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Extension Problem of Holomorphic Functions

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Abstract

In the Summer Seminar at Tateyama on Several Complex Variables, 18th July 1994, Professor T. Ohsawa[19] posed the following problem.

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $H$ be a one-codimensional complex linear subspace of $\mathbb{C}^n$. For any bounded holomorphic function $g$ on $\Omega \cap H$, does there exist a bounded holomorphic function $f$ on $\Omega$ such that the restriction $f|\Omega \cap H$ of $f$ to $\Omega \cap H$ coincides with $g$ on $\Omega \cap H$?

We give a counterexample for Ohsawa's Problem of a connected subvariety $H$ instead of a single hyperplane, all holomorphic functions on which cannot be extended to the whole domain $\Omega$ with smooth boundary.

1 Introduction.

In the present paper, we investigate the problem of extending bounded holomorphic functions from one-codimensional subvarieties to ambient spaces.

At first, we give survey under what conditions the problem on bounded holomorphic expansion was already affirmatively solved:

Let $X$ be a complex space, $A$ be an analytic subset in $X$ and $Y$ be a complex space. We say that Oka's principle holds for $(X, A, Y)$ if the following assertion holds: Any holomorphic mapping $f$ of $A$ in $Y$ is extended to a holomorphic mapping of $X$ into $Y$ if and only if $f$ is extended to a continuous mapping of $X$ into $Y$.

In case that $X$ is a Stein space and $C$ is the complex plane, by Cartan - Serre's theorem, Oka's principle holds for $(X, A, \mathbb{C}^*)$. In case that $X$ is a Stein manifold and $L$ is an abelian complex Lie group, Kajiwara[16] proved that Oka's principle holds for $(X, A, L)$. Kajiwara-Kazama[17] generalized the above result in proving that Oka's principle holds for $(X, A, L)$ in case that $X$ is a Stein space and that $L$ is a complex Lie group with parameter space in a complex Banach space.

H. Alexander[4] considered the problem in case that $H$ is a Rudin variety in the unit polydisk $\Delta^N$ of $\mathbb{C}^N$.

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M. Henkin-P. L. Polyakov[14] and P. L. Polyakov[21] gave the theories on the extension problem in case that $H$ is an analytic curve in general position in a polydisc in $\mathbb{C}^n$.

G. M. Henkin[12] proved the problem in case that $\Omega$ is a strictly pseudoconvex domain and $H$ is an analytic closed submanifold in general position in $\Omega$, i.e., $H = \bar{H} \cap \Omega$ where $\bar{H}$ is a submanifold in a neighborhood of $\Omega$ and intersects $\partial D$ transversally.

K. Adachi[2] proved that Henkin’s results are still valid when $\Omega$ is a pseudoconvex domain with smooth boundary and $H$ is a subvariety where $\partial H \cap \Omega$ consists of strictly pseudoconvex boundary points of $\Omega$.

The author[23] gave a counterexample for the Ohsawa’s problem in case that $\Omega$ is an unbounded weakly pseudoconvex domain, the boundary of which is not smooth. Moreover, H. Hamada and the author[10] gave a counterexample in case that $\Omega$ is bounded but has not smooth boundary, using Sibony’s domain.

The aim of the present paper is to give a counterexample for the Ohsawa’s problem of a connected subvariety $H$ instead of a single hyperplane, all holomorphic functions on which cannot be extended to the whole domain $\Omega$ with smooth boundary. The boundary of the subvariety $H$ consists of strictly pseudoconvex boundary points of $\Omega$, but $H$ is not in general position in a pseudoconvex domain $\Omega$.

\section{Main Results.}

Let $\Delta(z, r)$ be the disk with center $z$ and semiradius $r$ in the complex plane. The unit disk $\Delta(0, 1)$ is denoted by $\Delta$.

\textbf{Lemma 2.1 (Sibony[22])} Let $\{a_\nu\}_{\nu=1}^\infty$ be a sequence of points without cluster point in $\Delta$ such that each point of the unit circle $\partial \Delta$ is the nontangential limit of a subsequence of $\{a_\nu\}_{\nu=1}^\infty$. We define a function $\lambda : \Delta \to \mathbb{R} \cup \{-\infty\}$ by

$$
\lambda(z) = \sum_{\nu=2}^\infty \epsilon_\nu \log \left| \frac{z - a_\nu}{2} \right|
$$

where $\epsilon_\nu \not\equiv 0$ rapidly so that $\lambda \not\equiv -\infty$ and is subharmonic on $\Delta$. Further let $\psi : \Delta \to [0, 1)$ be the subharmonic function $\psi(z) = \exp(\lambda(z))$.

Define a pseudoconvex domain $U \subset \Delta^2$ by

$$
U = \{(z, w) \in \Delta^2; |w| < e^{-\psi(z)}\}.
$$

$U$ is a proper subdomain of $\Delta^2$ and all bounded holomorphic functions on $U$ is extended holomorphically to $\Delta^2$.

Moreover, he noted that there exist $0 < \eta, \zeta < 1$ so that if $(z, w)$ satisfies $|z| < \eta$, then $|w| < \zeta$.

\textbf{Lemma 2.2 (H. Hamada and M. Tsuji[10])} Let $w_0$ be a real number with $\zeta < w_0 < 1$. Then a bounded holomorphic function $1/(w - w_0)$ on $\{ (z, w) \in \mathbb{C}^2; z = 0 \} \cap U$ can not be extended bounded holomorphically to the domain $U$.  

$\psi(z)$
Lemma 2.3 Let \( \{\psi_k; k \geq 1\} \) be a sequence of \( C^\infty \) strictly subharmonic functions \( \psi_k \) on \( \mathbb{C} \) with \( \psi_k(z) \geq \psi_{k+1}(z) \) for each point \( z \in \mathbb{C} \) converging to a function \( \psi \). Let

\[
U_n = \{(z,w) \in \mathbb{C}^2; |z| < 1, \log |w| + \psi_n < 0\}.
\]

If the function \( 1/(w - w_0) \) on \( \{(z,w) \in \mathbb{C}^2; z = 0\} \cap U \) can be extended to a bounded holomorphic function \( G_n \) on \( U_n \), there exists a sequence \( C_n; n \geq 1 \) of positive numbers \( C_n \nearrow \infty \) such that \( |G_n(z,w)| \geq C_n \) for any \( (z,w) \in U_n \).

Proof. Since the sequence of domains \( U_n \subseteq U_{n+1}(n \geq 1) \) satisfies \( U = \cup_{n=1}^\infty U_n \) and \( 1/(w - w_0) \) cannot be extended bounded holomorphically to the domain \( U \) by Lemma 2.2 , we have \( C_n \nearrow \infty \) by the theory of normal families.

Lemma 2.4 (Fornaess and Sibony[8]) There exists a Reinhardt domain \( R \) in \( \mathbb{C}^2 \) with smooth boundary satisfying the following conditions:

1. \( R = \{(z,w) \in \mathbb{C}^2; |z| \leq 1, \log |w| + \phi(z) < 0\} \) for a smooth subharmonic function \( \varphi(z) = \varphi(|z|) \) on the open unit disc \( \Delta \) such that \( \varphi(z) \to +\infty \) as \( |z| \to 1 \).

2. The Laplacian of \( \varphi \) vanishes precisely on a sequence \( \{A_n; n \geq 1\} \) of disjoint annuli \( A_n = \{z \in \mathbb{C}; x_n - 2d_n < |z| < x_n + 2d_n\} \), where \( x_n + 3d_n = 1(n \geq 1) \) and \( x_n \nearrow 1 \) as \( n \to \infty \).

3. There exist positive integers \( p_n, q_n, \) and real constants \( a_n \) such that we have

\[
\varphi(z) = (p_n/q_n)|z| + a_n \text{ for any } z \in A_n.
\]

Fornaess and Sibony[8] constructed the following domain: Let \( \rho \) be a smooth nonnegative subharmonic function which vanishes precisely on \( \Delta(0,2) \) and which is strictly subharmonic when \( |z| > 2 \). For each \( n \geq 1 \), let \( V_n \) be an open set in \( \mathbb{C} \), \( K_n \) be a compact set in \( \mathbb{C} \) such that \( A_n \subseteq V_n \subseteq K_n \) and that \( K_n \cap K_m = \phi \) for \( 1 \leq n \leq m \). Let \( \sigma_n(z) \) be a \( C^\infty \) function on \( \mathbb{C} \) such that \( \sigma_n(z) \equiv 1 \) on \( V_n \) and the support of \( \sigma_n(z) \) is contained in \( K_n \).

Let \( \epsilon_n; n \geq 1 \) be a sequence of positive numbers \( \epsilon_n \searrow 0 \). We define a Hartogs domain

\[
B = \{(z,w) \in \mathbb{C}^2; \log |w| + \varphi_1(z) < 0\},
\]

where

\[
\varphi_1(z) = \varphi(z) + \sum_{n=1}^\infty \epsilon_n \sigma_n(z) \rho \left( \frac{z - x_n}{d_n} \right).
\]

For each \( n \geq 1 \), let \( M_n \) be a multiples of \( q_n \) and \( \chi_n \geq 0 \) be a \( C^\infty \) function on \( \mathbb{C} \) with compact support such that \( \chi_n(z) \geq 0 \) for any \( z \in \mathbb{C} \) and that \( \chi_n \equiv 1 \) in a neighborhood of \( \Delta(x_n, 2d_n) \). Let

\[
B' = \{(z,w) \in \mathbb{C}^2; |z| < 1, \log |w| + \varphi_2(z) < 0\},
\]

where

\[
\varphi_2(z) = \varphi_1(z) + \sum_n \chi_n \psi_n \left( \frac{z - x_n}{d_n} \right)/M_n.
\]

We can choose the \( M_n \)'s so large that \( B' \) has smooth boundary and is strictly pseudoconvex except in the set \( \{(z,w) \in \mathbb{C}^2; |z| = 1, |w| = 0\} \).

Define

\[
F(z) = \prod_{n=1}^\infty \frac{z - x_n}{1 - zz_n}.
\]
Then, there exist positive constants $c$ and $C$ such that, for $z \in \Delta(x_n, 2d_n)$,
\[
    c\frac{|z - x_n|}{d_n} \leq |F(z)| \leq C\frac{|z - x_n|}{d_n},
\]
and, for $z \notin \cup_{n \geq 1} \Delta(x_n, 2d_n)$, $|F(z)| > c$. Also we have $|F| < 1$ on $\Delta$.

Define a variety $V$ by
\[
    V = \{(z, w) \in \Delta \times \mathbb{C}; wF(z) = 0\} = \bigcup_{n=1}^{\infty} \{(z, w) \in \mathbb{C}^2; z = x_n\} \cup \{(z, w) \in \mathbb{C}^2; w = 0\},
\]
which is a connected subvariety, and a monomial $P_n$ in $(z, w) \in \mathbb{C}^2$ by
\[
    P_n = e^{a_n}z^{p_n}w^{q_n}.
\]

Since for $z \in \Delta(x_n, 2d_n)$, $\varphi_1(z) = (p_n/q_n) \log |z| + a_n$ it holds that
\[
    |P_n|^{M_n/q_n} < \exp(-\psi_n(z-x_n/d_n)) \leq \exp(-\psi(z-x_n/d_n)) \quad \text{on} \quad \{z = x_n\} \cap B'.
\]

Thus $|P_n|^{M_n/q_n} < \zeta < w_0$ on $\{z = x_n\} \cap B'$. As a result, a function on $V \cap B'$ given by
\[
    f(z, w) = \begin{cases} 
    1/(P_n^{M_n/q_n} - w_0) & \text{on} \quad \{(z, w) \in \mathbb{C}^2; z = x_n\} \cap B' \\
    -1/w_0 & \text{on} \quad \{(z, w) \in \mathbb{C}^2; w = 0\} \cap B'
    \end{cases}
\]
is a bounded holomorphic function on $V \cap B'$.

**Theorem 2.1** $f(z, w)$ can not be extended to bounded holomorphic function $G(z, w)$ on $B'$.

**Proof.** Let $B^{(n)} = \{(z, w) \in B'; |z - x_n|/d_n < 1\}$. We have a proper holomorphic map $\Phi_n: B^{(n)} \to U_n$,
\[
    \Phi_n : (z, w) \mapsto \left(\frac{z - x_n}{d_n}, P_n^{M_n/q_n}\right).
\]
The function $1/(w - w_0)$, which is regarded as defined on the set $\{(0, w) \in U_n\}$, can not be extended to a holomorphic function on $U_n$, the modulus of which at a point is less than $C_n$ by Lemma 2.3. If there is holomorphic function on $B_n$ with norm less than $C_n$, then by averaging the solutions over fibers of $\Phi_n$, we obtain a holomorphic function on $U_n$ with norm less than $C_n$.

So if $f(z, w)$ were extended to a bounded holomorphic function $G(z, w)$ on $B'$, we would have $\|G(z, w)\| \geq C_n$. Since $C_n \to +\infty$ as $n \to \infty$ and since the extended function $G(z, w)$ were bounded on $B'$, this is a contradiction.

It remains only to modify $B'$ near the unit circle $T \times \{0\}$ so that the resulting Hartogs domain is strictly pseudoconvex everywhere except at $(1,0)$. The following process is the same in [8]. Choose a smooth defining function $r(z, w)$ for $B'$ so that some root $-((-r)^{1/N}$ is strictly plurisubharmonic on $B'$. We write $-((-r)^{1/N} = -|\delta(z, |w|)|^{1/N}s(z, w)$, where $\delta$ is the signed distance function and
$s > 0$ is smooth on a neighborhood of the boundary of $B'$. Then we get a new strictly plurisubharmonic function $\rho$ by averaging:

$$\rho(z, |w|) = \frac{-1}{2\pi} \int_{0}^{2\pi} |\delta(z, |w|)|^{1/N_{S}}(Z, we^{i\theta})d\theta = -|\delta(z, |w|)|^{1/N}\tilde{s}(z, w),$$

where $\tilde{s}$ is smooth in a neighborhood of the boundary of $B'$ and is $> 0$.

Next, let $\gamma \geq 0$ be a smooth function on $\mathbb{C}$, strictly subharmonic away from 1 and vanishing only at 1. We can make $\gamma$ vanish sufficiently fast to infinite order at 1 so that the perturbation $\Omega$ to $B'$ will still be a counterexample to the Ohsawa's problem in case of variety by using the same example as for $B'$. Let $\Omega$ be defined by the inequality $\{(z, w) \in \mathbb{C}^{2}; \rho(z, |w|) + \gamma(z) < 0\}$. The domain $\Omega$ satisfies all conditions.

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References


