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THE SECOND PLURIGENUS OF SURFACE SINGULARITIES

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Introduction

Let (X, x) be a normal surface singularity over the complex number field $\mathbb{C}$ and $f\colon (M, A) \to (X, x)$ the minimal good resolution of the singularity (X, x), i.e., the smallest resolution for which an exceptional divisor A consists of non-singular curves intersecting transversally, with no three through one point. It is well known that there exists a unique minimal good resolution. Let $A = \bigcup_{i=1}^{k} A_i$ be the decomposition of the exceptional set A into irreducible components. The weighted dual graph of (X, x) is the graph such that each vertex of which represents a component of A weighted by the self-intersection number, while each edge connecting the vertices corresponding to A_i and A_j, $i \neq j$, corresponds to the point $A_i \bigcap A_j$. Giving the weighted dual graph is equivalent to giving the information of the genera of the A_i's and the intersection matrix $(A_i \cdot A_j)$. The geometric genus of the singularity (X, x) is defined by

$$p_g(X, x) = \dim_{\mathbb{C}} H^1(M, \mathcal{O}_M).$$

The m-th L^2-plurigenus of the singularity (X, x) is the integer $\delta_m(X, x)$ which was introduced in [Wt] and can be computed as

$$\delta_m(X, x) = \dim_{\mathbb{C}} H^0(M - A, \mathcal{O}_M(mK))/H^0(M, \mathcal{O}_M(mK + (m-1)A)),$$

where K denotes the canonical divisor on M. Note that $p_g(X, x) = \delta_1(X, x)$. The plurigenera of a Gorenstein surface singularity are determined by the weighted dual graph and p_g (cf. [O2]). In this paper we consider relations among the invariants δ_2, p_g, μ, τ and the modality of certain normal surface singularities, so "a singularity" always means a normal surface singularity over $\mathbb{C}$.

1. Preliminaries

(1.1) Let (X,x) be a surface singularity and $f\colon (M,A) \to (X,x)$ the minimal good resolution of the singularity (X,x). Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_M$-modules and D a divisor on M. We will use the following notation: $\mathcal{F}(D) = \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_M(D)$,

$$H^i(\mathcal{F}) = H^i(M,\mathcal{F}), \quad H^i_A(\mathcal{F}) = H^i_A(M,\mathcal{F}),$$
$$h^i(\mathcal{F}) = \dim_{\mathbb{C}} H^i(\mathcal{F}), \quad h^i_A(\mathcal{F}) = \dim_{\mathbb{C}} H^i_A(\mathcal{F}).$$

We denote by K the canonical divisor on M.

(1.2) We take the following characterization of minimally elliptic singularities as its definition.

Theorem 1.3 (Laufer [La1, Theorem 3.10]). *A singularity (X,x) is minimally elliptic if and only if (X,x) is an elliptic Gorenstein singularity.*

Theorem 1.4 (cf. [O1, O2]). *Let (X,x) be a singularity. Then*

$$\delta_2(X,x) = h^1_A(\mathcal{O}_M(2K+A)) = h^1(\mathcal{O}_M(-K-A)).$$

If (X,x) is a Gorenstein singularity with $p_g \geq 1$, then we have

$$\delta_2(X,x) = -(K+L_1)\cdot L_1/2 + p_g(X,x) = -K\cdot L_1 + \chi(\mathcal{O}_A) + p_g(X,x).$$

Corollary 1.5 (cf. [O1]). *Let (X,x) be a hypersurface (resp. complete intersection) minimally elliptic singularity. Then $\delta_2(X,x) \leq 4$ (resp. ≤ 5).*

(1.6) Let $\Omega^1_M\langle A\rangle$ be the sheaf of 1-forms with logarithmic poles along A, and $\mathcal{S}$ its dual. Then there are exact sequences (cf. [Wh3]):

(1.6.1)
$$0 \to \Omega^1_M \to \Omega^1_M\langle A\rangle \to \bigoplus_{i=1}^{k} \mathcal{O}_{A_i} \to 0;$$

(1.6.2)
$$0 \to \mathcal{S} \to \Theta_M \to \bigoplus_{i=1}^{k} \mathcal{O}_{A_i}(A_i) \to 0;$$

(1.6.3)
$$0 \to \Theta_M(-A) \to \mathcal{S} \to \Theta_A \to 0.$$

Corollary 1.7. *Let (X,x) be a singularity. Then $\delta_2(X,x) \geq h^1(\Theta_A)$.*

Proof. For a locally free sheaf $\mathcal{F}$ of rank 2 on M, $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_M}(\mathcal{F},\mathcal{O}_M) \otimes_{\mathcal{O}_M} \bigwedge^2 \mathcal{F}$. Thus we get isomorphisms $\Theta_M(-A) \cong \Omega^1_M(-K-A)$ and $\mathcal{S} \cong \Omega^1_M\langle A\rangle(-K-A)$. Then the exact sequences (1.6.1) and (1.6.3) give

$$h^1(\Theta_A) \cong h^1\left(\bigoplus_{i=1}^{k} \mathcal{O}_{A_i}(-K-A)\right). \tag{1.7.1}$$

From the following exact sequence

$$0 \to \mathcal{O}_A \to \bigoplus_{i=1}^{k} \mathcal{O}_{A_i} \to \bigoplus_{i<j} \mathcal{O}_{A_i \cap A_j} \to 0,$$

we have a surjective map

$$H^1(\mathcal{O}_A(-K-A)) \to H^1\left(\bigoplus_{i=1}^{k} \mathcal{O}_{A_i}(-K-A)\right).$$

By Theorem 1.4 and (1.7.1), we get

$$\delta_2(X,x) \geq h^1(\mathcal{O}_A(-K-A)) \geq h^1(\Theta_A). \qquad \square$$

(1.8) Note that $h^1(\Theta_A)$ is the tangent space of locally trivial deformation of A.

2. Equisingular deformations

(2.1) In this section, we discuss deformations. Let (X,x) be a singularity and $f\colon (M,A) \to (X,x)$ the minimal good resolution of (X,x). Let $A = \bigcup_{i=1}^{k} A_i$ be the decomposition into irreducible components. We denote by D_X the functor (cf. [Sc]) of deformations of a singularity (X,x). In [Wh2], Wahl introduced the equisingular functor ES_M of deformations of (M,A) to which all A_i lift, and which blow down to deformations of (X,x). A deformation of the singularity (X,x) is called an equisingular deformation if it is obtained from an equisingular deformation of (M,A). It is well known that a deformation of M blows down if and only if $h^1(\mathcal{O}_M)$ does not jump (cf. [Wh2, (4.3)]). Hence equisingular deformations preserve the geometric genera and the weighted dual graphs of singularities, and so the plurigenera of Gorenstein singularities (cf. Introduction). In [La2, La3, La4, La5], Laufer studied deformations of M in the analytic category. For a Gorenstein singularity (X,x), an equisingular deformation of

(M, A) induces a topologically constant deformation of (X, x), and the converse holds, too (see [La5, V, VI]).

By (1.6.2), We have the following exact sequence

$$0 \to H^1(\mathcal{S}) \to H^1(\Theta_M) \to H^1\left(\bigoplus_{i=1}^{k} \mathcal{O}_{A_i}(A_i)\right) \to 0.$$

There exists the versal deformation $\pi\colon \overline{M} \to (Q, o)$ of (M, A) with tangent space $T_{Q,o} \cong H^1(\Theta_M)$, and a submanifold (P, o) with tangent space $T_{P,o} \cong H^1(\mathcal{S})$ such that all of the A_i lift to above P and P is the maximal subspace of Q above which all of the A_i lift (cf. [La5, p. 26]).

Theorem 2.2 (Wahl [Wh2]). (1) *ES_M has a hull (in the sense of* [Sc]*) and the natural map $ES_M \to D_X$ is injective.*

(2) *If any deformation of (M, A) to which all A_i lift blows down to a deformation of (X, x), then $T(ES_M) = H^1(\mathcal{S})$, where $T(ES_M)$ denotes the tangent space of ES_M. If $p_g(X, x) \leq 1$, then this condition is satisfied.*

(2.3) Let $B = \mathbb{C}\{z_1, \dots, z_n\}$. Let (X, x) be a q-h singularity defined by an ideal $I \subset B$. Let us recall that the tangent space T^1_X of D_X is given by the exact sequence

$$\mathrm{Hom}_R(\Omega^1_B \otimes R, R) \to \mathrm{Hom}_R(I/I^2, R) \to T^1_X \to 0,$$

where $R = B/I$. Since $\mathrm{Hom}_R(I/I^2, R)$ is graded, so is T^1_X: we write as $T^1_X = \bigoplus_{i \in \mathbb{Z}} T^1_X(i)$.

Theorem 2.4 (Pinkham [P2, 4.6]). $T(ES_M) = \bigoplus_{i \geq 0} T^1_X(i)$.

Definition 2.5. A function $h \in \mathbb{C}\{z_0, z_1, z_2\} = \mathcal{O}_{\mathbb{C}^3,o}$ is called a quasi-homogeneous (q-h, for short) polynomial of degree d with weights $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}^3$, if

$$t^d h(z_0, z_1, z_2) = h(t^{\alpha_0} z_0, t^{\alpha_1} z_1, t^{\alpha_2} z_2)$$

for any $t \in \mathbb{C}$. We assume that α_0, α_1 and α_2 are relatively prime.

A function $h \in \mathcal{O}_{\mathbb{C}^3,o}$ is said to be semi-quasi-homogeneous (s-q-h, for short) of degree d with weights $(\alpha_0, \alpha_1, \alpha_2)$ if it is of the form $h = h_0 + h_1$, where h_0 is a q-h polynomial of degree d with weights $(\alpha_0, \alpha_1, \alpha_2)$ which defines an isolated singularity and all of the monomials of h_1 have degree strictly greater than d or $h_1 = 0$ (cf. [AGV, 12.1]). A singularity is said to be s-q-h if it is defined by a s-q-h function.

(2.6) Assume that $h \in \mathbb{C}\{z_0, z_1, z_2\} = \mathcal{O}_{\mathbb{C}^3,o}$ define an isolated singularity (X, o) at the origin. Let J_h be an ideal of $\mathcal{O}_{\mathbb{C}^3,o}$ generated by $\partial h/\partial z_0, \partial h/\partial z_1$ and $\partial h/\partial z_2$. $Q_h = \mathcal{O}_{\mathbb{C}^3,o}/J_h$ is called Jacobian algebra. Then we have $T^1_X \cong \mathcal{O}_{\mathbb{C}^3,o}/(h, J_h)$. It is well known that $(h, J_h) = J_h$ if and only if h is q-h (after a change of coodinates) (see [Sa]).

If h is a q-h polynomial of degree d with weights $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, then α induces a grading on $\mathcal{O}_{\mathbb{C}^3,o}$, and so on Q_h. Let $Q_h = \bigoplus_{i\geq 0} Q_h(i)$. Recall that a morphism of graded modules $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}((h)/(h^2), \mathcal{O}_X)$ has degree n if $\varphi(h)$ has degree $d+n$. Hence we have $T^1_X(i) \cong Q_h(i+d)$ (cf. (2.3)), and $T(ES_M) \cong \bigoplus_{i\geq d} Q_h(i)$. We see that a s-q-h singularity is a fibre in an equisingular deformation of a q-h singularity by Theorem 2.4 (cf. [AGV, Theorem 12.1]).

(2.7) We assume that the weighted dual graph of (X, x) is a star-shaped graph. Let us introduce some results of [TW].

We set $A = A_0 + \sum_{i=1}^{\beta} S_i$, where A_0 is the central curve, and S_i the branches. The curves of S_i are denoted by $A_{i,j}$, $1 \leq j \leq r_i$, where $A_0 \cdot A_{i,1} = A_{i,j} \cdot A_{i,j+1} = 1$ $(j = 1, \ldots, r_i - 1)$. Let $b_{i,j} = -A_{i,j} \cdot A_{i,j}$. For each branch S_i, positive integers e_i and d_i are defined by

$$d_i/e_i = b_{i,1} - \cfrac{1}{b_{i,2} - \cfrac{1}{\cdots - \cfrac{1}{b_{i,r_i}}}}$$

where $e_i < d_i$, and e_i and d_i are relatively prime. Let D be a divisor on A_0 such that $\mathcal{O}_{A_0}(D)$ is the conormal sheaf of A_0. We define a $\mathbb{Q}$-divisor C on A_0 and a graded ring R as follows: $C = D - \sum_{i=1}^{\beta} q_i P_i$, where $q_i = e_i/d_i$ and $P_i = A_0 \bigcap A_{i,1}$;

$$R = \bigoplus_{n\geq 0} H^0(\mathcal{O}_{A_0}(nC))T^n \subset \mathbb{C}(A_0)[T],$$

where $\mathbb{C}(A_0)$ is the field of rational functions of A_0, and T an indeterminate. Then Spec(R) is a q-h normal surface singularity, we denote by (Y, y), and the weighted dual graph of (Y, y) is the same as that of (X, x) (cf. [P1]).

By contracting the branches $S_1 \bigcup \cdots \bigcup S_\beta$, we get a normal surface M' with cyclic quotient singularities. Let $\Phi\colon (M', A') \to (X, x)$ be the morphism induced canonically, where A' is the image of A_0. We define a filtration on $\mathcal{O}_X$ by $F^n = \Phi_*\mathcal{O}_{M'}(-nA')$ for $n \in \mathbb{Z}$. Note that $F^n = \mathcal{O}_X$ for $n \leq 0$. Let $\mathcal{R} = \bigoplus_{n\in\mathbb{Z}} F^n T^n$ and $G = \bigoplus_{n\geq 0}(F^n/F^{n+1})T^n$. Then the natural map $\mathbb{C}[T^{-1}] \to \mathcal{R}$ defines a deformation of Spec(G) with general fibre isomorphic to (X, x), since $G \cong \mathcal{R}/T^{-1}\mathcal{R}$ and $\mathcal{O}_X \cong \mathcal{R}/(T^{-1} - a)\mathcal{R}$ for $a \in \mathbb{C} - \{0\}$ (cf. [TW, (5.15)]). By [TW,(6.3)], R is the normalization of G, and $R = G$ if and only if $p_g(Y,y) = p_g(X,x)$. By [Wh4, (1.12), (3.4)], (X,x) is a fibre in an equisingular deformation of (Y,y) if $p_g(Y,y) = p_g(X,x)$.

Proposition 2.8. *Let (X,x) be a minimally elliptic singularity with a star-shaped graph. Then there exist a q-h minimally elliptic singularity (Y,y) and an equisingular deformation $\pi\colon \overline{Y} \to \mathbb{C}$ of (Y,y) such that $X = \pi^{-1}(a)$ for $a \in \mathbb{C} - \{0\}$.*

Proof. We use the notation in (2.7). Since the weighted dual graph of (Y,y) is the same as that of (X,x), we see that (Y,y) is a minimally elliptic singularity. □

(2.9) Under the same notation as above, if (X,x) is a hypersurface minimally elliptic singularity, then so is (Y,y) by [La1, Theorem 3.13]. By Proposition 2.8 and (2.6), a hypersurface minimally elliptic singularity with star-shaped graph is a s-q-h singularity.

3. Hypersurface singularities

(3.1) We use the same notation as in Section 2. Let (X,x) be a Gorenstein singularity with contractible X. Let Z be a cycle such that $\mathcal{O}_M(K) \cong \mathcal{O}_M(-Z)$. If (X,x) is not a rational double point, then $Z \geq A$.

Let $\mathcal{C}$ be a sheaf on M defined by an exact sequence

$$0 \to \mathcal{C} \to \mathbb{C}_M \to \mathbb{C}_A \to 0.$$

If $Z \geq A$, then the exterior differentiation gives an exact sequence (cf. [Wh3, (1.5), (1.6)])

$$(3.1.1) \qquad 0 \to \mathcal{C} \to \mathcal{O}_M(-Z) \xrightarrow{d} \Omega^1_M\langle A\rangle(-Z) \xrightarrow{d} \Omega^2_M(-Z+A) \to 0.$$

As X is contractible, $H^i(\mathcal{C}) = 0$ for all i. Hence $H^i(\mathcal{O}_M(-Z)) \cong H^i(d\mathcal{O}_M(-Z))$ for all i. In particular, $H^i(d\mathcal{O}_M(-Z)) \cong H^i(\mathcal{O}_M(K)) = 0$ for $i \geq 1$.

(3.2) In the rest of this section, we always assume that (X,x) is a complete intersection singularity which is not a rational double point. Let $\mu(X,x)$ and $\tau(X,x)$ denote Milnor number and Tjurina number of (X,x), respectively. We need the following results of Greuel [Gr1, Gr2] (cf. [LS]).

Proposition 3.3. (1) *$\mu(X,x) = h^1_{\{x\}}(d\Omega^1_X)$, and $\tau(X,x) = h^1_{\{x\}}(\Omega^1_X)$* [Gr2, p. 168].

(2) *$H^q_{\{x\}}(\Omega^p_X) = 0$ for $p+q \leq 1$* [Gr2, Proposition 2.3].

(3) *The following sequences are exact* [Gr1, Satz 4.4]:

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to d\mathcal{O}_X \to 0;$$
$$0 \to d\mathcal{O}_X \to \Omega^1_X \to d\Omega^1_X \to 0.$$

(4) $H^0_{\{x\}}(d\Omega^1_X) = 0$ [Gr1, Lemma 4.5].

(3.4) From (3.1.1), we have an exact sequence

$$\begin{aligned} 0 \to H^1_A(d\mathcal{O}_M(-Z)) \to H^1_A(\Omega^1_M\langle A\rangle(K)) \to H^1_A(\mathcal{O}_M(2K+A)) \\ \to H^2_A(d\mathcal{O}_M(-Z)) \to H^2_A(\Omega^1_M\langle A\rangle(K)). \end{aligned}$$

By Theorem 1.4, we have $h^1_A(\mathcal{O}_M(2K+A)) = \delta_2(X,x)$. By the Serre duality, we have $h^1_A(\Omega^1_M\langle A\rangle(K)) = h^1(\mathcal{S})$. If we set

$$\rho = \dim_{\mathbb{C}} \ker\left(H^2_A(d\mathcal{O}_M(-Z)) \to H^2_A(\Omega^1_M\langle A\rangle(K))\right),$$

then we have

$$\delta_2(X,x) = h^1(\mathcal{S}) + \rho - h^1_A(d\mathcal{O}_M(-Z)). \tag{3.4.1}$$

We note that $h^1_A(d\mathcal{O}_M(-Z)) \le h^1(\mathcal{S})$.

Let $U = M - A \cong X - \{x\}$.

Lemma 3.5. $h^1_A(d\mathcal{O}_M(-Z)) = h^1_{\{x\}}(d\mathcal{O}_X) + p_g(X,x) - 1.$

Proof. From the exact sequence

$$0 \to H^0(d\mathcal{O}_M(-Z)) \to H^0(d\mathcal{O}_U) \to H^1_A(d\mathcal{O}_M(-Z)) \to 0,$$

and isomorphisms

$$H^0(d\mathcal{O}_M(-Z)) \cong H^0(\mathcal{O}_M(K)) \cong H^0(f_*\mathcal{O}_M(K)),$$

we see that

$$H^1_A(d\mathcal{O}_M(-Z)) \cong H^0(d\mathcal{O}_U)/H^0(f_*\mathcal{O}_M(K)). \tag{3.5.1}$$

Using (2) and (3) of Proposition 3.3, we obtain $H^0_{\{x\}}(d\mathcal{O}_X) = 0$ and hence

$$H^1_{\{x\}}(d\mathcal{O}_X) \cong H^0(d\mathcal{O}_U)/H^0(d\mathcal{O}_X). \tag{3.5.2}$$

Let $\mathcal{M}$ be an ideal sheaf of $\mathcal{O}_X$ which defines the singular point x. Note that $d\mathcal{O}_X \cong d\mathcal{M}$. Since X is contractible, we have

$$H^0(\mathcal{M}) \cong H^0(d\mathcal{M}) \cong H^0(d\mathcal{O}_X). \tag{3.5.3}$$

As (X,x) is a Gorenstein singularity with $p_g(X,x) \ge 1$, we have $f_*\mathcal{O}_M(K) \subset \mathcal{M}$. It is well known that $p_g(X,x) = \dim_{\mathbb{C}} H^0(\mathcal{O}_X)/H^0(f_*\mathcal{O}_M(K))$ for a Gorenstein singularity (X,x). From (3.5.1), (3.5.2) and (3.5.3), we have the following

$$\begin{aligned} h^1_A(d\mathcal{O}_M(-Z)) - h^1_{\{x\}}(d\mathcal{O}_X) &= \dim_{\mathbb{C}} H^0(d\mathcal{O}_X)/H^0(f_*\mathcal{O}_M(K)) \\ &= \dim_{\mathbb{C}} H^0(\mathcal{M})/H^0(f_*\mathcal{O}_M(K)) = p_g(X,x) - 1. \qquad \square \end{aligned}$$

Lemma 3.6. $\rho = \mu(X,x) - \tau(X,x) + h^1_{\{x\}}(d\mathcal{O}_X)$.

Proof. Since $H^1(d\mathcal{O}_M(-Z)) = H^2(d\mathcal{O}_M(-Z)) = 0$, we have

$$H^2_A(d\mathcal{O}_M(-Z)) \cong H^1(d\mathcal{O}_U) \cong H^2_{\{x\}}(d\mathcal{O}_X).$$

By the vanishing theorem of Wahl [Whl], $H^1(\Omega^1_M\langle A\rangle(K)) = 0$. Similarly, we get

$$H^2_A(\Omega^1_M\langle A\rangle(K)) \cong H^2_{\{x\}}(\Omega^1_X).$$

Then

$$\rho = \dim_{\mathbb{C}} \ker\left(H^2_{\{x\}}(d\mathcal{O}_X) \to H^2_{\{x\}}(\Omega^1_X)\right).$$

From Proposition 3.3, $H^0_{\{x\}}(d\Omega^1_X) = 0$ and we have an exact sequence

$$0 \to H^1_{\{x\}}(d\mathcal{O}_X) \to H^1_{\{x\}}(\Omega^1_X) \to H^1_{\{x\}}(d\Omega^1_X) \to H^2_{\{x\}}(d\mathcal{O}_X) \to H^2_{\{x\}}(\Omega^1_X),$$

and hence $\rho = \mu(X,x) - \tau(X,x) + h^1_{\{x\}}(d\mathcal{O}_X)$. □

Theorem 3.7. $\delta_2(X,x) = h^1(\mathcal{S}) + \mu(X,x) - \tau(X,x) - p_g(X,x) + 1$.

Proof. The theorem is immediately obtained from (3.4.1), Lemma 3.5 and Lemma 3.6. □

Corollary 3.8. *Let $\pi\colon \overline{X} \to T$ be an equisingular deformation of (X,x). We set $X_t = \pi^{-1}(t)$ for $t \in T$. Then*

$$\tau(X_t) \geq \mu(X,x) - \delta_2(X,x)$$

for any $t \in T$. In particular, if $p_g(X,x) = 1$, then $\tau(X_t) \geq \mu(X,x) - 5$.

Proof. We note that X_t is a complete intersection isolated singularity for any $t \in T$ (cf. [KS]). From (3.4) and Lemma 3.5, $h^1(\mathcal{S}) \geq p_g - 1$. By Theorem 3.7, we have that $\delta_2(X_t) \geq \mu(X_t) - \tau(X_t)$. By Theorem 1.4, δ_2 is determined by p_g and the weighted dual graph of the singularity, and so is μ by [St, (2.26)]. The property of the equisingular deformations implies that $\delta_2(X_t) = \delta_2(X,x)$ and $\mu(X_t) = \mu(X,x)$. Then we get the first formula. If $p_g(X,x) = 1$, then $\delta_2(X,x) \leq 5$ by Corollary 1.5. □

(3.9) For the remainder of this section, (X,o) denotes a hypersurface singularity defined by a function $h \in \mathbb{C}\{z_0, z_1, z_2\} = \mathcal{O}_{\mathbb{C}^3,o}$. It is well known that

$$\mu(X,o) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3,o}/J_h \quad\text{and}\quad \tau(X,o) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3,o}/(J_h, h),$$

and that $\mu(X,o) = \tau(X,o)$ if and only if h is q-h (after a change of coordinates).

We set $\mu = \mu(X,o)$. Let $\varphi_1, \dots, \varphi_\mu$ be functions in $\mathcal{O}_{\mathbb{C}^3,o}$ which induce $\mathbb{C}$-basis of $\mathcal{O}_{\mathbb{C}^3,o}/J_h$. Then we define a function $H(z,t) \in \mathbb{C}\{z_0, z_1, z_2, t_1, \dots, t_\mu\} = \mathcal{O}_{\mathbb{C}^3 \times \mathbb{C}^\mu, o}$ by

$$H(z,t) = h + \sum_{i=1}^{\mu} t_i \varphi_i,$$

and we set

$$Y(X,o) = \{\, (t_0) \in (\mathbb{C}^\mu, o) \mid \mu(H(z,t_0)) = \mu \,\},$$

where $\mu(H(z,t_0))$ denotes Milnor number of the singularity defined by $H(z,t_0)$. Then $Y(X,o)$ is an analytic subset of $(\mathbb{C}^\mu, o)$.

Definition 3.10. The modality $m(X,o)$ of the singularity (X,o) is the dimension of $Y(X,o)$ (cf. [Ga]). If (X,o) is defined by a quasi-homogeneous polynomial h of degree d, then the inner modality $m_0(X,o)$ of the singularity (X,o) is defined as the dimension of the vector space $\bigoplus_{i \geq d} Q_h(i)$ (cf. [YW]). Note that $m_0(X,o) \leq m(X,o)$ if (X,o) is a q-h singularity (see the proof of the follow).

Proposition 3.11. *If $p_g(X,o) = 1$, then $\delta_2(X,o) \leq m(X,o)$.*

If (X,o) is a q-h singularity, then $\delta_2(X,o) = m_0(X,o) \leq 4$.

Proof. Let $(\mathbb{C}^{\tau(X,o)}, o)$ be the versal deformation space of the singularity (X,o) and

$$p\colon (\mathbb{C}^{\mu(X,o)}, o) \to (\mathbb{C}^{\tau(X,o)}, o)$$

be a projection corresponding to the natural map of the tangent spaces

$$\mathcal{O}_{\mathbb{C}^3,o}/J_h \to \mathcal{O}_{\mathbb{C}^3,o}/(J_h, h).$$

There is a submanifold P of $(\mathbb{C}^{\tau(X,o)}, o)$ which represents ES_M. By the property of the equisingular deformations, $p^{-1}(P) \subset Y(X,o)$. By Theorem 2.2, we see that the dimension of $p^{-1}(P)$ is $h^1(\mathcal{S}) + \mu(X,o) - \tau(X,o)$. Hence

$$h^1(\mathcal{S}) + \mu(X,o) - \tau(X,o) \leq m(X,o).$$

From Theorem 3.7, we get $\delta_2(X,o) \leq m(X,o)$.

We assume that h is a q-h polynomial of degree d. Then Theorem 3.7 and 2.2, and (2.6) implies that $\delta_2(X,o) = h^1(\mathcal{S}) = \dim_{\mathbb{C}} \bigoplus_{i \geq d} Q_h(i) = m_0(X,o)$. By Corollary 1.5, $\delta_2(X,o) \leq 4$. $\square$

Remark 3.12. If the invariance of Milnor number implies the invariance of the topological type for two dimensional hypersurface singularities (cf. [LR]), then, in the proof above, we have $p^{-1}(P) = Y(X,o)$. In this case, $Y(X,o)$ is nonsingular, and $\delta_2(X,o) = m(X,o)$ holds.

Proposition 3.13. *Let (X,o) be a singularity defined by a s-q-h function $h \in \mathcal{O}_{\mathbb{C}^3,o}$ with weights $(1,1,1)$. Then $\delta_2(X,o) \geq m(X,o)$.*

Proof. We write $h = h_0 + h_1$ as in Definition 2.5. Let (X_0,o) be a singularity defined by h_0. Then by [GK], $m_0(X_0,o) = m(X_0,o)$. Hence we have that $\delta_2(X_0,o) \geq m(X_0,o)$ by [YW]. On the other hand, (X,o) is a fibre in an equisingular deformation of (X_0,o) by (2.6). Thus $\delta_2(X,o) = \delta_2(X_0,o)$. Since the modality is upper semi-continuous by [Ga], we have $\delta_2(X,o) = \delta_2(X_0,o) \geq m(X_0,o) \geq m(X,o)$. □

Proposition 3.14. *If $p_g(X,o) = 1$, $\delta_2(X,o) \leq 2$ and the weighted dual graph of (X,o) is a star-shaped graph, then $\delta_2(X,o) = m(X,o)$.*

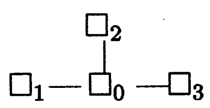
Proof. We know that (X,o) is a s-q-h singularity by (2.9). Let us use the notation in the proof of Proposition 3.13. Then $\delta_2(X,o) = \delta_2(X_0,o) = m(X_0,o)$ by Proposition 3.11, and $p_g(X,o) = 1$ Q-h hypersurface singularities with $p_g = 1$ and $m_0 \leq 4$ are listed in [YW]. The lists of all the singularities for which $m \leq 2$ are given in [AGV, 15.1]. Then we can see that for a s-q-h function of which the q-h part has inner modality $m_0 \leq 2$, we have $m = m_0$. Thus $m(X,o) = m_0(X_0,o) = \delta_2(X_0,o) = \delta_2(X,o)$. □

(3.15) We can classify the weighted dual graphs of minimally elliptic singularities with $\delta_2 \leq 2$. In the following, the symbol " ○ " corresponds to a component with self-intersection number -2 and " $\square_i$ " corresponds to a component A_i. We set $b_i = -A_i \cdot A_i$.

Proposition 3.16 (cf. [WO]). *Let (X,x) be a minimally elliptic singularity with $\delta_2(X,x) \leq 2$.*

(1) *$\delta_2(X,x) = 1$ if and only if (X,x) is a simple elliptic, cusp singularity or a singularity with the weighted dual graph*

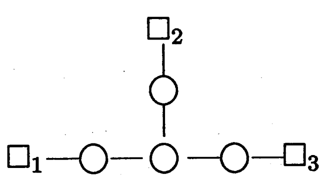
D_{b_1,b_2,b_3} :



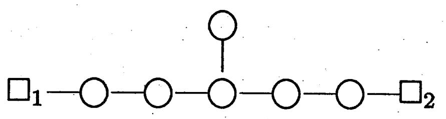
Where $b_0 = 1 < b_1 \leq b_2 \leq b_3$ *and* $1/b_1 + 1/b_2 + 1/b_3 < 1$.

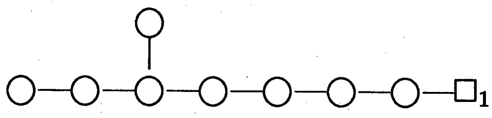
(2) *$\delta_2(X,x) = 2$ if and only if the weighted dual graph of (X,x) is one of the following.*

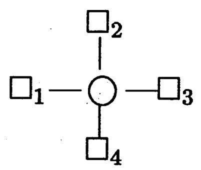
$\tilde{E}_6$:

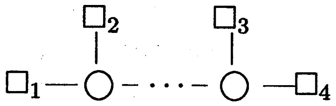


$2 \leq b_1 \leq b_2 \leq b_3, 2 < b_3$

$\tilde{E}_7$:  $2 \le b_1 \le b_2, 2 < b_2$

$\tilde{E}_8$:  $2 < b_1$

$\tilde{D}_4$:  $2 \le b_1 \le b_2 \le b_3 \le b_4, 2 < b_4$

$\tilde{D}_{i+4}$ $(i \ge 1)$:  $2 \le b_1 \le b_2, 2 \le b_3 \le b_4, 2 < b_4$
The number of " ○ " is $i+1$.

(3) *The list of the* (b_i) *corresponding to a hypersurface is the following.*

type	(b_i)
D_{b_1,b_2,b_3}	(2.3.7), (2.3.8), (2.3.9), (2.4.5), (2.4.6), (2.4.7), (2.5.5),(2.5.6) (3.3.4), (3.3.5), (3.3.6), (3.4.4), (3.4.5), (4.4.4)
$\tilde{E}_6$	(2.2.3), (2.2.4), (2.2.5), (2.3.3), (2.3.4), (3.3.3),
$\tilde{E}_7$	(2.3), (2.4), (2.5), (3.3), (3.4)
$\tilde{E}_8$	(3), (4), (5)
$\tilde{D}_4$	(2.2.2.3), (2.2.2.4), (2.2.2.5), (2.2.3.3) (2.2.3.4), (2.3.3.3)
$\tilde{D}_{i+4}$ $(i \ge 1)$	(2.2.2.3), (2.2.2.4), (2.2.2.5), (2.2.3.3) (2.3.2.3), (2.2.3.4), (2.3.2.4), (2.3.3.3)

Corollary 3.17. *Let* (X, o) *be a hypersurface singularity. Then* $\delta_2(X, o) = 1$ *if and only if* $m(X, o) = 1$.

Remark 3.18. Minimally elliptic singularities with $\delta_2 \le 2$ are Kodaira singularities (cf. [Kr]).

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