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GABOR, WAVELET AND CHIRPLET TRANSFORMS IN THE STUDY OF PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. Gabor transforms and Wilson bases are reviewed in the study of the boundedness of pseudodifferential operators in modulation spaces and of the decay of the singular values of compact pseudodifferential operators. Similarly, the wavelet transform is reviewed in the study of the L^2 boundedness of pseudodifferential operators. The 2-parameter Gabor transform and the 2-parameter wavelet transforms are generalized to 8-parameter chirplet transforms in the context of signal processing. Open questions concerning chirplets are addressed for possible applications to pseudodifferential operators and Fourier integral operators.

1. INTRODUCTION

Given a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, the Weyl correspondence [16], [17], [27], [43], defines a pseudodifferential operator

$$L_\sigma = \sigma(D, X) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

by the formula

$$(L_\sigma)f(x) = \iint e^{2\pi i(x-y)\cdot\xi} \sigma\big((x+y)/2, \xi\big) f(y)\, dy\, d\xi.$$

The Weyl representation is common in harmonic analysis and its relation with quantum theory. The representation of pseudodifferential operators as a generalization of partial differential operators is

$$\sigma(x, D)f(x) = \iint e^{2\pi i(x-y)\cdot\xi} \sigma(x, \xi) f(y)\, dy\, d\xi.$$

In fact, the name *pseudodifferential operator* [28] comes from its use in the study of parametrices of elliptic operators. A further generalization to Fourier integral operators [26]

$$Af(x) = \iint e^{i\phi(x,y,\xi)} a(x, y, \xi) f(y)\, dy\, d\xi.$$

was needed to obtain an asymptotic representation of the solutions of hyperbolic partial differential equations (see also [29]).

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To establish the continuity of the operator L_σ on certain spaces of functions with symbol σ in certain classes of functions, one often attempts to express the operator as a sum of simpler almost orthogonal operators or use other methods, see, for example, [5], [6], [7], [23], [24], [29], [30], [40]. In [7], the use of the Paley–Littlewood gives the best possible result which is much more precise than the results obtained by the use Cotlar's lemma under the hypotheses of the following theorem.

Theorem 1. *Let $n \leq 1$ and $N > n/2$ be two integers. Suppose that the symbol $\sigma(x, \xi)$ and all its derivatives $\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)$, for $|\alpha| \leq N$ and $|\beta| \leq N$, are continuous on $\mathbb{R}^n \times \mathbb{R}^n$ and verify*

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)| \leq C(1 + |\xi|)^{\delta(|\beta| - |\alpha|)}, \qquad |\alpha| \leq N, \quad |\beta| \leq N,$$

where $0 \leq \delta < 1$ and $C > 0$ are two constants. Then the operator $\sigma(x, D)$ is bounded on $L^2(\mathbb{R}^n)$.

In the context of operator theory, in [36] Yves Meyer has used Littlewood–Paley wavelets, which provide unconditional bases for the spaces $L^p(\mathbb{R}^n)$, to prove the $T(1)$ theorem of David and Journé and thus to derive the L^2 continuity of nonclassic pseudodifferential operators in a very simple way and study their symbolic calculus. For the L^p theory of pseudodifferential operators, see, for example, [15], [25], [41], [42].

Wavelets have also found applications in new trends in microlocal analysis [38], [39]. In [37], several microlocal spaces are considered. A windowed Fourier analysis of hyperbolic chirps is used as a digression to introduce a more involved hyperbolic paving of the time-frequency plane produced by the unfolding of an orthonormal wavelet basis, called an adapted basis.

In the context of signal processing, in [20] and [21] Gabor transforms are used to establish the continuity of pseudodifferential operators on modulation spaces and to study the decay of the singular values of compact pseudodifferential operators.

In [45] and [46], Kazuya Tachizawa has used Wilson's bases, which are unconditional bases for $L^2(\mathbb{R}^n)$ but not for $L^p(\mathbb{R}^n)$ with $p \neq 2$, to establish the boundedness of pseudodifferential operators on modulation spaces and to study the spectrum of these operators.

The Gabor transform, or short-time Fourier transform (STFT), produces a shift in time and in frequency, in the time-frequency plane, of a given analyzing compactly-supported window g. On the other hand, the wavelet transform (WT) produces a shift and a scaling in time, in the time domain, of a given analyzing mother wavelet ψ.

In [31] and [32], the STFT and WT are generalized to the chirplet transform by considering an 8-parameter transformation of the time-frequency plane. By restricting this generalization to planar subspaces one obtains useful transformations such as bowtie, dispersion, warblet, and perspective transforms. It is an open question whether results for Fourier integral operators would follow simply by the use of chirplet expansions.

The modulation space approach is not new in the sense that it is a type of coherent state expansion, and coherent states have been around a long time. The new aspect is that Feichtinger and Gröchenig [12], [13], [14], have shown that coherent states are concrete bases or basis-like systems for the modulation spaces, and, as a result, one has convenient equivalent norms for these spaces in terms of the basis coefficients and this fact can be applied to derive results on pseudodifferential operators. In one sense, this is similar to how wavelets have simplified some aspects of the Littlewood–Paley theory. Wavelets now provide concrete unconditional bases for a large class of function spaces. In this sense, Besov spaces are the wavelet analogue of the modulation spaces. And Yves Meyer has used wavelets to good effect in the study of integral operators. Wavelets are simpler and easier to use than, say, function-dependent atomic decompositions. The Gröchenig–Heil approach is complementary; it uses uniform decomposition of phase space instead of dyadic decompositions, but it is the basis-like aspect that makes things simple.

2. Preliminaries

We introduce general notation.

- Inner product:

$$< f, g >= \int f(x)\overline{g(x)}\, dx, \qquad f, g \in L^2(\mathbb{R}^n).$$

- Fourier and inverse Fourier transforms:

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int e^{-2\pi i x\cdot\xi} f(x)\, dx,$$

$$\check{f}(x) = \mathcal{F}^{-1}f(x) = \hat{f}(-x) = \int e^{-2\pi i x\cdot\xi} f(\xi)\, d\xi.$$

- Translation, T, and modulation, M:

$$T_y f(x) = f(x-y), \qquad M_\xi f(x) = e^{2\pi i x\cdot\xi} f(x), \qquad x, \xi \in \mathbb{R}^n.$$

- Dilation:

$$D_a f(x) = |a|^{-n/2} f(x/a), \qquad a \in \mathbb{R} \setminus \{0\}.$$

We notice that

$$\mathcal{F}(T_y f) = M_y \hat{f}, \qquad \mathcal{F}(M_\eta f) = T_\eta \hat{f}, \qquad M_\eta T_y = e^{2\pi i y\cdot\xi} T_y M_\eta,$$

and

$$\mathcal{F}(D_a f) = D_{1/a} \hat{f}.$$

- The short-time Fourier transform (STFT) of a function $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to a window $g \in \mathcal{S}(\mathbb{R}^n)$:

$$S_g(x,\xi) =< f, M_\xi T_x g >= \int e^{-2\pi i y\cdot\xi}\overline{g(y-x)}f(y)\,dy.$$

- The wavelet transform (WT) of a function $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to a window $g \in \mathcal{S}(\mathbb{R}^n)$ is

$$S'_g(x,a) = |a|^{-1/2}\int_{-\infty}^{\infty} g(y)\overline{g\left(\frac{x-y}{a}\right)}\,dy.$$

- The Schrödinger representation of the Heisenberg group $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$ on $L^2(\mathbb{R}^n)$:

$$\rho_0(x,\xi,\theta)f(y) = \theta\, e^{\pi i x\cdot\xi}\, e^{2\pi i y\cdot\xi} f(y+x).$$

- If the toral variable θ is unimportant, with $\alpha = (x,\xi)$, the reduced Schrödinger representation is

$$\begin{aligned}\rho_0(\alpha)f(y) &= \rho_0(x,\xi)f(y)\\ &= e^{\pi i x\cdot\xi}\, e^{2\pi i y\cdot\xi} f(y+x)\\ &= e^{\pi i x\cdot\xi} M_\xi T_{-x} f(y).\end{aligned}$$

- The radar ambiguity function, $A(f,g)$, of f and g:

$$\begin{aligned}A(f,g)(x,\xi) &=< \rho_0(x,\xi)f, g >\\ &= e^{-\pi i x\cdot\xi} S_g f(x,-\xi).\end{aligned}$$

- The Wigner distribution, $W(f,g)$, of f and g is the $\mathbb{R}^n \times \mathbb{R}^n$ Fourier transform of the ambiguity function:

$$W(f,g) = \mathcal{F}\big(A(f,g)\big).$$

  The mappings A and W are sesqui-unitary on $L^2(\mathbb{R}^n)$:

$$\begin{aligned}< A(f_1,g_1), A(f_2,g_2) > &=< f_1, f_2 >< g_2, g_1 >\\ &=< W(f_1,g_1), W(f_2,g_2) > .\end{aligned}$$

  The last equality is Moyal's identity.

Using the linear transformation $M : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$,

$$M = \begin{bmatrix} 0 & -1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix},$$

where each block is a multiple of the $n \times n$ identity matrix, we have

$$W\big(\rho_0(\alpha)f, \rho(\beta)g\big) = \rho\big(M(\alpha,\beta)g\big)W(f,g),$$

where ρ_0 and ρ are the Schrödinger representations on $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^{2n})$, respectively.

The pseudodifferential operator L_σ can be expressed in the following forms:

$$\begin{aligned} < L_\sigma f, g > &=< \widehat{\sigma}, A(g,f) > \\ &=< \sigma, W(g,f) >, \qquad f,g \in \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

3. Modulation spaces

Modulation spaces measure the joint time-frequency distribution of $f \in \mathcal{S}'(\mathbb{R}^n)$. A weight function on $\mathbb{R}^n$, $w : \mathbb{R}^n \to (0,+\infty)$, satisfies the inequality

$$w(x+y) \leq C(1+|x|)^s w(y), \qquad \text{for some } s \geq 0.$$

Definition 1. Given a weight function w on $\mathbb{R}^{2n}$, a window $g \in \mathcal{S}(\mathbb{R}^n)$, and two numbers p,q such that $1 \leq p,q \leq \infty$, we define the modulation space $M^w_{p,q}(\mathbb{R}^n)$ to be the Banach space of all distributions of slow growth, $f \in \mathcal{S}'(\mathbb{R}^n)$, for which the following norm is finite:

$$\|f\|_{M^w_{p,q}} = \left\{ \int \left[\int |S_g f(x,\xi)|^p w(x,\xi)^p \, dx \right]^{p/q} d\xi \right\}^{1/q},$$

with obvious modifications if p or $q = +\infty$.

If $w \equiv 1$, we write $M_{p,q}(\mathbb{R}^n)$, and if $w(x,\xi) = (1+|x|+|\xi|)^s$, we write $M^s_{p,q}(\mathbb{R}^n)$. If $w(\pm x, \pm\xi) = w(x,\xi)$, then $M^w_{p,p}(\mathbb{R}^n)$ is invariant under Fourier transformation since

$$|S_g f(x,\xi)| = |S_{\hat{g}} \hat{f}(\xi, -x)|.$$

The following well-known function spaces are modulation spaces.

1. $M_{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.
2. The weighted L^2-spaces, with weight $w(x,\xi) = w_s(x) = (1+|x|)^s$,
$$M^w_{2,2}(\mathbb{R}^n) = L^2_s(\mathbb{R}^n) = \{f; f(x)(1+|x|)^s \in L^2(\mathbb{R}^n)\}.$$
3. The Sobolev spaces, with weight $w(x,\xi) = w_s(\xi) = (1+|\xi|)^s$,
$$M^w_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n) = \{f; \hat{f}(\xi)(1+|\xi|)^s \in L^2(\mathbb{R}^n)\}.$$
4. The space with weight $w(x,\xi) = (1+|x|+|\xi|)^s$,
$$M^s_{2,2}(\mathbb{R}^n) = L^2_s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n).$$
5. The Feichtinger algebra: $M_{1,1} = S_0(\mathbb{R}^n)$.

It is to be noted that $M_{p,p}$ is not equal to L^p. The Feichtinger algebra $M_{1,1}$ is a very nice space. For example, it is an algebra under both pointwise multiplication and convolution. It is invariant under Fourier transform. It is the minimal Banach space that is isometrically invariant (using its own norm) under both translations and modulations.

It is contained in every L^p spaces, in the Fourier algebra (consisting of Fourier transforms of L^1 functions) and in the Wiener algebra (C_0, l^1) of continuous functions with norm

$$\|f\|_{(C_0,l^1)} = \sum \|f \cdot \chi_{[k,k+1)}\|_{L^\infty}.$$

The notation (C_0, l^1) is meant to imply that this space is an "amalgam space" of functions that are locally continuous and globally l^1.

Lemma 1. *Let the matrix $A \in \mathbb{R}^{n\times n}$ be symmetric, and the convolution T and multiplication U with the chirp $e^{-\pi i x\cdot Ax}$ be defined by*

$$\mathcal{F}(Tf)(\xi) = e^{-\pi i \xi\cdot A\xi}\hat{f}(\xi) = U\hat{f}(\xi).$$

Then T leaves $M^s_{p,q}(\mathbb{R}^n)$ invariant for each $s \geq 0$ and $1 \leq p, q \leq +\infty$.

Proof. See [16], p. 179 and [20]. □

4. Boundedness of pseudodifferential operators on modulation spaces

We represent a symbol σ as a superposition of time-frequency shifts by means of the following inversion formula.

Theorem 2. *If $\phi \in \mathcal{S}(\mathbb{R}^{2n})$ and $\|\phi\|_{L^2} = 1$, then*

$$\sigma = \iint S_\phi\sigma(\alpha,\beta) M_\beta T_\alpha \phi \, d\alpha \, d\beta.$$

If $\sigma \in M^w_{p,q}(\mathbb{R}^{2n})$ with $1 \leq p, q < \infty$, the integral converges in the norm of this space. If $p = \infty$ or $q = \infty$, or if $\sigma \in \mathcal{S}'(\mathbb{R}^{2n})$, the integral converges weakly.

As a consequence of this result we have the following representation of a pseudodifferential operator:

$$L_\sigma = \iint S_\phi\sigma(\alpha,\beta) L_{M_\beta T_\alpha \phi} \, d\alpha \, d\beta.$$

Thus, arbitrary operators L_σ can be studied in terms of the "elementary" pseudodifferential operators $L_{M_\beta T_\alpha \phi}$ and the STFT $S_\phi\sigma(\alpha,\beta)$ of the symbol σ.

In [20], K. Gröchenig and C. Heil have proved the following boundedness theorem for pseudodifferential operators on modulation spaces.

Theorem 3. *If the symbol $\sigma \in M_{\infty,1}(\mathbb{R}^{2n})$, then the pseudodifferential operator L_σ is a bounded mapping of $M_{p,p}(\mathbb{R}^{2n})$ onto itself for each p, $1 \leq p \leq \infty$, and there exists a constant C_p such that*

$$\|L_\sigma\|_{M_{p,p}\to M_{p,p}} \leq C_p \|\sigma\|_{M_{\infty,1}}.$$

Proof. The proof of the theorem follows from several lemmas to be found in [20] and included here, without proofs, for the convenience of the reader. □

We choose a fixed convenient window on $\mathbb{R}^{2n}$:

$$\begin{aligned}\phi(x,\xi) &= 2^n\, e^{-2\pi(x^2+\xi^2)}\\ &= W(\varphi,\varphi)(x,\xi), \qquad (x,\xi)\in\mathbb{R}^{2n},\end{aligned}$$

where φ is the Gaussian function

$$\varphi(x) = 2^{n/4}\, e^{-\pi x^2}, \qquad x\in\mathbb{R}^n.$$

For this window, the elementary operators $L_{M_\beta T_\alpha\phi}$ are rank-one projections as follows from next Lemma 2. For this purpose, we use the $4n\times 4n$ real matrix

$$N = \operatorname{diag}(-1,-1,1,1)M = \begin{bmatrix} 0 & 1/2 & 0 & 1/2\\ -1/2 & 0 & -1/2 & 0\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix},$$

and note that

$$(\alpha,\beta) = N(\xi,\eta) \Longleftrightarrow (-\alpha,\beta) = M(\xi,\eta).$$

Lemma 2. *Using the notation $(\alpha,\beta)=N(\xi,\eta)$ and noting that $\alpha\cdot\beta=-2[\xi,\eta]$, where*

$$[\xi,\eta] = \frac{1}{2}(\xi_2\cdot\eta_1 - \xi_1\cdot\eta_2), \qquad \text{for} \quad \xi=(\xi_1,\xi_2),\eta=(\eta_1,\eta_2)\in\mathbb{R}^{2n},$$

we have the following three formulae:

$$\begin{aligned} M_\beta T_\alpha\phi &= e^{-2\pi i[\xi,\eta]}W(\rho_0(\xi)\varphi,\rho_0(\eta)\varphi),\\ L_{M_\beta T_\alpha\phi}f &= e^{-2\pi i[\xi,\eta]} < f,\rho_0(\eta)\varphi > \rho_0(\xi)\varphi,\\ L_\sigma f &= \iint S_\phi(N(\xi,\eta))\, e^{-2\pi i[\xi,\eta]} < f,\rho_0(\eta)\varphi > \rho_0(\xi)\varphi\, d\xi\, d\eta.\end{aligned}$$

Lemma 3. *If $\sigma\in M_{\infty,1}(\mathbb{R}^{2n})$, then*

$$\sup_\eta \int_{\mathbb{R}^{2n}} |S_\phi\sigma(N(\xi,\eta))|\, d\xi \le \|\sigma\|_{M_{\infty,1}} < \infty$$

and

$$\sup_\xi \int_{\mathbb{R}^{2n}} |S_\phi\sigma(N(\xi,\eta))|\, d\eta \le \|\sigma\|_{M_{\infty,1}} < \infty.$$

Hence, the mapping T defined by

$$TG(\xi) = \int_{\mathbb{R}^{2n}} S_\phi\sigma(N(\xi,\eta))G(\eta)\, d\eta$$

is a bounded mapping of $L^p(\mathbb{R}^{2n})$ into itself for each $1\le p\le+\infty$, and

$$\|T\|_{L^p\to L^p} \le \|\sigma\|_{M_{\infty,1}}.$$

Finally, we need a result of H. G. Feichtinger and K. Gröchenig [12] which is stated in terms of the weighted mixed-norm space $L^w_{p,q}(\mathbb{R}^{2n})$:

$$\|F\|^w_{p,q} = \left\{ \int \left[\int |F(x,y)|^p w(x,y)^p \, dx \right]^{p/q} dy \right\}^{1/q}.$$

Proposition 1. *Let w be a weight function on $\mathbb{R}^{2n}$ and fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$. If $F \in L^w_{p,q}(\mathbb{R}^{2n})$, then there exists a constant $C = C(\varphi)$ such that*

$$f = \iint F(x,y)\rho_0(x,y)\varphi \, dx\, dy \in M^w_{p,q}(\mathbb{R}^n),$$

where the integral converges in $M^w_{p,q}(\mathbb{R}^n)$ for $p, q < +\infty$ and

$$\|f\|_{M^w_{p,q}} \leq C\|F\|_{L^w_{p,q}}.$$

These results combine to give the proof of Theorem 3.

We remark that L_σ is bounded on $L^2(\mathbb{R}^n) = M_{2,2}(\mathbb{R}^{2n})$ when $\sigma \in M_{\infty,1}(\mathbb{R}^{2n})$. In contrast to the traditional classes, membership in $M_{\infty,1}$ does not imply smoothness of σ and Theorem 3 applies to certain non-smooth symbols. On the other hand, since the Lipschitz classes $\Lambda^s(\mathbb{R}^{2n})$ for $s > 2n$, and hence $C^{2n+1}(\mathbb{R}^{2n})$ in particular, are embedded in $M_{\infty,1}(\mathbb{R}^{2n})$, Theorem 3 is an improvement in a certain direction of the classical Calderón–Vaillancourt theorem [5].

5. Asymptotic behavior of singular values of compact pseudodifferential operators

In signal processing, the decay properties of singular values of pseudodifferential operators L_σ may be a tool to determine the quality of time-frequency filters.

Let H be a separable Hilbert space and $\mathcal{B}(H)$ be the Banach space of bounded operators mapping H to H with operator norm $\|\cdot\|_{\mathcal{B}(H)}$.

The singular values $\{s_k(L)\}_{k=1}^\infty$ of a compact operator $L \in \mathcal{B}(H)$ are defined via the spectral theory. That is, since L is compact, $L^*L \geq 0$ has a discrete spectrum,

$$0 \leq \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq \cdots \to 0, \qquad \text{as} \quad k \to +\infty.$$

The singular values of L are $s_k = +\sqrt{\sigma_k}$. Since H is a Hilbert space, we have

$$s_k(L) = \inf\{\|L - T\|_{\mathcal{B}(H)};\ \operatorname{rank}(T) < k\}.$$

Definition 2. The Schatten class $\mathcal{I}_p$ is the set of compact operators $L \in \mathcal{B}(H)$ such that

$$\{s_k(L)\}_{k=1}^\infty \in l^p.$$

It is a Banach space under the norm

$$\|L\|_{\mathcal{I}_p} = \|\{s_k(L)\}\|_{l^p} = \begin{cases} \left[\sum_k s_k(L)^p\right]^{1/p}, & 1 \leq p < \infty, \\ \sup_k s_k(L) = s_1(L) = \|L\|_{\mathcal{B}(H)}, & p = \infty. \end{cases}$$

We have the following well-known Schatten classes.

1. $\mathcal{I}_\infty$ is the space of compact operators on H.
2. $\mathcal{I}_1$ is the space of trace-class operators on H.
3. $\mathcal{I}_2$ is the space of Hilbert–Schmidt operators on H.

The Schatten quasi-ideal $\mathcal{I}_{p,q}$ consists of compact operators whose singular values lie in the Lorentz space $l^{p,q}$ defined by

$$\|\{c_k\}\|_{l^{p,q}} = \left[\sum_{k=1}^{\infty} \left|k^{1/p-1/q} c_k\right|^q\right]^{1/q}.$$

The trace-class statement in Theorem 4 improves analogous results of Ingrid Daubechies [8] and Karlheinz Gröchenig [19], which require $s > 2n$.

The continuous-type expansion used in Theorem 2 leads to the following proposition proved in [19].

Proposition 2. *If the symbol $\sigma \in S_0(\mathbb{R}^{2n})$, then $L_\sigma \in \mathcal{I}_1$.*

This result is interpreted by a special case of embedding of $L^p_a \cap \mathcal{F}L^q_b$ into S_0 proved in [19].

Proposition 3.

1. *If $s > 2n$, then $L^2_s(\mathbb{R}^{2n}) \cap H^s(\mathbb{R}^{2n}) \subset S_0(\mathbb{R}^{2n})$.*
2. *If $s \le 2n$, then $L^2_s(\mathbb{R}^{2n}) \cap H^s(\mathbb{R}^{2n}) \not\subset S_0(\mathbb{R}^{2n})$.*

The main result of this section is the next theorem which is proved in [20].

Theorem 4. *If $\sigma \in L^2_s(\mathbb{R}^{2n}) \cap H^s(\mathbb{R}^{2n})$, with $s \ge 0$, then the singular values $\{s_k(L_\sigma)\}_{k=1}^{\infty}$ satisfy the asymptotic estimate*

$$s_k(L_\sigma) = O\big(k^{s/(2n)-1/2}\big), \qquad \text{as} \quad k \to +\infty.$$

Thus

$$L_\sigma \in \mathcal{I}_{2n/(n+s),\infty} \subset \mathcal{I}_p, \qquad \text{for each} \quad p > 2n/(n+s).$$

In particular, L_σ is trace-class if $s > d$.

It is convenient to use a discrete series expansion of the symbol because singular values can be estimated via finite-rank approximations. The following lemma found in [11] will be used.

Lemma 4. *If $L \in \mathcal{I}_2$, then*

$$s_{2K}(L)^2 \le \frac{1}{K} \sum_{k>K} s_k(L)^2 \le \frac{1}{K} \inf\big\{\|L - T\|^2_{\mathcal{I}_2};\ \operatorname{rank}(T) \le K\big\}.$$

The series expansions are based on the theory of Gabor frames [22]. Specifically, the following results are used.

(A) If $\phi = 2^n\, e^{-2\pi(x^2+y^2)} = W(\varphi,\varphi)$, where $\varphi(x) = 2^{n/4}\, e^{-\pi x^2}$ is the Gaussian function, and $a, b > 0$ are chosen so that $ab < 1$, then there exist constants $A, B > 0$ such that

$$A\|f\|_{L^2}^2 \le \sum_{m,n\in\mathbb{Z}^{2n}} |S_\phi f(na, mb)|^2 \le B\|f\|_{L^2}^2, \qquad \text{for all} \quad f \in L^2(\mathbb{R}^{2n}).$$

The collection $\{M_{mb}T_{na}\phi\}_{m,n\in\mathbb{Z}^{2n}}$ is called a Gabor frame for $L^2(\mathbb{R}^{2n})$.

(B) There exists a dual window $\gamma \in \mathcal{S}(\mathbb{R}^{2n})$ such that

$$\begin{aligned} f &= \sum_{m,n\in\mathbb{Z}^{2n}} S_\gamma f(na, mb) M_{mb} T_{na}\phi \\ &= \sum_{m,n\in\mathbb{Z}^{2n}} S_\phi f(na, mb) M_{mb} T_{na}\gamma, \qquad \text{for all} \quad f \in L^2(\mathbb{R}^{2n}), \end{aligned}$$

with unconditional convergence of the series in the L^2-norm.

(C) The frame expansion in (B) also holds for all $f \in M_{p,q}^w(\mathbb{R}^{2n})$, with unconditional convergence in the norm of $M_{p,q}^w$ if $1 \le p, q < \infty$, and weak convergence if p or $q = \infty$. Furthermore, the following is an equivalent norm [18] for $M_{p,q}^w(\mathbb{R}^{2n})$ for each $1 \le p, q \le \infty$:

$$\|f\|_{M_{p,q}^w} \sim \left\{ \sum_{m\in\mathbb{Z}^{2n}} \left[\sum_{n\in\mathbb{Z}^{2n}} |S_\gamma f(na, mb)|^p w(na, mb)^p \right]^{q/p} \right\}^{1/q}.$$

Finally, one uses the following property, which holds for arbitrary windows provided they are in the Wiener algebra:

$$\forall\ \{c_{mn}\} \in l^2, \qquad \Big\|\sum_{m,n} c_{mn} M_{ma} T_{na}\phi\Big\|_{L^2}^2 \le B \sum_{m,n} |c_{mn}|^2.$$

Putting all this together, we have a proof of Theorem 4.

Corollary 1. *If $\sigma \in L_s^2(\mathbb{R}^{2n}) \cap H^s(\mathbb{R}^{2n})$, with $s \ge 0$, then*

$$L_\sigma \in \mathcal{I}_{2n/(n+s),2}.$$

6. Pseudodifferential operators and Wilson bases

The Balian–Low theorem states that if the function $\psi \in L^2(\mathbb{R})$ generate a complete orthonormal basis of $L^2(\mathbb{R})$ of the form

$$e^{2\pi i l x}\psi(x-n), \qquad l, n \in \mathbb{Z},$$

then either $x\psi(x) \notin L^2(\mathbb{R})$ or $\xi\widehat{\psi}(\xi) \notin L^2(\mathbb{R})$.

By replacing the complex exponentials by sines and cosines [9], one obtains Wilson's bases as follows. One constructs a function $\psi : \mathbb{R} \to \mathbb{R}$ satisfying the inequalities

$$|\psi(x)| \le C\, e^{-a|x|} \qquad \text{and} \qquad |\widehat{\psi}(\xi)| \le C\, e^{-b|\xi|}, \qquad a, b > 0,$$

and such that the shifted and real-modulated functions

$$\begin{aligned}
\psi_{0,n}(x) &= \psi(x-n), \\
\psi_{l,n}(x) &= \sqrt{2}\,\psi(x-n/2)\cos(2\pi l n), \qquad l \neq 0, \quad l+n \in 2\mathbb{Z}, \\
\psi_{l,n}(x) &= \sqrt{2}\,\psi(x-n/2)\sin(2\pi l n), \qquad l \neq 0, \quad l+n \in 2\mathbb{Z}+1,
\end{aligned}$$

form an orthonormal basis for $L^2(\mathbb{R})$.

In [45] and [46], Kazuya Tachizawa uses an n-dimensional unconditional Wilson basis on modulation spaces $M^w_{p,q}(\mathbb{R}^n)$ to analyze global pseudodifferential operators whose Weyl symbol $\sigma(x,\xi)$ is such that

$$\sigma(x,\xi) \to \infty \quad \text{as} \quad |x|+|\xi| \to \infty.$$

An example is the symbol $|\xi|^2+|x|^2$ of the Schrödinger operator $-\Delta+|x|^2$. A second example is the operator with symbol of the form

$$\sigma(x,\xi) = <z>^2 + <z> (2+\sin <z>),$$

where $z=(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$ and $<z> = (1+|x|^2+|\xi|^2)^{1/2}$. The method based on the existence of a parametrix is not applicable to the operator with this symbol.

The results in [46] are based on the following observation. If $f(x)\in L^2(\mathbb{R}^n)$ is localized around $x_0\in\mathbb{R}^n$ and its Fourier transform $\hat{f}(\xi)$ is localized around $\xi_0\in\mathbb{R}^n$, then the function $\sigma(x,D)f(x)$ can be approximated by the function $\sigma(x_0,\xi_0)f(x)$. Moreover, if $\{f_i\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ and each f_i is localized in phase space, then one can get an approximate diagonalization of $\sigma(x,D)$ by $\{f_i\}$.

In [44], a tight Gabor frame $\{g_{mn}\}_{m,n\in\mathbb{Z}^n}$ of $L^2(\mathbb{R}^n)$ is used to give a new sufficient condition on the symbols to ensure that the singular values of the corresponding compact pseudodifferential operators belong to the Schatten quasi-ideal $\mathcal{I}_{p,q}$. Local orthonormal trigonometric bases [2] of the form

$$\psi_{mn}(x) = 2^{n/2} g(x-m)\prod_{i=1}^{n}\cos\big[(2n_i+1)\pi(x_i-m_i)/2\big], \qquad m,n\in\mathbb{Z}^n, x\in\mathbb{R}^n,$$

are used to obtain an approximate diagonalization of elliptic pseudodifferential operators similar to the diagonalization obtained in [46] by means of Wilson bases.

7. Pseudodifferential operators and wavelets

The reader is referred to [1] for definitions and references on wavelets. In this section we only mention one indirect use of wavelets in the study of the continuity of pseudodifferential operators in $L^2(\mathbb{R}^n)$.

The celebrated $T(1)$ theorem of David and Journé [10] has been used to prove boundedness of pseudodifferential operators. To state this theorem, Yves Meyer [36] defines a special class of singular integrals on an intermediary topological vector space V such that $\mathcal{D}(\mathbb{R}^n) \subset V \subset L^2(\mathbb{R}^n)$.

Definition 3. A continuous linear operator $T : V \to V'$ is associated to a singular integral operator if there exist an exponent $\gamma \in]0,1]$, two constants C_0 and C_1, and a function $K : \Omega \to \mathbb{C}$ such that

$$|K(x,y)| \leq C_0|x-y|^{-n}, \tag{1}$$

$$|K(x',y) - K(x,y)| \leq C_1|x'-x|^\gamma|x-y|^{-n-\gamma}, \qquad \text{if} \quad |x'-x| \leq \frac{1}{2}|x-y|, \tag{2}$$

$$|K(x,y') - K(x,y)| \leq C_1|y'-y|^\gamma|x-y|^{-n-\gamma}, \qquad \text{if} \quad |y'-y| \leq \frac{1}{2}|x-y|, \tag{3}$$

and

$$Tf(x) = \int K(x,y)f(y)\,dy$$

for every function $f \in V$ and every $x \notin \operatorname{supp} f$.

Theorem 5 (David–Journé). *Let T be as in Definition 3. Then a necessary and sufficient condition for T to extend by continuity to $L^2(\mathbb{R}^n)$ is that the following three conditions be satisfied:*

1. *$T(1)$ belongs to $BMO(\mathbb{R}^n)$,*
2. *${}^tT(1)$ belongs to $BMO(\mathbb{R}^n)$,*
3. *T is weakly continuous on $L^2(\mathbb{R}^n)$.*

A proof of the $T(1)$ theorem based on wavelets is found in [36]. It is enough to establish the L^2 continuity of T under the hypotheses (1)–(3) and 1–3. We may assume that V is the topological vector space of C^1 functions with compact support. We fix a multiresolution analysis V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R}^n)$ leading to a scaling function $\varphi(x)$ and wavelets

$$\psi_\lambda(x) := 2^{nj/2}\psi_{\varepsilon/2}(2^jx-k), \qquad \lambda = 2^{-j}k + 2^{-j-1}\varepsilon, \qquad \varepsilon \in \{0,1\}^n \backslash \{(0,\dots,0)\},$$

of class C^1 with compact support. We recall that the family $\{\varphi(x-k)\}$, $k \in \mathbb{Z}^n$, forms an orthonormal basis of V_0. The functions $\psi_\lambda(x)$ are real valued and the idea of the proof is to estimate the moduli $|t_{\lambda\lambda'}|$ of the matrix entries

$$t_{\lambda\lambda'} =< T\psi_\lambda, \psi_{\lambda'} >=< \psi_\lambda, T^t\psi_{\lambda'} >$$

of the operator T in the Hilbert wavelet basis. These matrices are characterized by the decrease of their entries as one moves away form the diagonal. The l^2 continuity of such matrices follows from a lemma of Schur.

A second proof of the $T(1)$ theorem based on the lemma of Cotlar and Stein is given in [36] which we quote for the convenience of the reader.

Lemma 5. *Let H be a Hilbert space and $T_j : H \to H$ for $j \in \mathbb{Z}$ be continuous linear operators and denote their adjoints by T_j^*. Suppose that there exists a sequence $\omega(j) \geq 0$ for $j \in \mathbb{Z}$ such that*

$$\sum_{j=-\infty}^{\infty} \sqrt{\omega(j)} < \infty,$$

and the following inequalities hold:

$$\|T_j^* T_k\| \leq \omega(j-k), \qquad \|T_j T_k^*\| \leq \omega(j-k), \qquad \text{for all } j, k \in \mathbb{Z}.$$

Then, pour all $x \in H$, the series

$$\sum_{j=-\infty}^{\infty} T_j(x)$$

converges in the H norm. Putting

$$T(x) = \sum_{j=-\infty}^{\infty} T_j(x)$$

we obtain

$$\|T\| \leq \sum_{j=-\infty}^{\infty} \sqrt{\omega(j)}.$$

Consider pseudodifferential operators T of the form

$$\begin{aligned} \sigma(x, D)f(x) &= \int e^{2\pi i x\cdot\xi} \sigma(x,\xi) \hat{f}(\xi)\, d\xi \\ &= \iint e^{2\pi i (x-y)\cdot\xi} \sigma(x,\xi) f(y)\, dy\, d\xi \\ &= \int K(x,y) f(y)\, dy \end{aligned}$$

with kernel

$$K(x,y) = \int e^{-2\pi i y\cdot\xi} \sigma(x,\xi)\, d\xi.$$

If both the symbol $\sigma(x,\xi)$ of T and the symbol of the adjoint T^* of T satisfy the inequalities

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)| \leq C(\alpha,\beta)|\xi|^{|\xi|-|\alpha|},$$

it follows by an application of the $T(1)$ theorem that T belongs to a subalgebra $\mathcal{A}_\infty$ of bounded linear operators from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

An operator in $\mathcal{A}_\infty$ can be described by its kernel, its symbol or its matrix in an orthonormal wavelet basis of Littlewood–Paley and this last property allows a symbolic calculus in $\mathcal{A}_\infty$.

The standard matrix realization $(t_{\lambda\lambda'})$ of the operator T in a wavelet basis has the obvious advantages of supplying a mechanism for the compression of operators that is simply a representation of the operator in as suitably chosen basis. In [4], the non-standard form of a kernel $K(x,y)$ is obtained by evaluating the expressions

$$\alpha_{II'} = \iint K(x,y)\psi_I(x)\psi_{I'}(y)\,dx\,dy, \quad \beta_{II'} = \iint K(x,y)\psi_I(x)\varphi_{I'}(y)\,dx\,dy,$$

$$\gamma_{II'} = \iint K(x,y)\varphi(x)\psi_{I'}(y)\,dx\,dy,$$

where the notation $I_{j,k} = [2^j(k-1), 2^j k]$ and

$$\psi_{j,k}(x) = \psi_{I_{j,k}} = \psi_I(x) = 2^{-1/2}\psi(2^{-j}x - k + 1)$$

is used, and similarly for $\varphi(x)$. The non-standard form gives exact estimates for pseudo-differential operators and Calderón–Zygmund operators and other translation invariant operators. The non-standard form has the numerical advantage of producing a much sparser matrix which requires only $O(N)$ operations to apply such an $N \times N$ matrix to a vector, depending on the particular operator as opposed to $O(N \log N)$ with the standard realization of the operator.

8. Chirplets

The chirplet transform, first proposed in [33], and developed in different directions [3], is an extension of the short-time Fourier transform and wavelet transform. Informally speaking, a chirplet is a piece of chirp (see Figure 1) in the same manner a wavelet could be loosely regarded as a piece of wave. Yves Meyer first coined the term *ondelette* from the French word "onde" (English "wave"), and the diminutive "ette". The closest English translation, *wavelet*, became the accepted word in papers written in the English language. Steve Mann and Simon Haykin have coined the French term *pépielette* (combining the diminutive with the French word "pépier" or "pépiement") to designate a "piece of chirp", and similarly in English they have coined the term *chirplet*.

In this and the next sections, we shall restrict attention to functions defined on $\mathbb{R}$. Small Latin letters will refer to time or physical domain and capital letters will refer to frequency or Fourier domain.

A general class of chirplets has been proposed by S. Mann and S. Haykin [34] by means of the two transformations

$$g_{wpq} = w(t)g\big(p(t)\big)\,e^{q(t)}$$

and

$$G_{WPQ} = W(f)G\big(P(f)\big)\,e^{Q(f)},$$

where w and W are the temporal and spectral windows, respectively, p and P are the physical and Fourier "perspective" (re-sampling or warping functions), and q and Q (if pure imaginary) are the modulation functions.

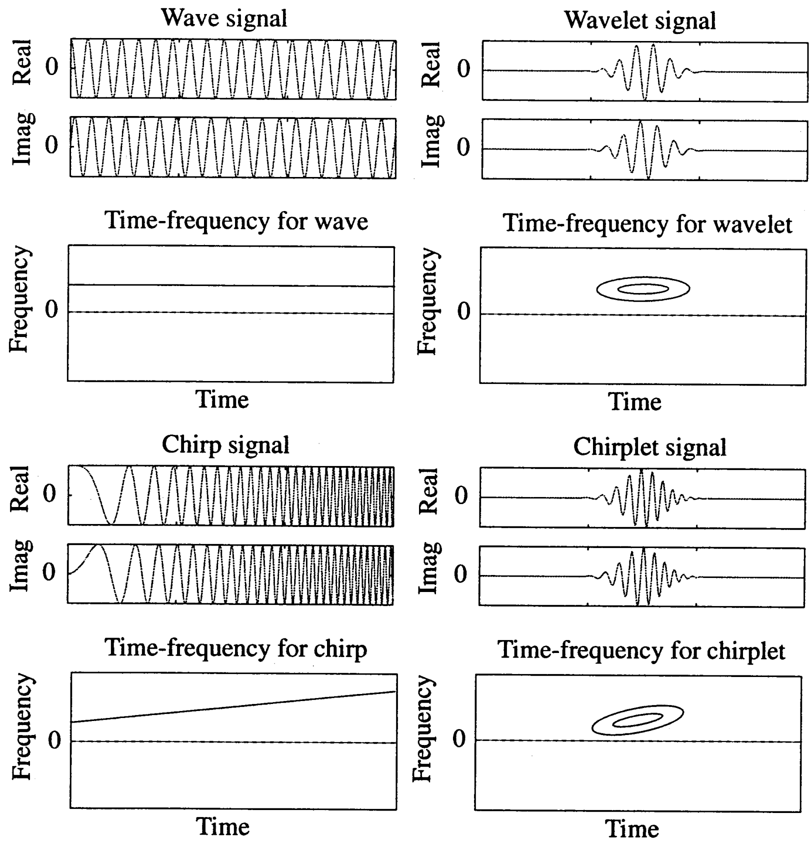

FIGURE 1. Wave, wavelet, chirp and chirplet.

Physical-domain windows are well-known to people working in the signal processing field. It has been found that windowing the data after Fourier transformation also proves to be very useful. Alternately, one may think of just convolving in the physical domain with the inverse Fourier transform of W. Thus, conceptually, both w and W are implemented in the physical domain: one windows with w and convolves with $\mathcal{F}^{-1}W$. Windowing by means of partitions of unity in the time and Fourier domains has been used in the study of pseudodifferential operators.

We now focus attention on an important distinction between time-frequency and time-scale methods.

8.1. **Modulates.** The one-dimensional time-frequency method may be regarded as expansions into (usually overdetermined) bases which may be regarded as translates and modulates of one primitive. In terms of the STFT, this primitive is the data window. Translates are time shifts and modulates are frequency shifts. A frequency shift by the amount f_c may be accomplished through multiplication by a complex exponential of frequency f_c. The Wigner distribution is an expansion of a signal into translates and modulates of itself (reversed). Hence the Wigner transform embodies this modulate class.

8.2. **Scaling.** The time-scale method (such as the wavelet transform) may be regarded as expansions into (usually overdetermined) bases which, for functions of one real variable, may be regarded as translates and dilates of one mother wavelet. Introducing the camera metaphor, where the 1-D mother wavelet is re-imaged, we see that the 1-D camera can be shifted along (translated) and zoomed in within a 2-D space. This combination of translation and zoom generates the family of wavelets as a collection of 1-D images of one mother wavelet.

The difference between the two phenomena may be best appreciated by a couple of examples. To those familiar with amateur (ham) radio, a "Type-J" modulated signal (single-sideband suppressed carrier) often has, associated with it, a *Donald Duck voice effect*; the voice is perhaps higher or lower in pitch than the original. But this pitch shift results in a very dissonant-sounding voice. The voice jiggles and the sound is very different from that of a tape recording being played at the wrong speed. The difference is that the jiggle is the result of a frequency-shift while the change in speed gives rise to a scale shift.

The Doppler effect is a process which belongs to scalings. For example, an automobile coming toward you with horns sounding an F-major 3rd interval will sound a higher pitch, but the harmonic structure of the sound is preserved, so that it still sounds a major 3rd interval. On the other hand, if one listens to a 10-meter band with a single-sideband receiver, a horn would likely sound very dissonant because each frequency component would be shifted by a certain number of herz, and the frequencies of the harmonic components would not likely end up being multiples of each other.

The chirplet is a generalization of this dichotomy which includes the modulates and the scalings as special cases. Thus the chirplet can be broken down into the same dichotomy: evolutive modulates and evolutive scalings.

8.3. **Evolutive modulates.** The family of q-chirplets is generated by a translation in frequency (modulation) which is allowed to vary in time, acting on a particular mother chirplet.

The Q-chirplets are generated by modulation in the Fourier domain: the Fourier transform of the mother chirplet is computed, and then acted on by the time-varying FM process, and then the result is brought back to the physical domain by inverse Fourier transformation. The Q-chirplets represent time shifts which are allowed to have a frequency dependency, in the same sense that the q-chirplets represent frequency shifts which are allowed to have a time dependency.

8.4. **Evolutive scalings.** The "physical perspectives" p represent projections. We may think of the mother chirplet as being projected onto a surface, or as existing on that surface and being imaged by a hypothetical camera. Now the surface orientation and camera angle give rise to a change in scale, which is different for different parts of the original time axis. In terms of the tape recorder metaphor, the physical perspectives ("p-chirplets") are the result of varying the tape speed while one mother chirplet is being played. We may view the mother chirplet as being subjected to arbitrary Doppler shifts, to generate the family of p chirplets.

The Fourier perspective (P-chirplets) are generated by arbitrary projections in the Fourier domain. Starting with a mother chirplet in the physical domain, we compute its Fourier transform, then perform a projective coordinate transformation in Fourier space, and then compute the inverse Fourier transform.

Two classes of chirplet transforms are introduced in [32] and are called metaplectic transforms (based on q-chirplets) and projective chirplet transforms (based on p-chirplets). Both are generalizations of the wavelet transform.

8.5. **Metaplectic chirplet transforms.** Metaplectic chirplet transform (MCT) is an inner product representation into the space of chirplets which may be generated from wavelets using multiplication by a complex exponential with a quadratic time term for q and convolution by a complex exponential with another quadratic in time for Q. It can be shown that these quadratic exponentials correspond, respectively, to shear in frequency and to shear in time, in the time-frequency plane. The MCT of a signal $s(\tau)$ generalizes the wavelet $|a|^{-1/2}g\big((\tau-t)/a\big)$ as follows:

$$\Gamma_s(t,f,a,q,Q) = |a|^{-1/2} < e^{2\pi i q\tau^2+t\tau}g(t) * e^{2\pi i Q\tau^2+f\tau}|s\big((\tau-b)/a\big)>,$$

where the wavelet parameter b is redundant and may be set to zero. Here and later we use the Dirac notation

$$< f|g > = < f(t)|g(t) > = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\,dt,$$

for the inner product of two functions $f, g \in L^2(\mathbb{R})$ in order to indicate dummy variables.

The algebraic structure of the metaplectic chirplet transform is that of the symplectic group [16]. In [35], it is applied to signal processing.

8.6. **Projective chirplet transforms.** The projective chirplet transform (PCT) is an inner product representation on projections of a wavelet family. The physical projective chirplet transform of s is the representation:

$$\Gamma_s(a,t,p) = < s(\tau)|g\left(\frac{a\tau+t}{p\tau+1}\right)/c > .$$

The Fourier projective chirplet of a signal s may be written:

$$\Gamma_s(1/a,f,r) = < s(\tau)|g\left(\frac{a\tau+f}{r\tau+1}\right)/C >,$$

where the Fourier perspective dilation parameter a is reciprocally related to the physical perspective dilation. Note that the physical and Fourier projective chirplet representations together also embody five parameters: t, f, a, p, r.

9. Coordinate transforms of the time-frequency plane

In previous sections, we have identified eight primitive coordinate transforms. We list the corresponding first six chirplet operators and their effect on a function $g(t)$ and on its Wigner time-frequency distribution $W_g(\tau, f)$. The last two transformations are given in the form of a scalar product.

- Translation in time:

$$T_t g(\tau) = g(\tau - t), \qquad W_g(\tau - t, \nu).$$

- Translation in frequency:

$$M_f g(\tau) = e^{2\pi i f \tau} g(\tau), \qquad W_g(\tau, \nu - f).$$

- Dilation in time:

$$D_a g(\tau) = |a|^{-1/2} g(a^{-1}\tau), \qquad W_g(a^{-1}\tau, a\nu).$$

- Dilation in frequency:

$$D_{1/a} g(\tau) = |a|^{1/2} \hat{g}(af), \qquad W_g(a\tau, a^{-1}\nu).$$

- Shear in time:

$$e^{\pi i \tau^2 Q} * g(\tau), \qquad W_g(\tau + Q\nu, \nu).$$

- Shear in frequency:

$$e^{\pi i \tau^2 q} g(\tau), \qquad W_g(\tau, \nu + q\tau).$$

- Perspective projection in the time domain:

$$\Gamma_s(a, t, p) =< s(\tau) | g\left(\frac{a\tau + t}{p\tau + 1}\right) / c > .$$

- Perspective projection in the frequency domain

$$\Gamma_s(1/a, f, r) =< s(\tau) | g\left(\frac{a\tau + f}{r\tau + 1}\right) / C > .$$

The six affine transformations of the time-frequency plane and the two perspective projections are illustrated in Figure 2.

In practice, from a computational, data storage, and display point of view, the 8-parameter chirplet transform is difficult to handle. Therefore, one considers subspaces of the entire parameter space. Planes are particularly attractive choices because of the ease with which they may be printed or displayed on a computer screen, and the fact that they

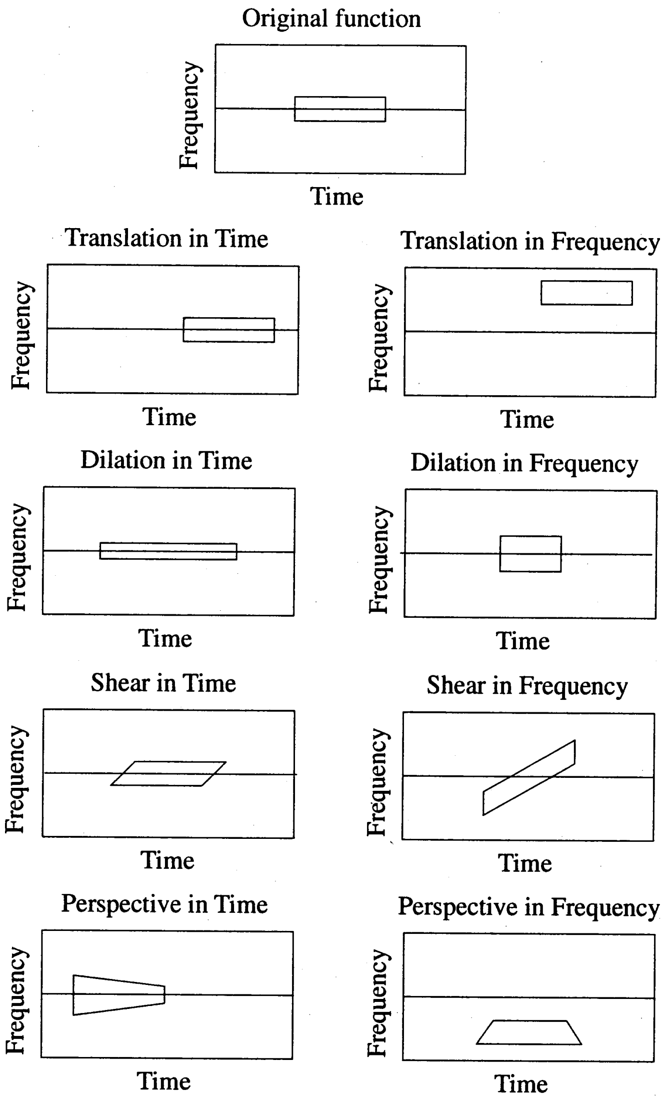

FIGURE 2. The six affine transformations and two perspective projection of the time-frequency plane.

lend themselves to admissible finite-energy signal representations. Known examples are the time-frequency (STFT) and time-scale (WT) planes discussed already.

9.1. **Bowtie chirplet.** The *bowtie* transform [32] is the three-parameter time-frequency-shear representation obtained by restricting the computation of the chirplet transform to the surface $x = (t, f, 1, 0, q, 1, 0, 0)$ in the eight-parameter coordinate transform. The bowtie transform is given by

$$B_s(t, f, q) = \int s(\tau)\overline{g(\tau - t)}\, e^{2\pi i f \tau}\, e^{\pi i q(\tau - t)^2}\, d\tau,$$

which is equivalent to a STFT with an added chirp component. Thus, in addition to measuring the time-frequency content of a signal, $B_s(t, f, q)$ also indicates its "chirpiness".

The bowtie chirplet has proven useful for studying the acceleration signature of Doppler-shifted radar signals. The extra shear parameter q of the bowtie transform makes it ideally suited to the analysis of signals of arbitrary piecewise-constant chirp rate, and thus to the analysis of constant acceleration targets.

9.2. **Dispersion chirplet.** Dispersion artifacts occur in signals acquired from media in which the wave propagation velocity varies with the frequency of the signal. The components of the signal are tilted in time-frequency. The dispersion transform of a signal s is the function on the two-parameter surface $x = (t, f, 1, Q(t, f), 0, 1, 0, 0)$ in the chirplet space:

$$D_s(t, f) = \sqrt{ip(t, f)} \int s(\tau) \left[e^{i\tau^2/2p(t,f)} * \overline{g(\tau - t)} \right] e^{-2\pi i f\tau} \, d\tau.$$

Since $Q(t, f)$ controls the tilt of basis elements (called *logon*) at the point (t, f) in time-frequency, it is called the tilt function. Generally, in situations where the dispersion characteristics of the medium are known or can be estimated, the dispersion transform yields a more concentrated time-frequency representation than either the WT or STFT.

9.3. **Warblet chirplet.** The *warblet* chirplet [35] is used to analyze signal of oscillating frequency. Examples of such signal are large pieces of ice floating in the ocean. Such a signal has a frequency that periodically rises and falls (much like the vibrato of musical instruments or the wail of a police or ambulance siren). Although the term "chirp" is most often used to describe a signal with monotonically rising or monotonically falling frequency, there is no requirement that the chirplets in the chirplet transform be monotone rising or falling. Looking closely at the physics of floating objects, one observes a somewhat sinusoidal evolution of the Doppler radar signals, suggesting that one should use some form of cyclostationary mother chirplet. When watching a cork bobbing up and down at the seaside, one notices that it moves around in a circle with a distinct periodicity. It moves up and down, but it also moves horizontally. The instantaneous frequency of the basic function is given by

$$f = \beta \cos(2\pi f_m t + p) + f_c,$$

where f_c is the center (carrier) frequency, p (which varies on the interval from 0 to 2π) is the relative position of one of the peak epochs in frequency, with respect to the origin, and f_m is the modulation frequency. Integrating, one gets the phase

$$\phi = \frac{\beta \sin(2\pi f_m t + p)}{f_m} + 2\pi f_c t,$$

which gives the family of chirplets defined by

$$g_{f_m,\beta,f_c} = A \, e^{i(\beta \sin(2\pi f_m t + p)/f_m + 2\pi f_c t)},$$

which may be appropriately windowed, such as with a Gaussian.

9.4. **Perspective-frequency transform.** A particular useful plane in image processing applications is one taken between the parameters of p_t (chirpiness due to perspective projection in the physical domain) and the frequency f [32]. Representing a signal on a set of modulated and then projected versions of one "mother chirplet", one finds that the camera orientation, with respect to almost-planar objects, could be readily determined, provided that those objects contained some form of periodicity. A practical example is, given an image photographed with the camera pointing up at a building, one notices that the windows in the building appear smaller at the top of the image than those at the bottom, even though they are all the same size in reality.

10. Conclusion and open questions

The chirplet transform allows for a unified framework for comparison of various time-frequency methods, because it embodies many other such methods as lower-dimensional subspaces in the chirplet analysis. For example, both the Short Time Fourier Transform and the wavelet transform are planar slices through an 8-parameter signal space.

Is there a multiresolution analysis for discrete and continuous chirplets? Do chirplets satisfy multiscale equations? Does one have scaling and wavelet-like chirplet functions? Do chirplets provide unconditional bases in some function spaces?

In the applications to pseudodifferential and Fourier integral operators, one is interested in compactly supported bases of orthogonal and biorthogonal chirplets in one and higher dimensions. Based on the success of the Gabor and wavelet transforms in the study of some classes of pseudodifferential operators in natural spaces, one would hope that new results could be easily obtained for other until now untractable operators by using appropriate coordinate transformations of the time-frequency plane.

One is also interested in the implementation of computationally efficient algorithms for real-time applications.

New embedding results between classical function spaces and modulation spaces would be useful for a better description of the latter spaces.

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