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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1998: 138-142</td>
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<td>Issue Date</td>
<td>1998-04</td>
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<td>URL</td>
<td><a href="http://hdl.handle.net/2433/61944">http://hdl.handle.net/2433/61944</a></td>
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<td>Type</td>
<td>Departmental Bulletin Paper</td>
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ROBIN FUNCTIONS FOR COMPLEX MANIFOLDS AND APPLICATIONS

Norman Levenberg and Hiroshi Yamaguchi

0. Introduction. In [Y] and later in [LY] the problem of the second variation of the Robin function for a smooth variation of domains in $C^n$ for $n \geq 2$ was studied. Precisely, let $\mathcal{D} = \cup_{t \in \mathbb{R}} (t, D(t)) \subset \mathbb{B} \times \mathbb{C}^n$ be a variation of domains $D(t)$ in $C^n$ each containing a fixed point $z_0$ and with $\partial D(t)$ of class $C^\infty$ for $t \in B := \{t \in \mathbb{C} : |t| < \rho\}$. We let $g(t, z)$ for $t \in B$ and $z \in \overline{D(t)}$ be the $R_{2n}$-Green function for the domain $D(t)$ with pole at $z_0$; i.e., $g(t, z)$ is harmonic in $D(t) \setminus \{z_0\}$, $g(t, z) = 0$ for $z \in \partial D(t)$, and $g(t, z) - \frac{1}{|z - z_0|^{2n-2}}$ is harmonic near $z_0$. We call

$$\lambda(t) := \lim_{z \to z_0} [g(t, z) - \frac{1}{|z - z_0|^{2n-2}}]$$

the Robin constant for $(D(t), z_0)$. Then

$$\frac{\partial^2 \lambda}{\partial t \partial \overline{t}}(t) = -c_n \int_{\partial D(t)} k_2(t, z)||\nabla_z g||^2 d\sigma_z - \frac{1}{4c_n} \int_{(D(t))} \sum_{a=1}^n \frac{\partial^2 g}{\partial t \partial \overline{z}_a}^2 dv_z. \quad (1)$$

Here, $c_n$ is a positive dimensional constant and

$$k_2(t, z) := ||\nabla_z \psi||^{-3} \left[ \frac{\partial^2 \psi}{\partial t \partial \overline{t}} ||\nabla_z \psi||^2 - 2 \Re \left\{ \frac{\partial \psi}{\partial t} \sum_{a=1}^n \frac{\partial \overline{\psi}}{\partial \overline{z}_a} \frac{\partial^2 \psi}{\partial t \partial \overline{z}_a} \right\} + \frac{1}{|z - z_0|^2} \Delta_z \psi \right],$$

where $\psi(t, z)$ is a defining function for $\mathcal{D}$, is the so-called Levi-curvature of $\partial \mathcal{D}$ at $(t, z)$; the numerator is the sum of the Levi-form of $\psi$ applied to the $n$ complex tangent vectors $(-\frac{\partial \psi}{\partial \overline{z}_1}, 0, \ldots, 0, \frac{\partial \psi}{\partial \overline{z}_n})$. In particular, if $\mathcal{D}$ is pseudoconvex (strictly pseudoconvex) at a point $(t, z)$ with $z \in \partial \mathcal{D}(t)$, it follows that $k_2(t, z) \geq 0$ ($k_2(t, z) > 0$) so that $-\lambda(t)$ is subharmonic in $B$. Given $D$ a bounded domain in $C^n$, we let $\Lambda(z)$ be the Robin constant for $(D, z)$. If we fix a point $\zeta_0 \in D$, for $\rho > 0$ sufficiently small and $a \in C^n$, the disk $\{\zeta = \zeta_0 + at, |t| < \rho\} := \zeta_0 + aB$ is contained in $D$. Under the biholomorphic mapping $T(t, z) = (t, z - at)$ of $B \times D$, we get the variation of domains $\mathcal{D} = T(B \times D)$ where each domain $D(t) := T(t, D) = D - at$ contains $\zeta_0$. Letting $\lambda(t) = \Lambda(\zeta_0 + at)$ denote the Robin constant for $(D(t), \zeta_0)$ and using (1) yields part of the following result, which was proved in [Y] and [LY].

Theorem. Let $D$ be a bounded pseudoconvex domain in $C^n$ with $C^2$ boundary. Then $\log(-\Lambda(z))$ and $-\Lambda(z)$ are real-analytic, strictly plurisubharmonic exhaustion functions for $D$.

In this note, we study a generalization of the second variation formula (1) to complex manifolds. We use our new formula to develop a “rigidity lemma” which allows us to construct, in certain cases, strictly plurisubharmonic exhaustion functions for Levi-pseudoconvex subdomains $D$ of complex manifolds; i.e., we use the Robin function to verify that $D$ is Stein. We remark that when we use the term pseudoconvex in describing certain complex manifolds or domains in complex manifolds, we always mean Levi-pseudoconvex.

1. The variation formula. Our general set-up is this: let $M$ be an $n$-dimensional complex manifold (compact or not) equipped with a Hermitian metric

$$ds^2 = \sum_{a,b=1}^n g_{ab} dz_a \otimes d\overline{z}_b$$

and let $\omega := i \sum_{a,b=1}^n g_{ab} dz_a \wedge d\overline{z}_b$ be the associated (real) $(1,1)$-form. As in the introduction, we take $n \geq 2$. We write $g^{ab} := (g_{ab})^{-1}$ for the elements of the inverse matrix to $(g_{ab})$. Given the standard operators $*, \partial, \overline{\partial}, d = \partial + \overline{\partial}$, $\delta := -\partial*$, we get the Laplacian operator

$$\Delta = \delta \overline{\partial} + \overline{\delta} \partial + \overline{\delta} \delta + \delta \delta,$$

which, in local coordinates acting on functions has the form

$$\Delta u = -2\left\{ \sum_{a,b=1}^n g^{ab} \frac{\partial^2 u}{\partial z_a \partial \overline{z}_b} + \frac{1}{2} \sum_{a,b=1}^n \left( \frac{\partial (G g^{ab})}{\partial z_a} \frac{\partial u}{\partial \overline{z}_b} + \frac{\partial (G g^{ab})}{\partial \overline{z}_a} \frac{\partial u}{\partial z_b} \right) \right\}$$
where $G := \det(g_{a\overline{b}})$. We remark that if $ds^2$ is Kähler, i.e., if $d\omega = 0$, then $\Delta u = -2 \sum_{a,b=1}^{n} g^{a\overline{b}} \frac{\partial^2 u}{\partial \overline{z}_a \partial z_b}$.

Given a nonnegative $C^\infty$ function $c = c(z)$ on $M$, we call a $C^\infty$ function $u$ on an open set $D \subset M$ a $c-$harmonic on $D$ if $\Delta u + cu = 0$ on $D$. In particular, if we fix a point $p_0 \in M$ and a coordinate neighborhood $U$ of $p_0$, we can find a $c-$harmonic function $Q_0$ in $U \setminus \{p_0\}$ satisfying

$$\lim_{p \to p_0} \frac{Q_0(p)}{d(p, p_0)^{2n-2}} = 1$$

where $d(p, p_0)$ is the geodesic distance (with respect to the metric $ds^2$) between $p$ and $p_0$. We call $Q_0$ a fundamental solution for $\Delta$ and $c$ at $p_0$. Fixing $p_0$ in a smoothly bounded domain $D \subset M$ and fixing a fundamental solution $Q_0$, the $c-$Green function $g$ for $(D, p_0)$ is the $c-$harmonic function in $D \setminus \{p_0\}$ satisfying $g = 0$ on $\partial D$ ($g$ is continuous up to $\partial D$) and $g(p) - Q_0(p)$ is regular at $p_0$. We note that, provided $c \neq 0$, the $c-$Green function always exists (cf. [NS]) and is nonnegative on $D$. Then

$$\lambda := \lim_{p \to p_0} [g(p) - Q_0(p)]$$

is called the $c-$Robin constant for $(D, p_0)$.

Now let $D = \cup_{t \in B}(t, D(t)) \subset B \times M$ be a variation of domains $D(t)$ in $M$ each containing a fixed point $p_0$ and with $\partial D(t)$ of class $C^\infty$ for $t \in B$. Let $g(t, z)$ be the $c-$Green function for $(D(t), p_0)$ and $\lambda(t)$ the corresponding $c-$Robin constant.

We have

$$\frac{\partial^2 \lambda}{\partial t \partial \overline{t}}(t) = -c_n \int_{\partial D(t)} k_2(t, z) \sum_{a,b=1}^{n} (g^{a\overline{b}} \frac{\partial g}{\partial \overline{z}_a} \frac{\partial g}{\partial z_b})d\sigma_z$$

$$-4c_n \{||\alpha\frac{\partial g}{\partial t}||^2_{\partial D(t)} + \frac{1}{2}||\sqrt{c}\frac{\partial g}{\partial t}||^2_{\partial D(t)} + \int \int_{D(t)} \left[ \frac{1}{it} \left( \frac{\partial}{\partial t} \frac{\partial}{\partial \overline{t}} \wedge \partial \ast \omega \right) + \frac{1}{2it} \right] \right\}$$

where $||f||^2_{\partial D(t)} = \int_{\partial D(t)} f \wedge \ast f \geq 0$, $d\sigma_z$ is the area element on $\partial D(t)$ with respect to the Hermitian metric, and

$$k_2(t, z) := \left[ \sum_{a,b=1}^{n} g^{a\overline{b}} \frac{\partial \psi}{\partial \overline{z}_a} \frac{\partial \psi}{\partial z_b} \right]^{-\frac{3}{2}} \left[ \sum_{a,b=1}^{n} g^{a\overline{b}} \frac{\partial \psi}{\partial \overline{z}_a} \frac{\partial \psi}{\partial z_b} \right] - 2R \left( \frac{\partial \psi}{\partial t} \left( \sum_{a,b=1}^{n} g^{a\overline{b}} \frac{\partial \psi}{\partial \overline{z}_a} \frac{\partial^2 \psi}{\partial z_b \partial \overline{t}} \right) + \frac{1}{it} \right] \left( \sum_{a,b=1}^{n} g^{a\overline{b}} \frac{\partial^2 \psi}{\partial z_a \partial \overline{z}_b} \right),$$

$\psi(t, z)$ being a defining function for $D$.

Note that if $D$ is pseudoconvex at a point $(t, z) \in \partial D$ with $z \in \partial D(t)$, then $k_2(t, z) \geq 0$. This follows since we can always choose local coordinates near a point $z \in M$ so that $g_{a\overline{b}}(z) = \delta_{a\overline{b}}$. A simple calculation shows that $\partial \ast \omega = 0$ if $ds^2$ is a Kähler metric; hence we have the following result.

**Corollary 1.1.** Suppose that $ds^2$ is a Kähler metric on $M$. Then

$$\frac{\partial^2 \lambda}{\partial t \partial \overline{t}}(t) = -c_n \int_{\partial D(t)} k_2(t, z) \sum_{a,b=1}^{n} (g^{a\overline{b}} \frac{\partial g}{\partial \overline{z}_a} \frac{\partial g}{\partial z_b})d\sigma_z - 4c_n \{||\alpha\frac{\partial g}{\partial t}||^2_{\partial D(t)} + \frac{1}{2}||\sqrt{c}\frac{\partial g}{\partial t}||^2_{\partial D(t)} \}.$$  \hspace{1cm} (1')

In particular, if $D$ is pseudoconvex in $B \times M$, then $-\lambda(t)$ is subharmonic on $B$.

**Remark 1.** Formula (1') is valid under the weaker assumption that the complex torsion of the metric $g_{a\overline{b}}$ vanishes. We do not discuss this notion here. Note that (1') reduces to (1) if $g_{a\overline{b}} = \delta_{a\overline{b}}$ and $c \equiv 0$.

We consider the same situation as in the corollary. From the variation formula (1') and continuity of $g(t, z)$ up to $\partial D(t)$, we get the following result.
Lemma 1.2 (rigidity). Assume $\mathcal{D}$ is pseudoconvex in $B \times M$, $ds^2$ is a Kähler metric on $M$ and that there exists $t_0 \in B$ such that $\frac{\partial^2 g}{\partial t \partial t}(t_0) = 0$. If $c(z) \neq 0$ on $D(t_0)$, then

$$\frac{\partial g}{\partial t}(t_0, z) \equiv 0 \text{ on } D(t_0).$$

Remark 2. The same conclusion is valid if we assume that $\partial D(t_0)$ has one strictly pseudoconvex boundary point (instead of (in addition to) assuming $c(z) \neq 0$ on $D(t_0)$). However, the importance of the above formulation of the rigidity lemma is that, as we will see below, the function $c$ gives us extra flexibility in order to deduce strict pseudoconvexity in certain cases.

2. Complex Lie groups. We apply the rigidity lemma to the study of complex Lie groups. Let $M$ be a complex Lie group of complex dimension $n$ with identity $e$ equipped with a Kähler metric $ds^2$ and let $c = c(z)$ be a nonnegative $C^\infty$ function on $M$. Let $D \subset M$ be a domain in $M$ with smooth boundary. For $z \in D$, let

$$D(z) := \{ wz^{-1} \in D : w \in D \} = D \cdot z^{-1}$$

be right-translation (multiplication) of $D$ by $z^{-1}$. Note that $D(z)$ is a smoothly bounded domain in $M$ which contains $e$ if $z \in D$; if $D$ and hence $D(z)$ is unbounded, the $c-$Green function for $(D(z), e)$ can be defined as a limit of $c-$Green functions for $(D_k(z), e)$ where $(D_k(z))$ are bounded domains with $D_k(z) \subset D_{k+1}(z)$ and $\cup D_k(z) = D(z)$. Let $\Lambda(z)$ denote the $c-$Robin constant for $(D(z), e)$ (we assume, a priori, that a fundamental solution $Q_0$ for $\Delta$ and $c \neq 0$ is fixed). Our first main result is the following.

Theorem 2.1. Suppose $D \subset M$ is pseudoconvex. Then

1. $-\Lambda(z)$ is a plurisubharmonic exhaustion function for $D$;
2. if $c > 0$, then $-\Lambda(z)$ is a strictly plurisubharmonic exhaustion function for $D$ if and only if $D$ is Stein; indeed, if the complex Hessian matrix $[\frac{\partial^2(-\Lambda)}{\partial z \partial \overline{z}}(\zeta)]$ has a zero eigenvalue with (geometric) multiplicity $k \geq 1$ at some point $\zeta \in D$, then the complex Hessian matrix of any plurisubharmonic exhaustion function $s(z)$ for $D$ has a zero eigenvalue with (geometric) multiplicity at least $k$ at each point $z \in D$.

We will sketch the proof of Theorem 2.1. First we remark that there exist $n$ linearly independent left-invariant holomorphic vector fields $X_1, ..., X_n$ such that $\text{Expt}X_j$, $j = 1, ..., n$ form local coordinates in a neighborhood $V$ of the identity $e \in M$; then $\text{Expt}X_j$, $j = 1, ..., n$ form local coordinates in a neighborhood $\mathcal{V}$ of $\zeta \in M$. If we fix a direction vector $\alpha$ and consider the complex disk $t \rightarrow \zeta + at$ for small $|t|$, we can assume that $\text{Expt}X_1 = \zeta + at$; for simplicity, we write $X := X_1$. This suggests, as in the variation of domains case described in the introduction, how to set up a variation of domains in the setting of the complex Lie group $M$. We note, for future use, that $t \rightarrow z \text{Expt}X$ is the unique integral curve to $X$ taking the value $z \in M$ for $t = 0$.

We now let $\zeta$ be a fixed point in $D$ and choose $B = \{ t \in C : |t| < \rho \}$ with $\rho$ sufficiently small so that

$$\eta := \zeta \text{Expt}X = \zeta + at \in D \text{ for all } t \in B.$$  

Let $T : B \times M \rightarrow B \times M$ via $T(t, z) = (t, F(t, z)) := (t, w)$ where $w = F(t, z) := z(\text{Expt}X)^{-1}$. Then $\mathcal{D} := T(B \times D)$ defines a variation of domains $D(t) := F(t, D) = \{ z(\text{Expt}X)^{-1} \in M : z \in D \} = D \cdot (\text{Expt}X)^{-1}$.

Let $g(t, w)$ be the $c-$Green function for $(D(t), e)$ and let $\lambda(t) := \Lambda(\text{Expt}X) \text{ for } t \in B$; this is the $c-$Robin constant for $(D(t), e)$ (note $e \in D(t)$ if $t \in B$ by (2)). Then

$$\sum_{j, k=1}^{n} \frac{\partial^2(-\Lambda)}{\partial \eta_j \partial \overline{\eta}_k}(\zeta)\alpha_j \overline{\alpha}_k = \frac{\partial^2(-\Lambda)}{\partial t \partial t}(\text{Expt}X)|_{t=0} = \frac{\partial^2(-\Lambda)}{\partial t \partial t}(0).$$

The plurisubharmonicity of $-\Lambda(z)$ now follows from Corollary 1.1 and the fact that $\mathcal{D} := T(B \times D)$ is the biholomorphic image of the pseudoconvex set $B \times D$; indeed, for each $t \in B$, the function $z = \phi(t, w) = (\phi_1(t, w), ..., \phi_n(t, w)) := w \zeta \text{Expt}X = F^{-1}(t, w)$ is the well-defined holomorphic inverse map of $z \rightarrow w = F(t, z)$ for all $w \in M$. Standard arguments show that $\Lambda(z) \rightarrow -\infty$ as $z \rightarrow z' \in \partial D$ which proves 1. of the theorem.
We will prove 2. in the case where $k = 1$; here, we use the assumption that $c > 0$ and apply the rigidity lemma. The key observation is the following.

**Claim:** Suppose that $\frac{\partial^2 G}{\partial t \partial \overline{z}}(0) = 0$.

a. $z \in D$ (resp. $\partial D$, $\overline{D}$) if and only if $z \exp t X \in D$ (resp. $\partial D$, $\overline{D}$) for all $t \in C$;

b. $D \cdot \cdot \cdot = D \cdot (z \exp t X)^{-1}$ (resp. $\partial D$, $\overline{D}$) for all $t \in C$ and for each $z \in M$.

To prove the claim, we apply the rigidity lemma to show that the left-invariant holomorphic vector field $X$ is a non-vanishing holomorphic vector field on $M$ satisfying the property that any integral curve $z(t)$ of $X$ with initial value $X(z_0)$ for $z_0 = z(0) \in \partial D$ remains in $\partial D$ for all $t \in C$. This is one implication in part a. of the claim for $\partial D$.

Recall that $z = \phi(t, w) = (\phi_1(t, w), ..., \phi_n(t, w)) := w \cdot \exp t X = F^{-1}(t, w)$ for all $w \in M$. Let $t \to \phi(t, w)$ be the (moving) image under $\phi$ of the identity element. Note that if $ds^2(t)$ denotes the pull-back of the metric $ds^2(w)$ under $F(t, z)$, then the Green function $G(t, z)$ for $D$ with pole at $\phi(t, w)$ (with respect to $ds^2(t)$) equals $g(t, w)$. The assumption that $\frac{\partial^2 G}{\partial t \partial \overline{z}}(0) = 0$ yields, by the rigidity lemma, $\frac{\partial G}{\partial t}(0, w) \equiv 0$ for $w \in \overline{D}$; this becomes

$$\frac{\partial G}{\partial t}(0, z) + \sum_{a=1}^{n} \frac{\partial G}{\partial z_{a}}(0, z) \frac{\partial \phi_{a}}{\partial t}(0, F(0, z)) + \frac{\partial G}{\partial \overline{z}_{a}}(0, z) \frac{\partial \overline{\phi}_{a}}{\partial t}(0, F(0, z)) = 0$$

for $z \in \overline{D}$. But $\frac{\partial G}{\partial t}(0, F(0, z)) = 0$ since $\phi(t, w) = (\phi_1(t, w), ..., \phi_n(t, w))$ is holomorphic in $t$; and $\frac{\partial G}{\partial t}(0, z) = 0$ for $z \in \partial D$ since $G(t, z) = 0$ for $z \in \partial D$ and $t \in B$. Thus

$$\sum_{a=1}^{n} \frac{\partial G}{\partial z_{a}}(0, z) \frac{\partial \phi_{a}}{\partial t}(0, F(0, z)) = 0$$

(4)

for $z \in \partial D$. Since $\phi(t, w)$ is defined for all $w \in M$, the vector field

$$Y(z) := \sum_{a=1}^{n} \frac{\partial \phi_{a}}{\partial t}(0, F(0, z)) \frac{\partial G}{\partial z_{a}}(0, z)$$

is a globally defined (on $M$) non-vanishing holomorphic vector field; using the fact that

$$\left(\frac{\partial G}{\partial z_{1}}(0, z), ..., \frac{\partial G}{\partial z_{n}}(0, z)\right)$$

is a (complex) normal vector to $\partial D$ at $z$, it can be shown that (4) implies that any integral curve $z(t)$ of $Y$ with initial value $Y(z_0)$ for $z_0 = z(0) \in \partial D$ remains in $\partial D$ for all $t \in C$. Thus, to verify the italicised statement, it suffices to show that $Y = X$.

Since $X$ is left-invariant, if $X(z) = \sum_{a=1}^{n} \eta_{a} \frac{\partial}{\partial z_{a}}$, then $\frac{\partial}{\partial t}(z \exp t X)_{a} |_{t=0} = \eta_{a}(z)$, $a = 1, ..., n$. But for

$$w = z \zeta^{-1},$$

$$\frac{\partial \phi_{a}}{\partial t}(0, F(0, z)) = \frac{\partial \phi_{a}}{\partial t}(0, w) = \frac{\partial }{\partial t}(w \zeta \exp t X)_{a} |_{t=0} = \eta_{a}(w \zeta) = \eta_{a}(z),$$

which gives the result.

The proof of the claim is now immediate. For example, to establish a. for $\partial D$; i.e., to show $z \in \partial D$ if and only if $z \exp t X \in \partial D$ for all $t \in C$, the "only if" direction has already been proved. Suppose now that $z \exp t X \in \partial D$ for all $t \in C$. Since

$$z = z(\exp t X)(\exp(-tX)) := z' \exp(-tX)$$

where $z' = z \exp t X \in \partial D$, the previous argument shows that $z \in \partial D$. Since $\partial D$ is a smooth, closed $(2n - 1)$-dimensional real hypersurface in $M$, the analogous results for $D$ and $\overline{D}$ follow from uniqueness of the integral curve $t \to z \exp t X$. Similarly we prove b. only for $\partial D$. Let $z_1 \in \partial D$ and $z \in M$. Since $z_1 \exp t X \in \partial D$ for all $t \in C$ from a. of the claim, the equation

$$z_1 z^{-1} = z_1(\exp t X)(\exp(-tX))z^{-1} = z_1(\exp t X)(z \exp t X)^{-1}$$
yields b. of the claim.

We can now finish the proof of 2. of the theorem. For a point $\zeta \in D$, let $a_i(\zeta)$, $i = 1, \ldots, n$ denote the eigenvalues of $\left[\frac{\partial^2(-\Lambda)}{\partial z_j \partial \overline{z}_k}(\zeta)\right]$. To prove 2. in the case $k = 1$, we suppose there exists a point $\zeta \in D$ with $a_1(\zeta) = 0$; without loss of generality, we can assume that $\zeta \text{Expt}X_1 = \zeta + \alpha t$ gives the direction of the corresponding eigenvector; i.e.,

$$\frac{\partial^2(-\Lambda)}{\partial \alpha^2}(\zeta + \alpha t)|_{t=0} = 0.$$ 

(5)

Taking $X = X_1$ in the previous claim, $D_1(t) := D \cdot (\zeta \text{Expt}X_1)^{-1}$ and $\lambda_1(t) := \lambda(\zeta \text{Expt}X_1)$, (5) becomes

$$\frac{\partial^2(-\Lambda)}{\partial t \partial \overline{t}}(0) = 0.$$ 

Then the integral curve $t \rightarrow \zeta \text{Expt}X_1$, $t \in C$, satisfies the conditions of the claim. In particular, if $z \in D$, then $D \cdot z^{-1} = D \cdot (\zeta \text{Expt}X_1)^{-1}$ for all $t \in C$ which implies that

$$\lambda(z \text{Expt}X_1) = \lambda(z) \text{ for all } t \in C.$$ 

But $-\Lambda$ is an exhaustion function for $D$; hence the image $C_{z}$ of the integral curve $t \rightarrow \zeta \text{Expt}X_1$, $t \in C$ is compactly contained in $D$ and $-\Lambda$ is constant on $C_{z}$. In particular, $-\Lambda$ is harmonic on $C_{z}$ for each $z \in D$; i.e., $\left[\frac{\partial^2(-\Lambda)}{\partial z_j \partial \overline{z}_k}(z)\right]$ has a zero eigenvalue $a_1(z)$ for each $z \in D$. But then if $s$ is any plurisubharmonic exhaustion function for $D$, $s$ is also subharmonic and entire on each complex curve $C_{z}$ and hence constant (and harmonic) on this curve, which implies that $\left[\frac{\partial^2s}{\partial z_j \partial \overline{z}_k}(z)\right]$ has a zero eigenvalue for each $z \in D$. In particular, $D$ is not Stein.

Remark 3. Note that if $M$ is a Stein manifold, then each pseudoconvex $D \subset M$ is Stein; this occurs, for example, if $M$ is a simply connected solvable Lie group or if $M$ is connected and semi-simple (cf. [GR]).

3. Complex homogeneous spaces. In this section, we let $M$ be a complex space with the property that there exists a complex Lie group $G \subset \text{Aut}M$ of complex dimension $n$ which acts transitiely on $M$. As prototypical examples, we can take $M = \mathbb{P}^N = \text{complex projective space}$, or, more generally, we can take $M = G(k, N) = \text{complex Grassmann manifold}$ (and $G = \text{Aut}M$). Let $D \subset M$ be a domain with smooth boundary. For $z \in M$, we let

$$D(z) := \{g \in G : g(z) \in D\}$$

be a (possibly unbounded) domain in $G$. Note that if $z \in D$, then the identity element $e$ of $G$ lies in $D(z)$. Thus if we let $ds^2$ be a Kähler metric on $G$ and let $c$ be a nonnegative smooth function on $G$, we can form the $c$-Robin constant $\lambda(z)$ for $(D(z), c)$ (recall that the $c$-Green function is defined by the usual exhaustion method for unbounded domains). Using the ideas and techniques from the previous section, we can prove the following result.

Theorem 3.1. Suppose $D$ is pseudoconvex in $M$. Then for $z \in D$, $D(z)$ is pseudoconvex in $G$ and $-\lambda(z)$ is a plurisubharmonic exhaustion function for $D$. Furthermore, if $c > 0$ in $G$ and $G$ is doubly transitive on $M$, then $-\lambda(z)$ is strictly plurisubharmonic; i.e., $D$ is Stein.

Recall that $G$ is doubly transitive on $M$ if for pairs of points $(a, b)$, $(c, d) \in M$, there exists $g \in G$ with $g(a) = c$ and $g(b) = d$. This is equivalent to the three point property of $(M, G)$: for each triple of points $a, b, c \in M$, there exists $g \in G$ with $g(a) = a$ and $g(b) = c$. Details of the proof of Theorem 3.1 will be given elsewhere.

References


Levenberg: Department of Mathematics, University of Auckland, Private Bag 92019 Auckland, NEW ZEALAND

Yamaguchi: Department of Mathematics, Shiga University, Otsu-City, Shiga 520, JAPAN