POLYNOMIAL HULLS WITH NO ANALYTIC STRUCTURE

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0. Introduction. Let $X$ be a compact set in $\mathbb{C}^N$ and $\hat{X}$ its polynomial hull:

$$\hat{X} := \{(z_1, ..., z_N) \in \mathbb{C}^N : |p(z_1, ..., z_N)| \leq |p|_X \text{ for all polynomials } p\},$$

where $|p|_X$ denotes the supremum norm of $p$ on $X$. If $X$ contains the boundary of an $H^\infty$ disk, i.e., if there exists a bounded, nonconstant holomorphic map $g = (g_1, ..., g_N)$ from the unit disk $\Delta$ in $\mathbb{C}$ into $\mathbb{C}^N$ with radial limit values $g^*(e^{i\theta})$ belonging to $X$ for a.e. $\theta$, then, by the maximum modulus principle, $\hat{X}$ contains the analytic disk $g(\Delta)$. In general, we say a set $S$ has analytic structure if it contains an analytic disk $g(\Delta)$. In this note, we discuss well-known examples of Stolzenberg [S] and Wermer [W] and recent modifications which show that a compact set can have non-trivial hull (i.e., $\hat{X} \neq X$) with $X$ (or at least $\hat{X} \setminus X$) containing no analytic structure. We remark that in both examples, the set $\hat{X}$ is constructed as a limit (in the Hausdorff metric) of compact subsets of analytic varieties in $\mathbb{C}^2$.

1. The Stolzenberg Example. Stolzenberg's set $X$ is a subset of the topological boundary of the bidisk $\Delta \times \Delta$ in $\mathbb{C}^2$ such that the origin $(0,0)$ lies in $\hat{X}$. However, the projection of the hull in each coordinate plane contains no nonempty open set; hence $\hat{X}$ contains no analytic structure. The rough idea of the Stolzenberg construction is, first of all, to take a countable dense set of points $\{a_j\}$ in the punctured disk $\{ t \in \mathbb{C} : 0 < |t| < 1 \}$ and form the algebraic varieties $C_j := \{(z, w) \in \mathbb{C}^2 : (z - a_j)(w - a_j) = 0\}$. These varieties avoid $(0,0)$ and have the property that each of the coordinate projections $\pi_z$ and $\pi_w$ of the union $U_j\cap(\Delta \times \Delta) = \{a_j\}$. Then a decreasing sequence of compact subsets $X_i$ of the topological boundary of the bidisk is constructed inductively so that $(0,0)$ lies in $X_i$ for each $i$ and $X_i \cap \Delta_i = \emptyset$ i.e., the hulls $\hat{X}_i$ avoid more and more of the algebraic varieties $C_j$. The intersection $X := \cap X_i$ is the desired set.

Remarks. Although the coordinate projections of $\hat{X}$ are nowhere dense, they have positive Lebesgue measure (as subsets of $\mathbb{R}^2$). This can be seen as follows: first of all, despite the lack of analytic structure in $\hat{X}$, (holomorphic) polynomials are not dense in the continuous (complex-valued) functions on $\hat{X}$, or, in the standard notation of uniform algebras, $P(\hat{X}) \neq C(\hat{X})$. Indeed, for any $p \in P(\hat{X})$, $|p|_X = |p|_X$; thus if $f \in C(\hat{X})$ satisfies $|f(0,0)| > |f|_X$ (such $f$ clearly exist), $f \notin P(\hat{X})$. Now if the coordinate projections of $\hat{X}$ have positive Lebesgue measure, by the Hartogs-Rosenthal theorem, the functions $\bar{\tau}$ and $\bar{w}$ are in $P(\hat{X})$; then, using the Stone-Weierstrass theorem, we get that $P(\hat{X}) = C(\hat{X})$, a contradiction.

Further Examples. By choosing $\{a_j\}$ a bit more carefully (in particular, to avoid an entire interval $[a,b]$ instead of just the origin), and by slightly modifying the construction of the sets $X_i$, Fornaess and the author proved the following.

Theorem 1 ([FL]). Let $D$ be a bounded domain in $\mathbb{C}^2$ with $\overline{D} = D$ and such that both coordinate projections of $D$ yield the unit disk. Let $0 < a < b < 1$. Then there exists a compact set $X \subset \partial D$ such that $\hat{X}$ contains no analytic structure but with $[a,b] \times [a,b] \subset X \setminus X$.

We remark that $[a,b] \times [a,b]$ is non-pluripolar in $\mathbb{C}^2$; i.e., if a plurisubharmonic function $u$ is equal to $-\infty$ on $[a,b] \times [a,b]$, then $u \equiv -\infty$.

Abstracting the concrete ideas in [FL], Duval and the author generalized Theorem 1.

Theorem 2 ([DL]). Let $D$ be a bounded domain in $\mathbb{C}^N$ with $\overline{D} = D$. Given $K \subset D$ with $K = \hat{K}$ (or $K \subset \overline{D}$ with $K = \overline{K} = K \cap \partial D$), there exists $X \subset \partial D$ compact with $K \subset \hat{X} \setminus X$ such that $\hat{X} \setminus X$ contains no analytic structure. In particular, if $K$ contains no analytic structure, then $\hat{X}$ contains no analytic structure.

As a corollary, by taking $K = \Gamma \times \ldots \times \Gamma$ ($N$ times) where $\Gamma$ is a Jordan arc in $\mathbb{C}$ with positive Lebesgue measure in $\mathbb{R}^2$, we get a compact set $X$ in $\partial D$ whose hull $\hat{X}$ contains no analytic structure but such that $\hat{X} \setminus X$ has positive Lebesgue measure in $\mathbb{R}^{2N}$.

Remarks. Intuitively, one might expect that if $\hat{X} \setminus X$ is nonempty but contains no analytic structure, then $\hat{X} \setminus X$ should still be "small" in some sense. The previous two theorems show that $\hat{X} \setminus X$ can still be quite
"large" in certain cases. The next result, due independently to Alexander and Sibony, shows that $\hat{X} \setminus X$ is always "large" when $\hat{X} \setminus X$ is nonempty but contains no analytic structure. Below, $h_2(S)$ denotes the Hausdorff 2−measure of a set $S$.

**Theorem 3 (Alexander [A1], Sibony [S1]).** Let $X \subset \mathbb{C}^N$ be compact and let $q \in \hat{X} \setminus X$. If there exists a neighborhood $U$ of $q$ in $\mathbb{C}^N$ with $h_2(\hat{X} \cap U) < +\infty$, then $\hat{X} \cap U$ is a one-dimensional analytic subvariety of $U$.

As a corollary, if $\hat{X} \setminus X \neq \emptyset$ and $\hat{X} \setminus X$ contains no analytic structure, then $h_2(\hat{X} \setminus X) = +\infty$.

2. **The Wermer Example.** In 1982, Wermer [W] constructed a compact set $X$ in $\partial \Delta \times \mathbb{C} \subset \mathbb{C}^2$; i.e., $\pi_z(X) = \partial \Delta$ (recall $\pi_z$ denotes the projection onto the first coordinate), with $\pi_z(X) = \overline{\Delta}$ and such that $\hat{X} \setminus X \subset \Delta \times \mathbb{C}$ does not contain any topological disk; i.e., there is no continuous nonconstant $g : \Delta \rightarrow \mathbb{C}^2$ with $g(\Delta) \subset \hat{X} \setminus X$. Clearly since $\pi_z(\hat{X} \setminus X) = \Delta$, the reason $\hat{X} \setminus X$ contains no analytic structure is not because of "small" coordinate projections as in the Stolzenberg example. Here, $\hat{X}$ is constructed as a limit (in the Hausdorff metric) of Riemann surfaces $\Sigma_n$ over $\overline{\Delta}$ which branch over more and more points. Starting with a countable dense set of points $\{a_j\}$ in $\Delta$, one chooses a sequence $\{c_j\}$ of positive numbers decreasing rapidly to 0 so that the graphs of the $2^n$−valued functions

$$g_n(z) := c_1 \sqrt{z - a_1} + c_2 (z - a_1) \sqrt{z - a_2} + \ldots + c_n (z - a_1) \cdots (z - a_{n-1}) \sqrt{z - a_n}$$

over $\overline{\Delta}$ form the desired Riemann surfaces $\Sigma_n$. To be precise, the actual construction done in [W] takes place over the disk of radius one-half centered at the origin in the $z$−plane; this yields the estimate $|a - b| < 1$ for $|a|, |b| < 1/2$.

**Remarks.** Although $\hat{X} \setminus X$ contains no analytic structure, remains some semblance of analyticity in this set. A result of Goldmann [G] shows that functions in the uniform algebra $P(X)$ behave like analytic functions in the sense that if $f \in P(X)$ vanishes on an open set $U$ (relative to $\hat{X}$), then $f$ vanishes identically. Such a uniform algebra is called an analytic algebra.

**Further Examples.** One can choose the parameters in the Wermer construction so that the intersection of $\hat{X} \setminus X$ with any analytic disk is "small".

**Theorem 4 ([L]).** There exist $X$ compact in $\partial \Delta \times \mathbb{C}$ with $\pi_z(\hat{X}) = \overline{\Delta}$ and such that $g(\Delta) \cap (\hat{X} \setminus X)$ is polar in $g(\Delta)$ for all $H^\infty$ disks $g$.

Note that in the Wermer example, we have no analytic structure in $\hat{X} \setminus X$; however, the set $X$ itself can contain lots of analytic disks. Indeed, we have the following "fattening lemma" of Alexander.

**Theorem 5 (Alexander [A2]).** There exists a Wermer-type set $X$ (compact in $\partial \Delta \times \mathbb{C}$ with $\pi_z(\hat{X}) = \overline{\Delta}$ and such that $\hat{X} \setminus X \subset \Delta \times \mathbb{C}$ contains no analytic structure) such that for all proper, closed subsets $\alpha$ of $\partial \Delta$ and all $M > 0$, setting

$$Z := X \cup \{(z, w) : z \in \alpha, |w| \leq M\},$$

we have $\hat{Z} \setminus Z = \hat{X} \setminus X$.

**Remarks.** One can also construct the Wermer set $\hat{X}$ as a decreasing intersection of the generalizedlemniscates

$$X_n := \{(z, w) : |z| \leq 1/2, |p_n(z, w)| \leq \epsilon_n\}$$

where $\{p_n\}$ are polynomials in $(z, w)$ which satisfy

1. $\Sigma_n = \{(z, w) : |z| \leq 1/2, p_n(z, w) = 0\}$;  
2. $p_n(z, w) = c_n z^{m_n} + R_n(z, w)$ where $\deg R_n < m_n := \deg p_n$;  
3. $\{\epsilon_n\}, \{\epsilon_n\}$ tend to 0 rapidly enough so that $X_{n+1} \subset X_n$ for all $n$ and $\hat{X} = \cap_n X_n$ (cf., [W]). Thus, from results in [LT], if

$$\lim_{n \to \infty} \left(\frac{\epsilon_n}{\epsilon_{n+1}}\right)^{1/m_n} = 0,$$

the set $\hat{X} \setminus X$ is pluripolar in $\mathbb{C}^2$ (see [L]).
In general, if \( X \) is compact in \( \partial \Delta \times \mathbb{C} \) with \( \pi_{z}(\hat{X}) = \overline{\Delta} \), then \( \hat{X} \setminus X \subset \Delta \times \mathbb{C} \) is pseudococoncave in the sense of Oka; i.e., \((\Delta \times \mathbb{C}) \setminus (\hat{X} \setminus X)\) is pseudococonvex. In the terminology of set-valued functions, \( \hat{X} \setminus X \) is the graph of an analytic multifunction over \( \Delta \) (cf. [SI]). Yamaguchi [Y] has shown in this setting that the function \( z \to \log C(\hat{X}_{z}) \), where \( \hat{X}_{z} := \{ w : (z, w) \in \hat{X} \} \) is the fiber of \( \hat{X} \) over \( z \) and \( C(S) \) denotes the logarithmic capacity of the compact set \( S \), is subharmonic on \( \Delta \). Thus, if there exists one \( z \in \Delta \) such that the fiber \( \hat{X}_{z} \) is non-polar in \( \mathbb{C} \), then \( \hat{X} \setminus X \) is non-pluripolar as a subset of \( \mathbb{C}^{2} \).

3. Final comments and open questions. Theorem 1 gives a concrete example of a compact set \( X \) with \( \hat{X} \setminus X \) being non-pluripolar without containing any analytic structure. It is unknown if the Wermer example can be modified in this manner.

1. **Does there exist \( X \) compact in \( \partial \Delta \times \mathbb{C} \) with \( \pi_{z}(\hat{X}) = \overline{\Delta} \) such that \( \hat{X} \setminus X \) contains no analytic structure but is non-pluripolar?**

   From the discussion in section 3, once \( \hat{X}_{z} \) is non-polar in \( \mathbb{C} \) for one \( z \in \Delta \), then \( \hat{X} \setminus X \) is non-pluripolar in \( \mathbb{C}^{2} \).

   Suppose \( S \subset \Delta \times \mathbb{C} \) is pseudococoncave. Sadullaev has shown [Sa] that \( S \) is pluripolar in \( \mathbb{C}^{2} \) if and only if each fiber \( S_{z} \) is polar ("only if" follows from Yamaguchi's result).

   2. **Let \( S \subset \Delta \times \mathbb{C} \) be pseudococoncave with each fiber \( S_{z} \) being polar. Is it true that for each \( r < 1 \), \( S' := S \setminus \{ |z| < r \} \) is complete pluripolar; i.e., there exists a plurisubharmonic in \( \{ |z| < r \} \times \mathbb{C} \) such that**

\[
S' = \{(z, w) : u(z, w) = -\infty\}?
\]

**Is it true that \( S \setminus \{ |z| \leq r \} \) is polynomially convex for each \( r < 1 \)?**

Recall that for the Stolzenberg example, \( P(\hat{X}) \neq C(\hat{X}) \). Recently, Izzo [I] has constructed an example of a compact set \( X \) in the unit sphere \( \partial B \) in \( \mathbb{C}^{3} \) which is polynomially convex \( (\hat{X} = X) \) but with \( P(X) \neq C(X) \). Note that a subset of the unit sphere \( \partial B \) in \( \mathbb{C}^{N} \) contains no analytic disk; thus there is no analytic obstruction to \( P(X) \) being dense in \( C(X) \). However, it is unknown if such an example can be constructed in \( \mathbb{C}^{2} \).

3. **Suppose \( X \subset \partial B \subset \mathbb{C}^{2} \) is compact and polynomially convex. Is \( P(X) = C(X) \)?**

We end this note by remarking that Alexander [A3] has recently constructed a compact set \( X \) in the unit torus \( \partial \Delta \times \partial \Delta \) in \( \mathbb{C}^{2} \) such that the origin \((0,0)\) lies in \( \hat{X} \) but such that \( \hat{X} \) contains no analytic structure.

References


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