数理解析研究所講究録
1036 巻 1998 年 46-64

A Time-Dependent Method for Inverse Scattering Problems

Hiroshi T. Ito （伊藤 宏）

Department of Mathematics, Kyoto University
Kyoto 606-01, JAPAN
e-mail : ito@kusm.kyoto-u.ac.jp

Abstract

We consider an inverse scattering problem, by using a time-dependent method, for the Dirac equation with a time-dependent electromagnetic field. The Fourier transform of the field is reconstructed from the scattering operator on a Lorentz invariant set

$$D := \{(\tau, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^3 ; |\tau| < c|\xi|\}. \tag{0.1}$$

in the dual space of the space-time. As corollaries of this result, we can reconstruct the electromagnetic field completely if it is a finite sum of fields each of which is a time-independent one by a suitable Lorentz transform, and we can also determine the field uniquely if the fields satisfies some exponential decay condition. Our assumptions and results are independent of a choice of inertial frames.

1 Introduction

To determine an unknown electromagnetic field we hit an particle, with the initial state ψ_i, into the field and observe the final state ψ_f. It is usually impossible to track the particle all the time. We know only the initial state and the final state.

Preparing various initial states, we obtain the map: $\psi_i \longmapsto \psi_f$. Roughly speaking, the map is called the scattering operator, whose precise definition will be given below. In this study the particle is supposed to obey a Dirac equation, which describes the motion of a relativistic particle with spin 1/2, for example, an electron or a positron.

The following is our problem.

Can we determine the field from the scattering operator ?

If the field is a time-independent one satisfying some short range condition, then it can be completely reconstructed from the scattering operator (see [Is], [It1], [J]). This means that the field can be determined by scattering experiments in the inertial frames in which the field is time-independent. On the other hand, taking account of the relativistic invariance of the Dirac equation, we expect that the field can be also determined by scattering experiments in the other inertial frames, in which the field may not be time-independent. Thus, it is important to treat the the Dirac equation with a time-dependent electromagnetic field if one investigates inverse scattering problems for Dirac equations.

We proceed our argument by fixing an inertial frame, in which t denotes the time-variable and x the space-variable. But, note that our assumptions and results are independent of a choice of inertial frames, which is proved in Section 3.

We begin with some explanation for our notatios. We denote by $\langle a, b\rangle_{\boldsymbol{R}^d}$ the usual inner product of a and b in $\boldsymbol{R}^d$ and may write $a \cdot b$ or $\langle a, b\rangle$ for simplicity. Moreover, the usual norm of $\boldsymbol{R}^d$ is denoted by the same symbol $|\cdot|$ for any d if no confusion occures. We also use the symbol $\langle T, u\rangle$ in place of $T(u)$ for a distribution T and a test function u.

The Dirac equation with an electromagnetic potential

$$A = (A^0, A^1, A^2, A^3) : \boldsymbol{R}_t \times \boldsymbol{R}^3_x \longrightarrow \boldsymbol{R}^4$$

is given by

$$(1.1) \quad i\frac{d}{dt}\Psi(t) = H_A(t)\Psi(t), \quad \Psi(t) \in \mathcal{H} := L^2(\boldsymbol{R}^3_x; \boldsymbol{C}^4),$$

$$H_A(t) = c\sum_{j=1}^{3} \alpha_j(D_j - A^j(t,x)) + \alpha_4 mc^2 - A^0(t,x)I_4,$$

where $c > 0$ is the velocity of light, $m \geq 0$ the rest mass of the particle, $D_j = -i\partial/\partial x_j$, and α_j's are 4×4 Hermitian matrices with the following properties:

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_4, \qquad 1 \leq j, k \leq 4,$$

where δ_{jk} is the Kronecker symbol and I_n is the $n \times n$ identity matrix.

Let $L \neq \{0\}$ be a subspace of $\boldsymbol{R}^4$. Then we denote by X_L the orthogonal projection of $X = (t, x) \in \boldsymbol{R}^4$ onto L and define a class of potentials:

$$S(L) := \left\{ A \in \mathcal{B}^1(\boldsymbol{R}^4, \boldsymbol{R}^4) \; ; \; \int_0^\infty g^L_A(r)dr < \infty \right\},$$

where $g^L_A(r) := \sup_{|X_L| \geq r} |A(X)|$ and $\mathcal{B}^1(\boldsymbol{R}^4, \boldsymbol{R}^4)$ is the space of $C^1(\boldsymbol{R}^4, \boldsymbol{R}^4)$-functions with bounded derivatives. If $A \in \mathcal{B}^1(\boldsymbol{R}^4, \boldsymbol{R}^4)$ satisfies the short range condition with respect to X_L-variable:

$$|A(X)| \leq K(1 + |X_L|)^{-\rho} \quad \text{on} \quad \boldsymbol{R}^4$$

for some $K > 0$ and some $\rho > 1$, then A belongs to $S(L)$.

We also say that A belongs to S if and only if A is decomposed as

$$A = \sum_{j=1}^{N} A_j, \quad A_j \in S(L_j),$$

for some N and for some subspaces L_j, $\quad 1 \leq j \leq N$.

If A belongs to S, the Dirac equation (1.1) has a unique unitary propagator $U_A(t,s)$, $s,\ t \in \boldsymbol{R}$:

$$i\frac{d}{dt}U_A(t,s) = H_A(t)U_A(t,s), \qquad U_A(s,s) = I.$$

Moreover, we have the following.

Proposition 1.1 *Let $A \in S$. Then the wave operators*

$$W_A^{\pm}(s) := s - \lim_{t \to \pm\infty} U_A(s,t) e^{-i(t-s)H_0}$$

exist for each $s \in \boldsymbol{R}$, where H_0 is the free Dirac operator:

$$H_0 = c \sum_{j=1}^{3} \alpha_j D_j + \alpha_4 mc^2.$$

Remark. The free Dirac operator H_0 is a self-adjoint operator with domain
$D(H_0) = H^1(\boldsymbol{R}^3; \boldsymbol{C}^4)$, the Sobolev space of order 1, and $U_0(t,s) = e^{-i(t-s)H_0}$.

The scattering operator is defined by

$$S_A(s) := W_A^+(s)^* W_A^-(s),$$

for each $s \in \boldsymbol{R}$. If some strong condition is imposed on the potential, the scattering operator is unitary in $\mathcal{H}$. But, it is not necessarily unitary under our weak assumption, $A \in S$.

The following useful relation follows immediately from definition:

$$S_A(s) = e^{-isH_0} S_A(0) e^{isH_0}, \quad s \in \boldsymbol{R}. \tag{1.2}$$

Thanks to this relation, we can know $S_A(s)$ for all $s \in \boldsymbol{R}$ if $S_A(s_0)$ is given for some s_0. The electromagnetic field strength F_A, determined by A, is defined by

$$F_A = (F_A^{jk})_{0 \le j < k \le 3} = \left(\frac{\partial A^k}{\partial x_j} - \frac{\partial A^j}{\partial x_k} \right)_{0 \le j < k \le 3} : \boldsymbol{R}^4 \longrightarrow \boldsymbol{R}^6,$$

where $x_0 = t$.

It should be recalled that the potential is not uniquely determined by the field and that it is not the potential but the field that can be an observable quantity. The following theorem shows that the scattering operator is also determined by the field not by the potential.

Theorem 1.2 *Let $A_{(1)}$ and $A_{(2)}$ be in S and suppose that*

$$A := A_{(2)} - A_{(1)} = \sum_{j=1}^{N} A_j, \quad A_j \in S(L_j)$$

with $\dim L_j \geq 2$, $1 \leq j \leq N$ and that $F_{A_{(1)}} = F_{A_{(2)}}$. Then $S_{A_{(1)}}(s) = S_{A_{(2)}}(s)$ for all $s \in \boldsymbol{R}$.

We next consider the inverse problem.

For a subspace $L \neq \{0\}$ of $\boldsymbol{R}^4$ a class $\widetilde{S}(L)$ of electromagnetic fields is defined in the same way as $S(L)$:

$$\widetilde{S}(L) := \left\{ F \in \mathcal{B}^0(\boldsymbol{R}^4, \boldsymbol{R}^6) \ ; \ \int_0^\infty g_F^L(r) dr < \infty \right\},$$

where $g_F^L(r) := \sup_{|X_L| \geq r} |F(X)|$ and $\mathcal{B}^0(\boldsymbol{R}^4, \boldsymbol{R}^6)$ is the space of bounded continuous functions from $\boldsymbol{R}^4$ to $\boldsymbol{R}^6$.

We denote by $\Xi = (\tau, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^3$ the dual variable of $X = (t, x)$ and define an open set D in $\boldsymbol{R}^4$ by

$$D := \{(\tau, \xi) \in \boldsymbol{R}^4; |\tau| < c|\xi|\}. \tag{1.3}$$

We denote by $\mathcal{S}'(\boldsymbol{R}^4; \boldsymbol{C}^6)$ the space of $\boldsymbol{C}^6$-valued tempered distributions and by $\widehat{F} \in \mathcal{S}'(\boldsymbol{R}^4; \boldsymbol{C}^6)$ the Fourier transform of $F \in \mathcal{S}'(\boldsymbol{R}^4; \boldsymbol{C}^6)$:

$$\widehat{F}(\tau, \xi) = (2\pi)^{-2} \iint e^{-it\tau - ix\cdot\xi} F(t, x) dt dx.$$

We also denote by $\widehat{F}|_\Omega$ the restriction of $\widehat{F}$ on an open set Ω, which is regarded as a distribution on Ω, i.e., $\widehat{F}|_\Omega \in \mathcal{D}'(\Omega)$.

Theorem 1.3 *Suppose the following: (i) $A = \sum_{j=1}^{N} A_j$ with $A_j \in S(L_j)$. (ii) $F_A = \sum_{k=1}^{N'} F_k$ with $F_k \in \widetilde{S}(L'_k)$.*

Then $\widehat{F_A}|_{D \setminus \Sigma}$ can be reconstructed from $S_A(0)$, where

$$\Sigma := \left(\bigcup_{\substack{1 \leq j \leq N \\ \dim L_j = 1}} L_j \right) \cup \left(\bigcup_{\substack{1 \leq k \leq N' \\ \dim L'_k = 1}} L'_k \right). \tag{1.4}$$

Remark. The decompositions in (i) and (ii) are not unique for a potential A and the field F_A, and the set Σ depends on the decompositions. Let $\mathcal{A}$ be the set of all decompositions of A and F_A such as (i) and (ii), and denote by Σ_a the Σ in (1.4) associated with a decomposition $a \in \mathcal{A}$. Then Σ in Theorem 1.3 can be replaced by

$$\widetilde{\Sigma} := \bigcap_{a \in \mathcal{A}} \Sigma_a.$$

This theorem tells us nothing about $\widehat{F_A}$ on D^c, the complement of D. If $\widehat{F_A}$ on D^c does not contributes to the scattering operator, this result is satisfactory one. But, this expectation is false.

Proposition 1.4 *Let $c = m = 1$ for simplicity and define a subset $D_1 \subset D^c$ by*

$$D_1 := \{\ (\tau, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^3\,;\ \tau > \sqrt{|\xi|^2 + 4}\,\}$$

and fix $\phi \in \mathcal{S}(\boldsymbol{R}_t \times \boldsymbol{R}_x^3\,;\ \boldsymbol{R})$ such that $\operatorname{supp}\widehat{\phi} \cap D_1 \neq \emptyset$. Let $S_{\lambda A}(0)$ be the scattering operator associated with the potential $\lambda A = (\lambda\phi, 0)$ for $\lambda \in \boldsymbol{R}$.

Then $S_{\lambda A}(0) \neq I$ for any small $\lambda \neq 0$.

Remark. We should notice that the proposition does not assert that the above potential can be reconstructed from the scattering operator. The contribution to the scattering operator will not necessarily imply the possibility of the reconstruction of the field. The author does not know whether the field F_A can be reconstructed completely from the scattering operator if the support of $\widehat{F_A}$ has an intersection with D^c as in the case of Schrödiger equations [We1]. The velocity of a relativistic particle cannot exceed that of light, though a nonrelativistic particle can have any speed. This is one of the most difference between a particle obeying a Dirac equation and one obeying a Schrödinger equation. The following intuitive argument shows that the field F with

$\mathrm{supp}\widehat{F} \subset D' := \{\, (\tau, \xi)\,;\, |\tau| > c|\xi|\, \}$ may be peculiar one from the point of view of the relativity. We write formaly

$$F(t,x) = (2\pi)^{-2} \int_{D'} F_{\tau,\xi}(t,x) d\tau d\xi,$$

where $F_{\tau,\xi}(t,x) := e^{it\tau + ix\cdot\xi} \widehat{F}(\tau,\xi)$ satisfies the wave equation

$$\frac{1}{\tau^2 |\xi|^{-2}} \frac{\partial^2}{\partial t^2} F_{\tau,\xi}(t,x) = \Delta_x F_{\tau,\xi}(t,x).$$

This implies that each component $F_{\tau,\xi}$ of the field F propagates with velocity $|\tau||\xi|^{-1} > c$, which contradicts the relativity.

But we can determine the field completely, if we impose some conditions on the field, as corollaries of Theorem 1.3.

Theorem 1.5 *Suppose (i) and (ii) in Theorem 1.3. Moreover, we assume* $\Sigma \cap D = \emptyset$ *and*

$$\textit{supp}\widehat{F_A} \setminus \{0\} \cap D^c = \emptyset. \tag{1.5}$$

Then F_A *can be completely reconstructed from* $S_A(0)$.

Using this theorem we can treat time-independent potentials.

Corollary 1.6 *Suppose (i) and (ii) in Theorem 1.3 with* $\Sigma \cap D = \emptyset$. *Moreover, suppose that for each* $k = 1, \cdots, N'$ *there exists* $V_k \in T := \{\, X = (t,x) \in \boldsymbol{R}^4\,;\, c|t| > |x| \}$ *such that*

$$F_k(sV_k + X) = F_k(X), \quad s \in \boldsymbol{R}, \quad X \in \boldsymbol{R}^4. \tag{1.6}$$

Then F_A *can be reconstructed completely from* $S_A(0)$.

Remark 1. In Section 3 we will show that each F_k satisfying (1.6) is time-independent on a suitable inertial frame determined by Vk.

Remark 2. If $A \in S$ is independent of t and satisfies

$$|A(x)| + |F_A(x)| \le K(1+|x|)^{-\rho} \quad \text{on } \boldsymbol{R}^3, \tag{1.7}$$

for some constants $K > 0$ and $\rho > 1$. Then the above corollary shows that the elecromagnetic field $F_A(x)$ can be completely reconstructed from the scattering operator $S_A(0)$. This has been already obtained under different conditions in [It1], [J] and [Is]. Roughly speaking, the decay rate of the potential is supposed to be $\rho > 3$ in [It1], $\rho > 3/2$ in [J] and $\rho > 2$ in [Is]. However, the decay condition on the field is not imposed in [J], and the magnetic field is not treated in [Is].

The next theorem shows that the field is uniquely determined by the scattering operator under some exponential decay condition.

Theorem 1.7 *Let $A \in S$.*
(1) Suppose there exists $V \in S^3$, S^3 being the unit sphere in $\boldsymbol{R}^4$, such that

$$(1.8) \qquad |F_A(X)| \leq Ke^{-\delta|\langle V,X\rangle_{R^4}|} \quad \textit{on } \boldsymbol{R}^4,$$

for some constants $K > 0$ and $\delta > 0$.
Then $\widehat{F_A}|_{R^4\setminus L}$ is determined by $S_A(0)$, where L is the one-dimensional subspace spanned by V.
(2) Suppose ,in addition to the assumption of (1), that there exist $V' \in S^3$ linearly indepndent of V and a bounded function g satisfying $g(t)(1+t)^{-1} \in L^1((0,\infty))$ such that

$$(1.9) \qquad |F_A(X)| \leq g(|\langle V',X\rangle_{R^4}|)e^{-\delta|\langle V,X\rangle_{R^4}|} \quad \textit{on } \boldsymbol{R}^4,$$

Then F_A is uniquely determined by $S_A(0)$.

Remark. Roughly speaking, (2) implies that if F_A satisfies

$$|F_A(X)| \leq K(1+|\langle V_1,X\rangle_{R^4}|)^{-\rho}e^{-\delta|\langle V_2,X\rangle_{R^4}|} \quad \text{on} \quad \boldsymbol{R}^4$$

for some linearly independent $V_1, V_2 \in \boldsymbol{R}^4 \cap S^3$ and for some $K > 0$, $\rho > 0$ and $\delta > 0$, then F_A is completely determined by the scattering operator.

It is an important problem in physics as well as in mathematics whether the external field can be determined from the scattering operator. In the case of Schrödinger operators with time-independent potentials, it is known, since Faddeev [F], that the potential can be reconstructed from the high-energy behavior of the scattering matrices (see, for example, [Is-K], [Ne], [Sa], [Wa], [Ni]). The proofs are based on a stationary representation of the scattering matrices and some resolvent estimates at the high-energy range. Using a similar stationary method, the author [It1] has proved that the electromagnetic fileld can be reconstructed from the high-energy behavior of the scattering matrices of the Dirac operator with a time-independent potential. In [Is] Isozaki has obtained a similar result as well as a uniqueness result for the fixed energy problem by a different method.

On the other hand, Enss and Weder [E-We] have found a new method, a time-dependent method (a geometric method), to reconstruct the potential from the high-energy asymptotics of the scattering operator in the case of Schrödinger operators without magnetic fields. Since their method is simple, it can be applicable to many cases. Recently, Arians [A1] has applied their method to reconstruct the electromagnetic field for the Schrödinger operator with a time-independent electromagnetic potential. See also [A2, We2, We3]. On the other hand, Weder [We1] has shown that the potential can be completely reconstructed from the scattering operator for the Schrödinger equations with a time-dependendent potential. For Dirac operators with a time-independent potential Jung [J] has reconstructed the electromagnetic field by using the geometric method. Some of his proofs are applicable to the case of time-dependent potentials (Proposition 2.2).

Furthermore, our results seem to be related with [St] and [R-S], in which inverse scattering problems for wave equations with time-dependent potentials are treated.

2 Sketch of the Proof of Theorem 1.3

To explain how we reconstruct the electromagnetic field from the scattering operator, we give a sketch of the proof of Theorem 1.3. For further details of the proof, see [it2].

To do so, we have to prepare some notations.
For a subspace L we define a set $C(L) \subset S^2$ by

$$C(L) := \{\, \omega \in S^2 \,;\, \widetilde{\omega} \in L^{\perp} \},$$

where $\widetilde{\omega} := (1, c\omega) \in \boldsymbol{R}^4$. We sometimes write $C(V) = C(L)$ if L is the one-dimensional subspace spanned by $V \neq 0 \in \boldsymbol{R}^4$. If $V = (v_0, v) \in \boldsymbol{R} \times \boldsymbol{R}^3$,

$$C(V) = \{\, \omega \in S^2 \,;\, \langle v, \omega \rangle_{R^3} = -v_0/c \}.$$

Hence, $C(V)$ is a circle on S^2 if $V \in D$, one point $\{ -v_0 v/c|v|^2 \}$ on S^2 if $V \in \overline{D} \setminus D$ and empty if $V \notin \overline{D}$. Moreover, it is easy to see that if $V_1,\ V_2 \in \overline{D}$, then

$$C(V_1) = C(V_2) \quad \Longleftrightarrow \quad V_1 \,//\, V_2. \tag{2.1}$$

On the other hand,

$$C(L) = \bigcap_{j=1}^{N} C(V_j)$$

if $\{ V_1, \cdots, V_N \}$ is a basis of L. Therefore, the number of elements of $C(L)$ is at most two if $\dim L=2$. Moreover, it is at most one and zero if $\dim L=3$ and $\dim L=4$, respectively, since $\dim L^{\perp} = 1$ and $\dim L^{\perp} = 0$, respectively.

For $A = \sum_{j=1}^{N} A_j \in S$ with $A_j \in S(L_j)$ we define

$$C_A := \bigcup_{j=1}^{N} C(L_j).$$

Then line integrals

$$K_A^{\omega}(\eta) := \int_{-\infty}^{\infty} \langle \widetilde{\omega}, A(t\widetilde{\omega} + \eta) \rangle_{R^4}\, dt, \quad \eta \in \boldsymbol{R}_t \times \boldsymbol{R}_x^3, \tag{2.2}$$

is well-defined if $\omega \in S^2 \setminus C_A$. The matrix $\alpha \cdot \omega := \sum_{j=1}^{3} \alpha_j \omega_j$ for $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$ has eigenvalues 1 and -1 with multiplicity two, respectively. Thus, $P_\pm(\omega) := \frac{1}{2}(1 \pm \alpha \cdot \omega)$ is the eigenprojection associated with the eigenvalue ± 1 of $\alpha \cdot \omega$.

The proof of Theorem 1.3 is based on the following proposition.

Proposition 2.1 *Let $A \in S$ and $s \in \boldsymbol{R}$. Then*

$$(2.3) \quad w - \lim_{E \to +\infty} e^{-iE\omega \cdot x} P_\pm S_A(s) P_\pm e^{iE\omega \cdot x} = e^{iK_A^{\pm\omega}(s,x)} P_\pm(\omega)$$

if $\pm\omega \in S^2 \setminus C_A$. Here, $e^{\pm iE\omega \cdot x}$ and $e^{iK_A^{\pm\omega}(s,x)}$ are multiplication operators.

Remark1. We can also show that

$$(2.4) \qquad w - \lim_{E \to +\infty} e^{-iE\omega \cdot x} P_\pm S_A(s) P_\mp e^{iE\omega \cdot x} = 0$$

for each $s \in \boldsymbol{R}$ if $\omega, \ -\omega \in S^2 \setminus C_A$.

Remark2. If the potential A is a time-independent and short-range potential, the result has been already proved by Jung [J], and his proof is applicable to our case with a slight modification.

Lemma 2.2 *Let $f \in \mathcal{H}$.*
(i) Suppose $\omega \in S^2 \setminus C_A$. Then there exists $E_0 > 0$ such that

$$(2.5) \qquad \lim_{t \to \pm\infty} \|(W_A^\pm(0) - U_A(0,t) e^{-itH_0}) P_+ e^{iE\omega \cdot x} f\| = 0$$

uniformly in $E \geq E_0$.
(ii) Suppose $-\omega \in S^2 \setminus C_A$. Then there exists $E_0 > 0$ such that

$$(2.6) \qquad \lim_{t \to \pm\infty} \|(W_A^\pm(0) - U_A(0,t) e^{-itH_0}) P_- e^{iE\omega \cdot x} f\| = 0$$

uniformly in $E \geq E_0$.

The following lemma follows from this immediately.

Lemma 2.3 *Let $f,\ g \in \mathcal{H}$ and let $\pm\omega \in S^2 \setminus C_A$. Then*

$$\lim_{t \to +\infty} \left(e^{-iE\omega\cdot x} P_\pm e^{itH_0} U_A(t,-t) e^{itH_0} P_\pm e^{iE\omega\cdot x} f, g\right)$$
$$= \left(e^{-iE\omega\cdot x} P_\pm S_A(0) P_\pm e^{iE\omega\cdot x} f, g\right)$$

uniformly in $E \geq E_0$.

We define

$$Q(t,x) := H_A(t) - H_0 = -c \sum_{j=1}^{3} \alpha_j A^j(t,x) - A^0(t,x) I_4,$$

$$W_\omega(t,x) := \langle \widetilde{\omega}, A(t, x+ct\omega)\rangle_{R^4} P_+(\omega) + \langle \widetilde{(-\omega)}, A(t, x-ct\omega)\rangle_{R^4} P_-(\omega).$$

Lemma 2.4 *For each $t > 0$ and each $\omega \in S^2$, one has*

$$s - \lim_{E \to +\infty} e^{-iE\omega\cdot x} e^{itH_0} U_A(t,-t) e^{itH_0} e^{iE\omega\cdot x} = e^{i \int_{-t}^{t} W_\omega(s,x) ds}. \tag{2.7}$$

In the momentum space it can be easily seen that

$$s - \lim_{E \to +\infty} e^{-iE\omega\cdot x} P_\pm e^{iE\omega\cdot x} = \frac{1}{2}(I \pm \alpha \cdot \omega) = P_\pm(\omega). \tag{2.8}$$

From this together with Lemma 2.4 it follows that

$$\lim_{E \to +\infty} e^{-iE\omega\cdot x} P_\pm e^{itH_0} U_A(t,-t) e^{itH_0} P_\pm e^{iE\omega\cdot x} = e^{i \int_{-t}^{t} \langle \widetilde{(\pm\omega)}, A(s, x \pm cs\omega)\rangle_{R^4} ds} P_\pm(\omega.) \tag{2.9}$$

Combinating this with Lemma 2.3, we can obtain Proposition 2.1 for $s = 0$. The other cases can be treated by using (1.2).

Here we give the idea of the proof of Theorem 1.3 for a simple case, $A = (\phi, 0)$, $\phi \in \mathcal{S}$. In this case, the right hand side of (2.3) determines

$$K_A^\omega(\eta) := \int_{-\infty}^{\infty} \phi(t\widetilde{\omega} + \eta) dt,$$

since $K_A^\omega(\eta) \to 0$ as $|\eta| \to \infty$ with $\langle \eta, \widetilde{\omega} \rangle = 0$. Thus the Fourier transform and the inverse Fourier transform yield

$$\begin{aligned} \widehat{\phi}(\Xi) &= (2\pi)^{-2}\sqrt{1+c^2}\int_{\Pi_{\widetilde{\omega}}} e^{-i\langle \eta, \Xi \rangle} K_A^\omega(\eta) d\eta, \quad \text{for} \quad \Xi \in \Pi_{\widetilde{\omega}}, \\ K_A^\omega(\eta) &= (2\pi\sqrt{1+c^2})^{-1}\int_{\Pi_{\widetilde{\omega}}} e^{i\langle \eta, \Xi \rangle} \widehat{\phi}(\Xi) d\Xi, \quad \text{for} \quad \eta \in \Pi_{\widetilde{\omega}}, \end{aligned}$$

where $\Pi_\theta := \{ \eta \in \boldsymbol{R}^4 \ ; \langle \theta, \eta \rangle_{R^4} = 0 \}$. On the other hand, we can easily verify that

$$\bigcup_{\omega \in S^2} \Pi_{\widetilde{\omega}} = \overline{D}, \text{ the closure of } D.$$

Therefore it follows that the only $\widehat{\phi}|_{\overline{D}}$ is reconstructed from the right hand side of (2.3). In the case of Schrödinger equations, similar line integrals $\int_{-\infty}^{\infty} \phi(\eta_0, t\omega + \eta')dt$, $\eta = (\eta_0, \eta') \in \boldsymbol{R} \times \boldsymbol{R}^3$ are appeared [We], which are also obtained from $cK_A^\omega(\eta)$ by $c \to +\infty$. By the same argument as above, we can see that these line integrals determine the whole $\widehat{\phi}$. Therefore the potential ϕ is completely reconstructed in the case of Schrödinger equations.

Now we define

$$\widetilde{C_A} := C_A \cup \left(\bigcup_{k=1}^{N'} C(L'_k) \right).$$

Lemma 2.5 *Let $\omega \in S^2 \setminus \widetilde{C_A}$ and $\theta \in S^3$ with $\langle \widetilde{\omega}, \theta \rangle_{R^4} = 0$, and let $\eta \in \boldsymbol{R}^4$. Then one has*

$$(2.10) \quad \frac{d}{ds} K_A^\omega(s\theta + \eta)\,|_{s=0} = -\int_{-\infty}^{+\infty} \langle \widetilde{\omega} \wedge \theta, F_A(t\widetilde{\omega} + \eta) \rangle_{R^6}\, dt,$$

where $X \wedge Y = (X_jY_k - X_kY_j)_{0 \le j < \kappa \le 3} \in \boldsymbol{R}^6$ for $X = (X_0, \cdots, X_3)$ and $Y = (Y_0, \cdots, Y_3)$.

Remark. Since $\widetilde{\omega} \in S^2 \setminus \widetilde{C_A}$, $F_A(t\widetilde{\omega} + \eta)$ is integrable with respect to $t \in \boldsymbol{R}$.

Proof. By Stokes' theorem we have

$$K_A^\omega(\eta)-K_A^\omega(\eta+s_0\theta)=\int_0^{s_0}ds\int_{-\infty}^{\infty}\langle\widetilde{\omega}\wedge\theta,F_A(t\widetilde{\omega}+s\theta+\eta)\rangle_{R^6}\,dt,$$

from which the lemma follows immediately. □

Since

$$\frac{d}{ds}\exp\left(-iK_A^\omega(s\theta+\eta)\right)=-i\frac{d}{ds}K_A^\omega(s\theta+\eta)\cdot\exp\left(-iK_A^\omega(s\theta+\eta)\right)$$

and $\widetilde{\omega}\wedge\widetilde{\omega}=0$, we can conclude from Proposition 2.1 and Lemma 2.5 that for each $\eta\in\boldsymbol{R}^4$, $\omega\in S^2\setminus\widetilde{C_A}$ and $\theta\in S^3$ the integral

$$\int_{-\infty}^{+\infty}\langle\widetilde{\omega}\wedge\theta,F_A(t\widetilde{\omega}+\eta)\rangle_{R^6}\,dt \tag{2.11}$$

can be constructed from the scattering operator $S_A(0)$ (see (1.2)).

Proof of Theorem 1.3. For a while we fix $\Xi_0=(\tau_0,\xi_0)\in D\setminus\Sigma$. Since $C(\Xi_0)\cap\widetilde{C_A}$ is a finite or empty set due to (2.1) , we can take $\{\omega_j^0\}_{j=1}^3\subset C(\Xi_0)\setminus\widetilde{C_A}$ with $\omega_j^0\neq\omega_k^0$ if $j\neq k$. Then $\{\,\widetilde{\omega}_j^0\,\}_{j=1}^3$ are linearly independent in $\boldsymbol{R}^4$, where $\widetilde{\omega}_j^0=(1,c\omega_j^0)$. We also take $\widetilde{\omega}_0^0$ from $\boldsymbol{R}^4$ so that $\{\,\widetilde{\omega}_j^0\,\}_{j=0}^3$ is a basis of $\boldsymbol{R}^4$. (Here we has abused the notation $\widetilde{\omega}_0^0$, which need not be expressed as $(1,c\omega_0^0)$ for some $\omega_0^0\in S^2$, to simplify notations below.) It should be notice that $\{\,\widetilde{\omega}_j^0\wedge\widetilde{\omega}_k^0\,\}_{0\le k<j\le 3}$ is also a basis of $\boldsymbol{R}^6$.

We first assume $F_A\in L^1(\boldsymbol{R}_t\times\boldsymbol{R}_x^3)$. Noting that $\langle\Xi_0,\widetilde{\omega}_j^0\rangle_{R^4}=0$ for $j=1,2,3$, we have, for each $0\le k<j\le 3$,

$$\begin{aligned}&\left\langle\widetilde{\omega}_j^0\wedge\widetilde{\omega}_k^0,\widehat{F_A}(\Xi_0)\right\rangle_{R^6}\\&=(2\pi)^{-2}\sqrt{1+c^2}\int_{\Pi_{\widetilde{\omega}_j^0}}e^{-i\langle\eta,\Xi_0\rangle}d\eta\int_{-\infty}^{\infty}\left\langle\widetilde{\omega}_j^0\wedge\widetilde{\omega}_k^0,F_A(t\widetilde{\omega}_j^0+\eta)\right\rangle_{R^6}dt.\end{aligned}$$

Since $\{\,\widetilde{\omega}_j^0\wedge\widetilde{\omega}_k^0\,\}_{0\le k<j\le 3}$ is a basis of $\boldsymbol{R}^6$ and since the integral with respect to t in the right hand side is determined by the scattering operator, $\widehat{F_A}(\Xi_0)$ is determined by the scattering operator for each $\Xi_0\in D\setminus\Sigma$.

If F_A does not belong to $L^1(\boldsymbol{R}_t\times\boldsymbol{R}_x^3)$, more complicated arguments are needed as [St], since $\widehat{F_A}$ should be regarded as a distribution. □

3 Relativistic invariance

A Lorentz transformation $\Lambda : \boldsymbol{R}^4 \longrightarrow \boldsymbol{R}^4$ is a linear map preserving the Lorentz metric,

$$\langle X, X' \rangle_{LM} := c^2 x_0 x_0' - \sum_{j=1}^{3} x_j x_j',$$

where $X = (x_0, \cdots, x_3)$, etc. This condition is equivalent to

$$(3.1) \qquad {}^t\Lambda G \Lambda = G, \quad \text{where} \quad G := \begin{pmatrix} c^2 I_1 & O \\ O & -I_3 \end{pmatrix}.$$

A Poincaré transformation $\Pi(\Lambda, a)$,

$$(3.2) \qquad \boldsymbol{R}^4 \ni X \longmapsto X' = \Lambda X + a,$$

is the combination of a Lorentz transformation Λ and a space-time translation by $a \in \boldsymbol{R}^4$ and is associated with changing an inertial frame with coordinate $X = {}^t(t, x)$ into another inertial frame with coordinate $X' = {}^t(t', x',)$ defined by (3.2).

Let $\Lambda = (\Lambda_{jk})_{0 \le j,k \le 3}$. Then it is seen from (3.1) that $\Lambda_{00} \ge 1$ or $\Lambda_{00} \le -1$. The latter case contain the time-reversal, and the argument is more complicated. So we assume the former case. Before proceeding our argument, we give typical examples. It is known that any Lorentz transformation Λ with $\det\Lambda = 1$ and $\Lambda_{00} \ge 1$ can be written as a product of these.

Example 1. **Rotation in the space.**

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \quad R \in SO(3).$$

Example 2. **Lorentz boost**
Let I be a inertial frame with coordinates (t, x), and let I' be the inertial frame with (t', x') moving with relative velocity $(v, 0, 0)$, $|v| < c$, to I. Then

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} t \\ x \end{pmatrix},$$

where

$$t' = \frac{t - (v/c^2)x_1}{\sqrt{1 - v^2/c^2}}, \quad x_1' = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}},$$
$$x_2' = x_2, \qquad x_3' = x_3.$$

By a Poincaré transformation $\Pi(\Lambda, a)$ the Dirac equation

$$i\frac{\partial}{\partial t}\Psi(t,x)$$
$$= \left[c\sum_{j=1}^{3}\alpha_j\left(\frac{1}{i}\frac{\partial}{\partial x_j} - c^{-1}A^j(t,x)\right) + \alpha_4 mc^2 + A^0(t,x)I_4\right]\Psi(t,x)$$

in the old inertial frame is converted into the Dirac equation

$$i\frac{\partial}{\partial t'}\Psi'(t',x')$$
$$= \left[c\sum_{j=1}^{3}\alpha_j\left(\frac{1}{i}\frac{\partial}{\partial x_j'} - c^{-1}A^{*j}(t',x')\right) + \alpha_4 mc^2 + A^{*0}(t',x')I_4\right]\Psi'(t',x')$$

in the new inertial frame, where $\Psi'(t',x') = L(\Lambda)\Psi(t,x)$ for some invertible 4×4 matrix $L(\Lambda)$ determined by only Λ, and $A^*(t',x') = \Lambda A(t,x)$.(Here remark that the constants in front of A^j, etc., are different from those before for our convenience.) Therefore each component of the potential $A^*(X')$ in the new frame is written as a linear combination of those of $A(\Lambda^{-1}(X'-a))$ (see, e.g., [Tha]). Let $V_1, \cdots, V_n$ be a basis of a subspace L. Then there exists a constant $K > 0$ such that

$$K^{-1}|X_L| \le \sum_{j=1}^{n} |\langle V_j, X\rangle| \le K|X_L|.$$

Hence we can see that $A^* \in S({}^t\Lambda^{-1}L)$ if $A \in S(L)$. Each component $F_{A^*}^{*lm}(X')$ of the electromagnetic field $F_{A^*}^*(X')$ in the new frame is also written as a linear combination of components $F_A^{jk}(\Lambda^{-1}(X'-a))$, $0 \le j < k \le 3$, of $F_A(\Lambda^{-1}(X'-a))$. Thus, carrying out the Fourier transform, we have

$$\widehat{F^{*lm}_{A^*}}(\Xi) = \sum_{0\le j<k\le 3} c_{jk} e^{-i\langle \Xi, a\rangle} \widehat{F^{jk}_A}({}^t\Lambda\Xi), \quad \Xi \in \boldsymbol{R}^4,$$

where c_{jk}'s are constants determined by Λ. Hence, $F^*_{A^*} \in \widetilde{S}({}^t\Lambda^{-1}L)$ if $F_A \in \widetilde{S}(L)$. On the other hand, by virtue of (3.1) we see that $\Lambda^{-1}G^{-1t}\Lambda^{-1} = G^{-1}$. Then it follows that

$$\langle {}^t\Lambda^{-1}\Xi, {}^t\Lambda^{-1}\Xi' \rangle_{LM^*} = \langle \Xi, \Xi' \rangle_{LM^*},$$

where

$$\langle \Xi, \Xi' \rangle_{LM^*} := \xi_0\xi_0' - c^2 \sum_{j=1}^{3} \xi_j \xi_j'$$

for $\Xi = (\xi_0, \cdots, \xi_3)$, etc. Thus, we can see that

$${}^t\Lambda^{-1}D = D.$$

Namely, the set D in the dual space of the space-time is invariant under Poincaré transformations in the space-time. We also see that D_1 in Proposition 1.4 is invariant.

After all we can conclude that the statements of Theorems 1.2, 1.3, 1.5, 1.7 and Corollary 1.6 does not depend on the choice of the inertial frame.

In the last of this section we show that each field F_k in Corollary 1.6 is regarded as a time-independent one on a suitable inertial frame. Let $V = V_k$ is timelike;

$$V \in T := \{\, (t, x) \in \boldsymbol{R}^4 \, ; \, c|t| > |x| \,\}$$

with $\langle V, V \rangle_{LM} = c^2$. Then it is known that there exists a Lorentz transform $\Lambda : \boldsymbol{R}_t \times \boldsymbol{R}^3_x \longrightarrow \boldsymbol{R}_{t'} \times \boldsymbol{R}^3_{x'}$ such that $\Lambda V = {}^t(1, 0, 0, 0)$. Therefore $F_k(\Lambda^{-1}\cdot)$ is t'-independent.

References

[A1] Arians, S., Geometric approach to inverse scattering for the Schrödinger equation with magnetic and electric potentials, *J. Math. Phys.* **38** (1997), 2761-2773.

[A2] Arians, S., Geometric approach to inverse scattering for Hydrogen-like systems in a homogeneous magnetic field, preprint.

[E-We] Enss, V., Weder, R., The geometrical approach to multidimensional inverse scattering, *J. Math. Phys.* **36** (1995), 3902-3921.

[F] Faddeev, L. D., Uniqueness of the inverse scattering problem, *Vestn. Leningr. Univ.* **7**(1956), 126-130.

[H] Hörmander, L., "The Analysis of Partial Differential Operators I", Springer Verlag, 1989.

[Is] Isozaki, H., Inverse scattering theory for Dirac operators, *Ann. Inst. H. Poincaré* **66** (1997), 237-270.

[Is-K] Isozaki, H., Kitada, H., Scattering matrices for two-body Schrödinger operators, *Sci. Papers Collede Arts Sci. Univ. Tokyo* **35** (1985), 81-107.

[It1] Ito, H. T., High-energy behavior of the scattering amplitude for a Dirac operators, *Pub. RIMS, Kyoto Univ.* **31** (1995), 1107-1133.

[it2] Ito, H. T., An inverse scattering problem for Dirac equations with time-dependent electromagnetic potentials, prerint.

[J] Jung, W., Geometrical approach to inverse scattering for the Dirac equation, *J. Math. Phys.* **38** (1997), 39-48.

[Ne] Newton, R. G., " Inverse Schrödinger Scattering in Three Dimensional", Springer Verlag, 1989.

[Ni] Nicoleau, F., A stationary approach to inverse scattering for Schrödinger operators with first order perturbation, *Comm. Part. Diff. Eq.* **22** (1997),527-553.

[R-S] Ramm, A. G., Sjöstrand, J., An inverse problem for the wave equation, *Math. Z.* **206**(1991), 119-130.

[Sa] Saitō, Y., An asymptotic behavior of the S-matrix and the inverse scattering problem, *J. Math. Phys.* **25**(1984), 3105-3111.

[St] Stefanov, P. D., Uniqueness of the multi-dimensional inverse scattering problem for time dependent potentials, *Math. Z.* **201** (1989), 541-559.

[Tha] Thaller, B., "The Dirac Equation", Springer Verlag, 1992.

[Wa] Wang, X. P., On the uniqueness of inverse scattering for N-body systems, *Inverse Problems* **10** (1994), 765-784.

[We1] Weder, R., Inverse scattering for N-body systems with time-dependent potentials, *in* "Inverse Problems of Wave Propagation and Diffraction", Eds. Chavent, G. and Sabatier, P. C., Lecture Notes in Physics 486, Springer Verlag,1997.

[We2] Weder, R., Multidimensional inverse scattering in an elecric field, *J. Funct. Anal.* **139** (1996), 441-465.

[We3] Weder, R., Inverse scattering for the nonlinear Schrödinger equation, *preprint* 1997.