \cdot a^+(k) + \overline{g}(k)e^{-ik\cdot q} \cdot a(k)\} \tag{6}$$

The time dependence of is defined by letting the original interaction H_I, given by (5), (6), evolve under the free Hamiltonian H_o and then performing the time rescaling (2.1a). A simple calculation shows that this is equivalent to replace the operators $a_\lambda^\pm(t,k)$ in (6) by the *rescaled fields*

$$a_\lambda(t,k) = \frac{1}{\lambda} e^{i(\omega(k)+kp)t/\lambda^2} e^{-ikq} a(k) \tag{7}$$

We will consider the limit of the correlation functions

$$\lim_{\lambda\to 0} \langle a_\lambda^{\epsilon_N}(t_N,k_N) a_\lambda^{\epsilon_{N-1}}(t_{N-1},k_{N-1}) \dots a_\lambda^{\epsilon_1}(t_1,k_1)\rangle \tag{8}$$

where $\varepsilon = \{\epsilon_N, ..., \epsilon_1\} \in \{1,0\}^N$, $\epsilon \in \{1,0\}$ ($\epsilon = 0$ for a, $\epsilon = 1$ for a^+) and $\langle\cdot\rangle$ denotes the Gibbs state of the reservoir at inverse temperature β, i.e. the mean zero Boson Gaussian state with pair correlations vanishing on the off-diagonal terms and, on the diagonal ones, equal to

$$\langle a_k a_{k'}^\dagger \rangle = \frac{\delta(k-k')}{1 - e^{-\beta\omega_k}} \tag{9}$$

$$\langle a_{k'}^\dagger a_k \rangle = \frac{\delta(k-k')}{e^{\beta\omega_k} - 1} \tag{10}$$

(the other correlators can be calculated using Gaussianity). By our assumption one only needs to consider the case $N = 2n$. If the number of creators is equal to the number of annihilators, one can consider the partition $\sigma(\varepsilon)$ of ε into pairs of 0 and 1, that corresponds to the expansion of the Gaussian expectation of

$$b^{\epsilon_N}(t_N,k_N) b^{\epsilon_{N-1}}(t_{N-1},k_{N-1}) \dots b^{\epsilon_1}(t_1,k_1)$$

into sums of products of pairs of creators and annihilators. An arbitrary partition of this kind corresponds to some Feynmann diagram. The main result is the following: in the stochastic limit only the partitions that correspond to halfplanar noncrossing diagrams survive. These partitions will be called nontrivial. The simplest context in which these diagrams arise is that of the algebra of free creation-annihilation operators with commutation relations

$$A_i A_j^\dagger = \delta_{ij}.$$

After the stochastic limit we find a generalization of this algebra which is based on the same diagrams. In particular the Bose statistics becomes a generalization of the Boltzmannian (or Free) statistics. Furhter analysis of this algebra and of the corresponding statistics is a subject of particular interest and should serve as a fundament for the investigation of the limit dynamics.

In the present work we prove convergence of these correlators and show that in the stochastic limit we have non-trivial cancellations as a consequence of which in the limit the crossing diagrams vanish. More precisely we show that the above limit exists and has the form

$$\langle b^{\epsilon_N}(t_N,k_N)b^{\epsilon_{N-1}}(t_{N-1},k_{N-1})\dots b^{\epsilon_1}(t_1,k_1)\rangle$$

THEOREM 1. *The limit temperature correlation functions exist always and*

i) if the number of creators is not equal to the number of annihilators, then the above correlator is equal to zero (even before the limit);

ii) if the number of creators is equal to the number of annihilators ($N = 2n$), then the limit (8) is equal to the following sum over the nontrivial partitions

$$\sum_{\sigma(\varepsilon)}\prod_{h=1}^{n}\delta(k_{m'_h}-k_{m_h})c_{m_h m'_h}(k_{m_h})2\pi\delta(t_{m'_h}-t_{m_h})$$

$$\delta\left(\omega(k_{m_h})+k_{m_h}p+\sum_{\alpha}(-1)^{\varepsilon_\alpha}\chi_{(m_\alpha,m'_\alpha)}(m_h)k_{m_\alpha}\cdot k_{m_h}-\varepsilon_h k^2_{m_h}\right) \tag{11}$$

where $\{(m'_j,m_j): j=1,\dots,n\}$ is the unique non-crossing partition of $\{1,\dots,2n\}$ associated with ε and $\chi_{(m_\alpha,m'_\alpha)}(m_h)$ is equal to 1 if m_h is between m_α and m'_α, while it is equal to 0 otherwise. The indices m'_h corresponds to annihilators, m_h corresponds to creators, and

$$c_{m_h m'_h}(k)=\frac{1}{1-e^{-\beta\omega_k}},\quad m'_h>m_h$$

$$c_{m_h m'_h}(k)=\frac{1}{e^{\beta\omega_k}-1},\quad m'_h<m_h$$

$(-1)^{\varepsilon_h}=1$ for $m'_h>m_h$ and $(-1)^{\varepsilon_h}=-1$ for $m'_h<m_h$.

(4) Proof of the result for the 2– and 4–point correlat ors

In order to explain the main idea we shall prove the statement of Theorem (1) in the simplest examples, i.e. the 2–point and the 4–point correlators. For the 2–point correlator one has:

$$\langle b_t(k_1)b_\tau^+(k_2)\rangle = \lim_{\lambda\to 0}\langle\frac{1}{\lambda}\, a_{t/\lambda^2}(k_1)\frac{1}{\lambda}\, a^+_{\tau/\lambda^2}(k_2)\rangle =$$

$$= \lim_{\lambda\to 0}\frac{1}{\lambda^2}\,\langle e^{it/\lambda^2(\omega(k_1)+k_1p)}e^{-iq(k_1-k_2)}e^{-i\frac{\tau}{\lambda^2}(\omega(k_2)+k_2p)}\rangle\langle a_{k_1}a^+_{k_2}\rangle \tag{0}$$

Using the formulae (3.9), (3.10) we get

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}\, e^{i\frac{t-\tau}{\lambda^2}\,(\omega(k_1)+k_1p)}\frac{\delta(k_1-k_2)}{1-e^{-\beta\omega(k_1)}}$$

Using the module extension of the limit formula of [AcLuVo93]:

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}\, e^{\frac{it}{\lambda^2}\,(\omega(k)+kp)} = 2\pi\delta(\omega(k)+kp)\delta(t) \tag{1}$$

we get 2–point correlator

$$\langle b_t(k_1)b_\tau^+(k_2)\rangle = 2\pi\delta(t-\tau)\delta(\omega(k_1)+k_1p)\cdot\frac{\delta(k_1-k_2)}{1-e^{-\beta\omega(k_1)}} \tag{2}$$

Let us now investigate the following 2–point correlator

$$\langle b_\tau^+(k_2)b_t(k_1)\rangle = \lim_{\lambda\to 0}\frac{1}{\lambda^2}\,\langle a^+_{k_2}e^{ik_2q}e^{-i\frac{\tau}{\lambda^2}\,(\omega(k_2)+k_2p)}e^{\frac{it}{\lambda^2}\,(\omega(k_1)+k_1p)}e^{-ik_1q}a_{k_1}\rangle$$

Using the commutation relation for Weyl operators

$$e^{i\alpha p}e^{i\beta q} = e^{i\beta q}e^{i\alpha q}e^{i\alpha\beta} \tag{3}$$

where $[p,q]=-i$ we get for (0)

$$\lim_{\lambda\to 0}\frac{1}{\lambda^2}\,\frac{\delta(k_2-k_1)}{e^{\beta\omega(k_1)}-1}\, e^{i\frac{t-\tau}{\lambda^2}(\omega(k_1)+k_1p-k_1^2)}$$

Using formula (1) we get

$$\langle b_\tau^+(k_2)b_t(k_1)\rangle = 2\pi\delta(t-\tau)\delta(\omega(k_1)+k_1p-k_1^2)\frac{\delta(k_2-k_1)}{e^{\beta\omega(k_1)}-1} \tag{4}$$

Let us now calculate the 4–point correlator

$$\langle b_{t_1}(k_1)b_{t_2}(k_2)b^+_{t_2'}(k_2')b^+_{t_1'}(k_1')\rangle \tag{5}$$

By Gaussianity and (3.9), (3.10) we get

$$\langle a_{k_1} a_{k_2} a^+_{k'_2} a^+_{k'_1} \rangle = \frac{1}{1-e^{-\beta\omega(k_1)}} \frac{1}{1-e^{-\beta\omega(k_2)}}$$

$$\cdot(\delta(k_2-k'_2)\delta(k_1-k'_1)+\delta(k_1-k'_2)\delta(k_2-k'_1)) \tag{6}$$

Formula (6) for the bosonic correlator $\langle a_{k_1} a_{k_2} a^+_{k'_2} a^+_{k'_1} \rangle$ contains two terms proportional to δ–functions that correspond to two Wick diagrams. Let us calculate the first term, that is proportional to $\delta(k_1-k'_1)\delta(k_2-k'_2)$. We have

$$1 \text{ st term } = \lim_{\lambda\to 0} \frac{1}{1-e^{-\beta\omega(k_1)}} \frac{1}{1-e^{-\beta\omega(k_2)}} \delta(k_1-k'_1)\delta(k_2-k'_2)$$

$$\frac{1}{\lambda^4} e^{i\frac{t_1-t'_1}{\lambda^2}(\omega(k_1)+k_1p)} e^{i\frac{t_2-t'_2}{\lambda^2}(\omega(k_2)+k_2p)} e^{i\frac{t_2-t'_2}{\lambda^2} k_1k_2} \tag{7}$$

Using formula (1) we get

$$1- \text{ st term } = (2\pi)^2 \frac{1}{1-e^{-\beta\omega(k_1)}} \frac{1}{1-e^{-\beta\omega(k_2)}}$$

$$\delta(k_1-k'_1)\delta(k_2-k'_2)\delta(t_1-t'_1)\delta(t_2-t'_2)\delta(\omega(k_1)+k_1p)\delta(\omega(k_2)+k_2p+k_1k_2) \tag{8}$$

Let us calculate the second term of correlator, that is proportional to $\delta(k_1-k'_2)\delta(k_2-k'_1)$. We have

$$2- \text{ nd term } = \lim_{\lambda\to 0} \frac{1}{1-e^{-\beta\omega(k_1)}} \frac{1}{1-e^{-\beta\omega(k_2)}} \delta(k_1-k'_2)\delta(k_2-k'_1) \cdot$$

$$\frac{1}{\lambda^4} e^{i\frac{t_1-t'_2}{\lambda^2}(\omega(k_1)+k_1p)} e^{i\frac{t_2-t'_1}{\lambda^2}(\omega(k_2)+k_2p)} e^{i\frac{t_2-t'_2}{\lambda^2} k_1k_2} = 0$$

according to formula (1) and the Riemann–Lebesgue Lemma (cf. [AcLuVo97c] for more details in the Fock case). We get therefore that the 4–point correlator is given by formula (8)

(5) The vanishing of the crossing diagrams: general case

We follow the pattern of the proof given in [Gou96] and [AcLuVo97c] and we shall introduce the necessary modifications due to temperature. To calculate the correlators in the stochastic limit we recall that the 2-parameter family of Weyl operator $W(a,b)$ $(a,b \in {}^d)$ is defined by

$$W(a,b) = e^{i(a\cdot p + b\cdot q)}$$

The unitary operators $W(a,b)$ satisfy

$$W(a,b) = e^{ia\cdot p} e^{ib\cdot q} e^{-ia\cdot b/2} = e^{ib\cdot q} e^{ia\cdot p} e^{ia\cdot b/2}$$

$$W(a_1,b_1)W(a_2,b_2) = W(a_1+a_2, b_1+b_2)\exp\{\frac{i}{2}(a_1\cdot b_2 - a_2\cdot b_1)\} \tag{1a}$$

$$W(a_1,b_1)...W(a_n,b_n) = W(\sum_j a_j, \sum_j b_j)\exp\{\frac{i}{2}\sum_{j<l}(a_j\cdot b_l - a_l\cdot b_j)\} \tag{1b}$$

$$W(a,b)^+ = W(-a,-b) \tag{1c}$$

Under the free system evolution we have

$$p_t = p \quad , \quad q_t = q + tp$$

so the Weyl operators evolve as

$$e^{itp^2} W(a,b) e^{-itp^2} = e^{i(a\cdot p_t + b\cdot q_t)} = e^{i((a+tb)p + b\cdot q)} = W(a+tb, b)$$

Recalling that the rescaled field operators (3.7) are

$$a_\lambda(t,k) = \frac{1}{\lambda} e^{i(\omega(k)+kp)t/\lambda^2} e^{-ikq} a(k) \tag{2}$$

we will consider the limit temperature correlation functions,

$$\langle b^{\epsilon_N}(t_N,k_N) b^{\epsilon_{N-1}}(t_{N-1},k_{N-1}) \dots b^{\epsilon_1}(t_1,k_1)\rangle =$$

$$= \lim_{\lambda\to 0}\langle a_\lambda^{\epsilon_N}(t_N,k_N) a_\lambda^{\epsilon_{N-1}}(t_{N-1},k_{N-1}) \dots a_\lambda^{\epsilon_1}(t_1,k_1)\rangle$$

Here $\varepsilon = \{\epsilon_N, ..., \epsilon_1\} \in \{1,0\}^N$, $\epsilon \in \{1,0\}$ ($\epsilon = 0$ for a and $\epsilon = 1$ for a^+). For $N = 2n$ one can consider the partition $\sigma(\varepsilon)$ of ε into pairs of 0 and 1, that correspond to Wick partition of

$$b^{\epsilon_N}(t_N,k_N) b^{\epsilon_{N-1}}(t_{N-1},k_{N-1}) \dots b^{\epsilon_1}(t_1,k_1)$$

to pairs of creators and annihilators. An arbitrary partition of this kind corresponds to some Wick diagram. We will be interested in partitions, that correspond to halfplanar noncrossing diagrams. We will call these partitions nontrivial.

THEOREM (1) The limit temperature correlation functions exist always and

i) if N is odd, then the above limit is equal to zero;
ii) if $N = 2n$, then the above limit, i.e. the limit

$$\lim_{\lambda\to 0}\langle a_\lambda^{\epsilon_{2n}}(t_{2n},k_{2n})a_\lambda^{\epsilon_{2n-1}}(t_{2n-1},k_{2n-1})\dots a_\lambda^{\epsilon_1}(t_1,k_1)\rangle \tag{5}$$

is equal to zero if ε is trivial; is equal to

$$\sum_{\sigma(\varepsilon)}\prod_{h=1}^{n}\delta(k_{m'_h}-k_{m_h})c_{m_h m'_h}(k_{m_h})2\pi\delta(t_{m'_h}-t_{m_h})$$

$$\delta\left(\omega(k_{m_h})+k_{m_h}p+\sum_\alpha(-1)^{\varepsilon_\alpha}\chi_{(m_\alpha,m'_\alpha)}(m_h)k_{m_\alpha}\cdot k_{m_h}-\frac{1-(-1)^{\varepsilon_h}}{2}k_{m_h}^2\right) \tag{6}$$

where $\{(m'_j,m_j):j=1,\dots,n\}$ is the unique non-crossing partition of $\{1,\dots,2n\}$ associated with ε. description Here the index m'_h corresponds to an annihilator; m_h to a creator and

$$c_{m_h m'_h}(k)=\frac{1}{1-e^{-\beta\omega_k}},\quad m'_h>m_h$$

$$c_{m_h m'_h}(k)=\frac{1}{e^{\beta\omega_k}-1},\quad m'_h<m_h$$

$(-1)^{\varepsilon_h}=1$ for $m'_h>m_h$ and $(-1)^{\varepsilon_h}=-1$ for $m'_h<m_h$.

Proof. From (2) and the identity

$$e^{i\alpha p}e^{i\beta q}=e^{i(\alpha p+\beta q)}e^{i\frac{1}{2}\alpha\beta}$$

we deduce

$$a_{t,k}^\epsilon\equiv\frac{1}{\lambda}\exp i(-1)^\epsilon\{\frac{t}{\lambda^2}(\omega(k)+kp)-kq-\frac{t}{2\lambda^2}k^2\}a^\epsilon(k). \tag{4}$$

For $\varepsilon=\{\epsilon_{2n},\dots,\epsilon_1\}\in\{1,0\}^{2n}$ non–trivial, we have

$$\langle\prod_{j=1}^{2n}a_{t_j,k_j}^{\epsilon_j}\rangle=$$

$$\prod_{j=1}^{2n}\left\{\frac{1}{\lambda}\exp i(-1)^{\epsilon_j}\{\frac{t_j}{\lambda^2}(\omega(k_j)+k_jp)-k_jq-\frac{t_j}{2\lambda^2}k_j^2\}\right\}\langle\prod_{h=1}^{2n}a^{\epsilon_h}(k_h)\rangle \tag{6}$$

but

$$\langle\prod_{h=1}^{2n}a^{\epsilon_n}(k_h)\rangle=\sum_{\{m'_h\neq m_h\}}\prod_{h=1}^{n}\delta(k_{m'_h}-k_{m_h})c_{m_h m'_h}(k_{m_h}) \tag{7}$$

that is, we sum over all possible pair contractions of annihilator–creator indices $\{(m'_h, m_h) : h = 1, \dots, n\}$. All operators in these products are ordered from the right to the left. Therefore we may write

$$\langle \prod_{j=1}^{2n} a^{\epsilon_j}_{t_j,k_j} \rangle =$$

$$\prod_{j=1}^{2n} \left\{ \frac{1}{\lambda} \exp i(-1)^{\epsilon_j} \{ \frac{t_j}{\lambda^2}(\omega(k_j) + k_j p) - k_j q - \frac{t_j}{2\lambda^2} k_j^2 \} \right\}$$

$$\sum_{\{m'_h \neq m_h\}} \prod_{h=1}^{n} \delta(k_{m'_h} - k_{m_h}) c_{m_h m'_h}(k_{m_h}) \tag{8}$$

Now, using the rules for multiplying Weyl operators and our product convention, we have that

$$\prod_{j=1}^{2n} \left\{ \frac{1}{\lambda} \exp i(-1)^{\epsilon_j} \{ \frac{t_j}{\lambda^2}(\omega(k_j) + k_j p) - k_j q - \frac{t_j}{2\lambda^2} k_j^2 \} \right\} =$$

$$= \exp\{ \frac{i}{2} \sum_{1 \leq j < l \leq 2n} (-1)^{\epsilon_j + \epsilon_l} k_j \cdot k_l \frac{t_j - t_l}{\lambda^2} \}$$

$$\left(\frac{1}{\lambda} \right)^{2n} \exp i \sum_{j=1}^{2n} (-1)^{\epsilon_j} \{ \frac{t_j}{\lambda^2}(\omega(k_j) + k_j p) - k_j q - \frac{t_j}{2\lambda^2} k_j^2 \} \tag{9}$$

the phase factor is then

$$\frac{i}{2} \sum_{l=1}^{2n} \sum_{j<l} (-1)^{\epsilon_j + \epsilon_l} k_j \cdot k_l (t_j - t_l)$$

and, using that the m'_h run over half of the $2n$ indices l and the m_h run over the other half, $(-1)^{\epsilon_{m'_h}} = 1$ and $(-1)^{\epsilon_{m_h}} = -1$)

$$= \frac{i}{2} \sum_{h=1}^{n} \left\{ \sum_{1 \leq j < m'_h} (-1)^{\epsilon_j} k_j \cdot k_{m'_h} (t_j - t_{m'_h}) - \sum_{1 \leq j < m_h} (-1)^{\epsilon_j} k_j \cdot k_{m_h} (t_j - t_{m_h}) \right\} =$$

$$= \frac{i}{2} \sum_{h=1}^{n} \left\{ \sum_{\alpha}^{m'_\alpha < m'_h} k_{m'_\alpha} \cdot k_{m'_h} (t_{m'_\alpha} - t_{m'_h}) - \sum_{\beta}^{m_\beta < m'_h} k_{m_\beta} \cdot k_{m'_h} (t_{m_\beta} - t_{m'_h}) \right.$$

$$\left. - \sum_{\gamma}^{m'_\gamma < m_h} k_{m'_\gamma} \cdot k_{m_h} (t_{m'_\gamma} - t_{m_h}) + \sum_{\delta}^{m_\delta < m_h} k_{m_\delta} \cdot k_{m_h} (t_{m_\delta} - t_{m_h}) \right\} = \frac{i}{2} \sum_{h=1}^{n} (I_h + II_h) \tag{11}$$

We use that $k_{m_h} = k_{m'_h}$. Putting together the first term with the third and the second with the fourth we get

$$I_h = \sum_{\alpha}^{m'_\alpha < m'_h} k_{m_\alpha} \cdot k_{m_h}(t_{m'_\alpha} - t_{m'_h}) - \sum_{\gamma}^{m'_\gamma < m_h} k_{m_\gamma} \cdot k_{m_h}(t_{m'_\gamma} - t_{m_h}) =$$

$$= \sum_{\alpha}^{m'_\alpha < m'_h} k_{m_\alpha} \cdot k_{m_h}(t_{m'_\alpha} - t_{m_h}) + \sum_{\alpha}^{m'_\alpha < m'_h} k_{m_\alpha} \cdot k_{m_h}(t_{m_h} - t_{m'_h}) - \sum_{\gamma}^{m'_\gamma < m_h} k_{m_\gamma} \cdot k_{m_h}(t_{m'_\gamma} - t_{m_h}) =$$

$$= \sum_{\alpha}^{m_h < m'_\alpha < m'_h} k_{m_\alpha} \cdot k_{m_h}(t_{m'_\alpha} - t_{m_h}) + \sum_{\alpha}^{m'_\alpha < m'_h} k_{m_\alpha} \cdot k_{m_h}(t_{m_h} - t_{m'_h})$$

for $m'_h > m_h$ and

$$I_h = - \sum_{\alpha}^{m'_h < m'_\alpha < m_h} k_{m_\alpha} \cdot k_{m_h}(t_{m'_\alpha} - t_{m_h}) + \sum_{\alpha}^{m'_\alpha < m'_h} k_{m_\alpha} \cdot k_{m_h}(t_{m_h} - t_{m'_h}) + k_{m_h} \cdot k_{m_h}(t_{m_h} - t_{m'_h})$$

for $m'_h < m_h$. For the sum of the second and the fourth term we get

$$-II_h = \sum_{\beta}^{m_\beta < m'_h} k_{m_\beta} \cdot k_{m_h}(t_{m_\beta} - t_{m'_h}) - \sum_{\delta}^{m_\delta < m_h} k_{m_\delta} \cdot k_{m_h}(t_{m_\delta} - t_{m_h}) =$$

$$\sum_{\beta}^{m_\beta < m'_h} k_{m_\beta} \cdot k_{m_h}(t_{m_\beta} - t_{m'_h}) - \sum_{\delta}^{m_\delta < m_h} k_{m_\delta} \cdot k_{m_h}(t_{m_\delta} - t_{m'_h}) - \sum_{\delta}^{m_\delta < m_h} k_{m_\delta} \cdot k_{m_h}(t_{m'_h} - t_{m_h}) =$$

$$\sum_{\beta}^{m_h < m_\beta < m'_h} k_{m_\beta} \cdot k_{m_h}(t_{m_\beta} - t_{m'_h}) + \sum_{\delta}^{m_\delta < m_h} k_{m_\delta} \cdot k_{m_h}(t_{m_h} - t_{m'_h}) + k_{m_h} \cdot k_{m_h}(t_{m_h} - t_{m'_h})$$

for $m'_h > m_h$ and

$$-II_h = - \sum_{\beta}^{m'_h < m_\beta < m_h} k_{m_\beta} \cdot k_{m_h}(t_{m_\beta} - t_{m'_h}) + \sum_{\delta}^{m_\delta < m_h} k_{m_\delta} \cdot k_{m_h}(t_{m_h} - t_{m'_h})$$

for $m'_h < m_h$. For (11) we get

$$I_h + II_h = \sum_{\alpha}^{m_h < m'_\alpha < m'_h} k_{m_\alpha} \cdot k_{m_h}(t_{m'_\alpha} - t_{m_h}) + \sum_{\alpha}^{m'_\alpha < m'_h} k_{m_\alpha} \cdot k_{m_h}(t_{m_h} - t_{m'_h}) -$$

$$- \sum_{\beta}^{m_h < m_\beta < m'_h} k_{m_\beta} \cdot k_{m_h}(t_{m_\beta} - t_{m'_h}) - \sum_{\beta}^{m_\beta < m_h} k_{m_\beta} \cdot k_{m_h}(t_{m_h} - t_{m'_h}) -$$

$$-k_{m_h}\cdot k_{m_h}(t_{m_h}-t_{m'_h}) \tag{12}$$

for $m'_h > m_h$ and

$$I_h+II_h=-\sum_{\alpha}^{m'_h<m'_\alpha<m_h} k_{m_\alpha}\cdot k_{m_h}(t_{m'_\alpha}-t_{m_h})+\sum_{\alpha}^{m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}(t_{m_h}-t_{m'_h})+$$

$$+\sum_{\beta}^{m'_h<m_\beta<m_h} k_{m_\beta}\cdot k_{m_h}(t_{m_\beta}-t_{m'_h})-\sum_{\beta}^{m_\beta<m_h} k_{m_\beta}\cdot k_{m_h}(t_{m_h}-t_{m'_h})+$$

$$+k_{m_h}\cdot k_{m_h}(t_{m_h}-t_{m'_h})$$

for $m'_h < m_h$. Let us now investigate the following term in (9)

$$\left(\frac{1}{\lambda}\right)^{2n}\exp i\sum_{j=1}^{2n}(-1)^{\epsilon_j}\{\frac{t_j}{\lambda^2}(\omega(k_j)+k_jp)-k_jq-\frac{t_j}{2\lambda^2}k_j^2\}\sum_{\{m'_h\neq m_h\}}\prod_{h=1}^{n}\delta(k_{m'_h}-k_{m_h})c_{m_hm'_h}(k_{m_h})$$

Notice that

$$\sum_{1\leq l\leq 2n}(-1)^{\epsilon_l}t_lk_l=-\sum_{1\leq h\leq n}(t_{m_h}-t_{m'_h})k_{m_h} \tag{10}$$

$$\sum_{1\leq l\leq 2n}(-1)^{\epsilon_l}k_lq=0$$

because $k_{m_h}=k_{m'_h}$. We get for the term in (9)

$$\left(\frac{1}{\lambda}\right)^{2n}\exp -i\sum_{1\leq h\leq n}\frac{t_{m_h}-t_{m'_h}}{\lambda^2}\left(\omega(k_{m_h})+k_{m_h}p-\frac{1}{2}k_{m_h}^2\right)$$

$$\sum_{\{m'_h\neq m_h\}}\prod_{h=1}^{n}\delta(k_{m'_h}-k_{m_h})c_{m_hm'_h}(k_{m_h})$$

With the change of variables

$$\begin{cases} u_{m_h}=t_{m_h}\\ v_{m_h}=t_{m_h}-t_{m'_h}\end{cases} \tag{13}$$

obtain the following lemma.

LEMMA 1. The correlator equals to

$$\langle\prod_{j=1}^{2n}a_{t_j,k_j}^{\epsilon_j}\rangle=\exp\frac{i}{2}\frac{1}{\lambda^2}\sum_{h=1}^{n}\left\{I_h+II_h\right\}$$

$$\left(\frac{1}{\lambda}\right)^{2n} \exp -i \sum_{1\le h\le n} \frac{v_{m_h}}{\lambda^2}\left(\omega(k_{m_h}) + k_{m_h}p - \frac{1}{2}k^2_{m_h}\right)$$

$$\sum_{\{m'_h \neq m_h\}} \prod_{h=1}^{n} \delta(k_{m'_h} - k_{m_h}) c_{m_h m'_h}(k_{m_h}) \tag{14}$$

The phase factor in (14) is equal to

$$\sum_{\alpha}^{m_h<m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}(-v_{m_\alpha}+u_{m_\alpha}-u_{m_h}) + \sum_{\alpha}^{m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h} -$$

$$-\sum_{\beta}^{m_h<m_\beta<m'_h} k_{m_\beta}\cdot k_{m_h}(v_{m_h}+u_{m_\beta}-u_{m_h}) - \sum_{\beta}^{m_\beta<m_h} k_{m_\beta}\cdot k_{m_h}v_{m_h} - k_{m_h}\cdot k_{m_h}v_{m_h} \tag{15}$$

for $m'_h > m_h$ and

$$-\sum_{\alpha}^{m'_h<m'_\alpha<m_h} k_{m_\alpha}\cdot k_{m_h}(-v_{m_\alpha}+u_{m_\alpha}-u_{m_h}) + \sum_{\alpha}^{m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h} +$$

$$+\sum_{\beta}^{m'_h<m_\beta<m_h} k_{m_\beta}\cdot k_{m_h}(v_{m_h}+u_{m_\beta}-u_{m_h}) - \sum_{\beta}^{m_\beta<m_h} k_{m_\beta}\cdot k_{m_h}v_{m_h} + k_{m_h}\cdot k_{m_h}v_{m_h}$$

for $m'_h < m_h$. The Riemann-Lebesgue lemma implies that the oscillatory factors of the type $\exp ik^2u/\lambda^2$ cause the associated term to vanish in the limit $\lambda \to 0$. Therefore, in this limit, a partition $\{(m_h, m'_h)\}$ survives in (14) if and only if, for each fixed $h = 1, ..., n$ and for any α

$$m_h < m_\alpha < m'_h \Leftrightarrow m_h < m'_\alpha < m'_h \tag{16}$$

or

$$m_h > m_\alpha > m'_h \Leftrightarrow m_h > m'_\alpha > m'_h \tag{16}$$

i.e. if and only if it is a non crossing partition. This means that only the non–trivial sequences $\varepsilon = \{\epsilon_{2n}, ..., \epsilon_1\} \in \{1,0\}^{2n}$ give a non trivial contribution in the limit. Denoting $\{(m_h, m'_h)\}$ the unique pair partition associated to such a sequence, the corresponding value of the phase term (15) is

$$\sum_{\alpha}^{m_h<m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}(-v_{m_\alpha}-v_{m_h}) + \sum_{\alpha}^{m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h} -$$

$$-\sum_{\beta}^{m_\beta<m_h} k_{m_\beta}\cdot k_{m_h}v_{m_h} - k_{m_h}\cdot k_{m_h}v_{m_h} \tag{17}$$

for $m'_h > m_h$ and

$$-\sum_{\alpha}^{m'_h<m'_\alpha<m_h} k_{m_\alpha}\cdot k_{m_h}(-v_{m_\alpha}-v_{m_h}) + \sum_{\alpha}^{m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h} -$$

$$-\sum_{\beta}^{m_\beta<m_h} k_{m_\beta}\cdot k_{m_h}v_{m_h} + k_{m_h}\cdot k_{m_h}v_{m_h}$$

for $m'_h < m_h$.

Let us investigate the calculated phase term. We have for $m'_h > m_h$

$$\sum_{\alpha}^{m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h} = \sum_{\alpha}^{m_h<m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h} + \sum_{\alpha}^{m'_\alpha\leq m_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h}$$

Because $m'_\alpha \neq m_h$, we have for the last term

$$\sum_{\alpha}^{m'_\alpha\leq m_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h} = \sum_{\alpha}^{m'_\alpha<m_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h}$$

Therefore the phase term is equal to

$$-\sum_{\alpha}^{m_h<m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}v_{m_\alpha} + \sum_{\alpha}^{m'_\alpha<m_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h} - \sum_{\beta}^{m_\beta<m_h} k_{m_\beta}\cdot k_{m_h}v_{m_h} - k_{m_h}\cdot k_{m_h}v_{m_h}$$

For the case $m'_h < m_h$ due to the non crossing condition we have

$$-\sum_{\alpha}^{m'_h<m'_\alpha<m_h} k_{m_\alpha}\cdot k_{m_h}(-v_{m_\alpha}-v_{m_h}) = -\sum_{\alpha}^{m'_h<m_\alpha<m_h} k_{m_\alpha}\cdot k_{m_h}(-v_{m_\alpha}-v_{m_h})$$

Therefore the phase term is equal to

$$\sum_{\alpha}^{m'_h<m'_\alpha<m_h} k_{m_\alpha}\cdot k_{m_h}v_{m_\alpha} + \sum_{\alpha}^{m'_\alpha<m'_h} k_{m_\alpha}\cdot k_{m_h}v_{m_h} - \sum_{\beta}^{m_\beta<m'_h} k_{m_\beta}\cdot k_{m_h}v_{m_h} + k_{m_h}\cdot k_{m_h}v_{m_h}$$

Let us denote the phase term as

$$I_h + II_h = \Phi_h - (-1)^{\varepsilon_h} k_{m_h}\cdot k_{m_h}v_{m_h}$$

Here $(-1)^{\varepsilon_h} = 1$ for $m'_h > m_h$ and $(-1)^{\varepsilon_h} = -1$ for $m'_h < m_h$. One can get for the phase term the formula

$$\Phi_h = -\sum_{\alpha\in(m_h,m'_h)or(m'_h,m_h)} (-1)^{\varepsilon_h} k_{m_\alpha}\cdot k_{m_h}v_{m_\alpha} - \sum_{\alpha:h\in(m_\alpha,m'_\alpha)or(m'_\alpha,m_\alpha)} (-1)^{\varepsilon_\alpha} k_{m_\alpha}\cdot k_{m_h}v_{m_h}$$

$$\sum_{1\le h\le n} \Phi_h = -2 \sum_{1\le h\le n} \sum_{\alpha: h\in(m_\alpha, m'_\alpha) or (m'_\alpha, m_\alpha)} (-1)^{\varepsilon_\alpha} k_{m_\alpha} \cdot k_{m_h} v_{m_h} =$$

$$= -2 \sum_{1\le h\le n} \sum_\alpha (-1)^{\varepsilon_\alpha} \chi_{(m_\alpha, m'_\alpha)}(m_h) k_{m_\alpha} \cdot k_{m_h} v_{m_h}$$

Here $\chi_{(m_\alpha, m'_\alpha)}$ is the indicator of the interval (m_α, m'_α) or (m'_α, m_α). We have proved the following lemma.

LEMMA 2. The noncrossing part of the correlator is equal to

$$\left(\frac{1}{\lambda}\right)^{2n} \exp -i \sum_{1\le h\le n} \frac{v_{m_h}}{\lambda^2}((\omega(k_{m_h}) + k_{m_h} p) + \sum_\alpha (-1)^{\varepsilon_\alpha} \chi_{(m_\alpha, m'_\alpha)}(m_h) k_{m_\alpha} \cdot k_{m_h} -$$

$$-\frac{1}{2} k_{m_h}^2 + \frac{1}{2}(-1)^{\varepsilon_h} k_{m_h}^2) \sum_{\{m'_h \neq m_h\}} \prod_{h=1}^{n} \delta(k_{m'_h} - k_{m_h}) c_{m_h m'_h}(k_{m_h})$$

Using the Riemann-Lebesgue lemma and keeping only noncrossing partition we get that the correlator from the statement of the theorem namely that the limit

$$\lim_{\lambda\to 0} \langle a_\lambda^{\epsilon_{2n}}(t_{2n}, k_{2n}) a_\lambda^{\epsilon_{2n-1}}(t_{2n-1}, k_{2n-1}) \dots a_\lambda^{\epsilon_1}(t_1, k_1)\rangle \tag{5}$$

in nontrivial case is equal to

$$\sum_{\{m'_h \neq m_h\}} \prod_{h=1}^{n} \delta(k_{m'_h} - k_{m_h}) c_{m_h m'_h}(k_{m_h}) 2\pi \delta(t_{m'_h} - t_{m_h})$$

$$\delta\left(\omega(k_{m_h}) + k_{m_h} p + \sum_\alpha (-1)^{\varepsilon_\alpha} \chi_{(m_\alpha, m'_\alpha)}(m_h) k_{m_\alpha} \cdot k_{m_h} - \frac{1-(-1)^{\varepsilon_h}}{2} k_{m_h}^2\right) \tag{6}$$

where $\{(m'_j, m_j) : j = 1, \dots, n\}$ is the unique non-crossing partition of $\{1, \dots, 2n\}$ associated with ε. The theorem is proven.

(6) The hot free algebra

In analogy with [AcLuVo97c] now we want to condensate the apparently complicated expression (6) of the correlators into a simple and easy to use set of algebraic rules.

LEMMA 1. The correlators of the previous theorem are satisfied if we take $b_t(k)$ equal to the sum of free independent noises

$$b_t(k) = b_1(t,k) + b_2^+(t,k) \tag{1}$$

where b_i satisfy the following hot free algebra relations

$$b_1(t,k_1)b_1^+(\tau,k_2) = 2\pi\delta(t-\tau)\delta(\omega(k_1)+k_1p)\frac{\delta(k_1-k_2)}{1-e^{-\beta\omega(k_1)}}$$

$$b_2(t,k_1)b_2^+(\tau,k_2) = 2\pi\delta(t-\tau)\delta(\omega(k_1)+k_1(p-k_1))\frac{\delta(k_1-k_2)}{e^{\beta\omega(k_1)}-1}$$

$$b_1b_2^+ = b_2b_1^+ = 0$$

$$b_1(t,k)p = (p+k)b_1(t,k)$$

$$b_2(t,k)p = (p-k)b_2(t,k)$$

and take the functional $\langle\cdot\rangle$ to be the expectation with respect to the free product of the two Fock vectors. In terms of the master field (1) this corresponds to the mean zero gaussian field with covariance

$$\langle b_t^+(k)b_{t'}(k')\rangle = \frac{1}{1-e^{-\beta\omega_k}}\,\delta(t-t')\delta(k-k')$$

$$\langle b_t(k)b_{t'}^+(k')\rangle = \frac{1}{e^{\beta\omega_k}-1}\,\delta(t-t')\delta(k-k')$$

Idea of the proof. The fields b_i of the hot free algebra arise as the stochastic limit of the Araki-Woods standard identification of the GNS representation of a boson field algebra, associated to a Gaussian equilibrium state, with the tensor product of a Fock and an anti Fock representation. To construct such a representation we introduce two independent bosonic fields $c_1(k)$, $c_2(k)$

$$[c_i(k), c_j^+(k')] = \delta_{ij}\delta(k-k')$$

such that every $c_i(k)$ acts in the Fock representation. We then consider the operators

$$a(k) = \sqrt{m(k)}c_1(k) + \sqrt{m(k)-1}c_2^+(k)$$

$$a^+(k) = \sqrt{m(k)}c_1^+(k) + \sqrt{m(k)-1}c_2(k)$$

Clearly

$$[a(k), a^+(k')] = \delta(k-k')$$

and, for the vacuum expectation we get

$$\langle a(k)a^{+}(k')\rangle = m(k)\delta(k-k')$$

Taking

$$m(k) = \frac{1}{1-e^{-\beta\omega_k}}$$

we get the thermal state (9), (10).

The stochastic limit of the rescaled operator (3.7) will then be

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\,e^{i\frac{t}{\lambda^2}(\omega(k)+kp)}e^{-ikq}a_k =$$

$$= \lim_{\lambda\to 0}\frac{1}{\lambda}\,e^{i\frac{t}{\lambda^2}(\omega(k)+kp)}e^{-ikq}\sqrt{m(k)}c_1(k) + \lim_{\lambda\to 0}\frac{1}{\lambda}\,e^{i\frac{t}{\lambda^2}(\omega(k)+kp)}e^{-ikq}\sqrt{m(k)-1}c_2^{+}(k)$$

where now the two limits are in the Fock representation. But from [AcLu92] we know that such limits give rise to QED Hilbert module white noises. So it is natural to expect that the master field in the temperature case shall be the sum of two such white noises $b_1(t,k)$, $b_2^{+}(t,k)$. So that the above limit is equal to

$$b(t,k) = b_1(t,k) + b_2^{+}(t,k)$$

in agreement with (1). It remains to be checked that Boson independence of the fields before the limit becomes free independence of the master field after the limit, i.e. $b_1 b_2^{+} = b_2 b_1^{+} = 0$.

The proof is done by computing the correlation functions using the commutation relations listed above and comparing the result with (3.9). For example, using the calculations made in section (4) for the 2–point correlators for b and the relation (1), we have

$$\langle b_t(k)b_\tau^{+}(k')\rangle = \langle b_1(t,k)b_1^{+}(t,k')\rangle + \langle b_2^{+}(t,k)b_2(t,k')\rangle = \langle b_1(t,k)b_1^{+}(\tau,k')\rangle$$

Therefore

$$\langle b_1(t,k_1)b_1^{+}(\tau,k_2)\rangle = 2\pi\delta(t-\tau)\delta(\omega(k_1)+k_1p)\frac{\delta(k_1-k_2)}{1-e^{-\beta\omega_k}}$$

Similarly using

$$\langle b_\tau^{+}(k_2)b_t(k_1)\rangle = \langle b_2(\tau,k_2)b_2^{+}(t,k_1)\rangle$$

we get

$$\langle b_2(t,k_1)b_2^{+}(\tau,k_2)\rangle = 2\pi\delta(t-\tau)\delta(\omega(k_1)+k_1(p-k_1))\frac{\delta(k_2-k_1)}{e^{\beta\omega(k_1)}-1}$$

Moreover it is easy to see that the pairings $b(t_{m'_h},k_{m'_h})b^{+}(t_{m_h},k_{m_h})$ and $b^{+}(t_{m_h},k_{m_h})b(t_{m'_h},k_{m'_h})$ give rise to the factor

$$\delta(k_{m'_h}-k_{m_h})c_{m_h m'_h}(k_{m_h})2\pi\delta(t_{m'_h}-t_{m_h})\delta\left(\omega(k_{m_h})+k_{m_h}p-\frac{1-(-1)^{\varepsilon_h}}{2}k_{m_h}^2\right)$$

and the last relation gives the term $\sum_\alpha (-1)^{\varepsilon_\alpha} \chi_{(m_\alpha, m'_\alpha)}(m_h)$ in the phase shift.

Remark. We conjecture that, in analogy with the result of Skeide [Ske97] for the Fock case, also in this case the structure of interacting Hilbert module defined by Lemma (1) above can be reduced to the single structure of Hilbert module by a proper choice of the left and right multiplication. This would be the finite temperature analogue of the QED Hilbert module.

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