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Three dimensional hypersurface purely elliptic singularities

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**Abstract**

Three dimensional hypersurface purely elliptic singularities are classified into three classes according to the shape of their Newton boundaries.

A lot of examples of their defining equations are obtained from the defining equations of hypersurface simple $K3$ singularities. There are at least 95 types for the defining equations of hypersurface purely elliptic singularities of the type $(0, 1)$ or $(0, 0)$.

**1 Introduction**

In the theory of normal two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the most reasonable class of singularities after rational double points. They are characterized as two-dimensional purely elliptic singularities of $(0,1)$-type and of $(0,0)$-type, respectively. What are natural generalizations in three-dimensional case of simple elliptic singularities. The notion of a simple $K3$ singularity was defined in [4] as a three-dimensional isolated Gorenstein purely elliptic singularity of $(0,2)$-type. A simple $K3$ singularity is characterized as a normal three-dimensional isolated singularity such that the exceptional set of any $Q$-factorial terminal modification is a three-dimensional $K3$ surface (see [4]). Here we are interested in three-dimensional hypersurface purely elliptic singularities of $(0,i)$-type for $i = 0$ or $i = 1$. Let $f \in \mathbb{C}[z_0, z_1, z_2, z_3]$ be a polynomial which

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is non-degenerate with respect to its Newton boundary $\Gamma(f)$ in the sense of [5], and whose zero locus $X = \{f = 0\}$ in $\mathbb{C}^4$ has an isolated singularity at the origin $0 \in \mathbb{C}^4$. Then the condition for the singularity $(X, x)$ to be a purely elliptic singularity of $(0,0)$-type is given by a property of the Newton boundary of $\Gamma(f)$ of $f$.

In this paper, we give the method to obtain the principal parts of defining equations, which define three-dimensional hypersurface purely elliptic singularities of $(0,i)$-type for $i = 0$ to $i = 1$.

## 2 Preliminaries

In this section, we recall some definitions and results from [1], [4] and [6].

First we define the plurigenera $\delta_m, m \in \mathbb{N}$, for normal isolated singularities and define purely elliptic singularities. Let $(X, x)$ be a normal isolated singularity in an $n$-dimensional analytic space $X$, and $\pi : (M, E) \rightarrow (X, x)$ a good resolution. In the following, we assume that $X$ is a sufficiently small Stein neighbourhood of $x$.

**DEFINITION.** ([6]) Let $(X, x)$ be a normal isolated singularity. For any positive integer $m$,

$$\delta_m(X, x) := \dim_{\mathbb{C}} \Gamma(X - \{x\}, \mathcal{O}(mK))/L^{2/m}(X - \{x\}),$$

where $K$ is the canonical line bundle on $X - \{x\}$.

Then $\delta_m$ is finite and does not depend on the choice of a Stein neighborhood.

**DEFINITION.** ([6]) A singularity $(X, x)$ is said to be purely elliptic if $\delta_m = 1$ for every $m \in \mathbb{N}$.

When $X$ is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities, i.e., there is a nowhere vanishing holomorphic 2-form on $X - \{x\}$ (see [2]). But in higher dimension, purely elliptic singularities are not always quasi-Gorenstein (see [3]).

In the following, we assume that $(X, x)$ is quasi-Gorenstein. Let $E = \bigcup E_i$ be the decomposition of the exceptional set $E$ into its irreducible components, and write

$$K_M = \pi^* K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} m_j E_J$$

with $m_i \geq 0, m_j \geq 0$. Ishii [1] defined the essential part of the exceptional set $E$ as $E_j = \sum_{j \in J} m_j E_J$, and showed that if $(X, x)$ is purely elliptic, then $m_j = 1$ for all $j \in J$. 


DEFINITION. ([1],[6]) A quasi-Gorenstein purely elliptic singularity \((X,x)\) is of \((0,i)\)-type if \(H^{n-1}(E_J, \mathcal{O}_E)\) consists of the \((0,i)\)-Hodge component \(H^{0,i}(E_J)\), where
\[
C \simeq H^{n-1}(E_J, \mathcal{O}_E) = Gr^0_{p}^{n-1}(E_J) = \bigoplus_{i=1}^{n-1} H^{0,i}(E_J)
\]
n-dimensional quasi-Gorenstein purely elliptic singularities are classified into \(2n\) classes, including the condition that the singularity is Cohen-Macaulay or not.

Next we consider the case where \((X,x)\) is a hypersurface singularity defined by a non-degenerate polynomial \(f = \sum a_{\nu}z^{\nu} \in \mathbb{C}[z_0, z_1, \ldots, z_n]\) and \(x = 0 \in \mathbb{C}^{n+1}\). Recall that the Newton boundary \(\Gamma(f)\) of \(f\) is the union of the compact faces of \(\Gamma_+(f)\), where \(\Gamma_+(f)\) is the convex hull of \(\bigcup_{a_{\nu} \neq 0} (\nu + \mathbb{R}_{\mathbb{O}}^{n+1})\) in \(\mathbb{R}^{n+1}\). For any face \(\Delta\) of \(\Gamma_+(f)\), set \(f_{\Delta} := \sum_{\nu \in \Gamma_\nu} a_\nu z^{\nu}\). We say \(f\) to be nondegenerate, if
\[
\frac{\partial f_{\Delta}}{\partial z_0} = \frac{\partial f_{\Delta}}{\partial z_1} = \cdots = \frac{\partial f_{\Delta}}{\partial z_n} = 0
\]
has no solution in \((\mathbb{C}^*)^{n+1}\) for any face \(\Delta\). Where \(f\) is nondegenerate, the condition for \((X,x)\) to be a purely elliptic singularity of \((0,i)\)-type is given as follows:

**THEOREM 2.1** Let \(f\) be a nondegenerate polynomial and suppose \(X = \{f = 0\}\) has an isolated singularity at \(x = 0 \in \mathbb{C}^{n+1}\).

(1) \((X,x)\) is purely elliptic if and only if \((1,1,\ldots,1) \in \Gamma(f)\).

(2) Let \(n = 3\) and let \(\Delta_0\) be the face of \(\Gamma(f)\) consisting the point \((1,1,1,1)\) in the relative interior of \(\Delta_0\). Then we have

(i) \((X,x)\) is a singularity of \((0,2)\)-type if and only if \(\dim_{\mathbb{R}} \Delta_0 = 3\).

(ii) \((X,x)\) is a singularity of \((0,1)\)-type if and only if \(\dim_{\mathbb{R}} \Delta_0 = 2\).

(iii) \((X,x)\) is a singularity of \((0,0)\)-type if and only if \(\dim_{\mathbb{R}} \Delta_0 = 1\) or \(\dim_{\mathbb{R}} \Delta_0 = 0\).

### 3 Principal parts

In this section, we give examples of the principal parts of hypersurface purely elliptic singularities of \((0,i)\)-type defined by a nondegenerate polynomial for \(i = 0\) to \(i = 1\).

**EXAMPLE.** Let \((X,x)\) be the hypersurface purely elliptic singularity
\[
xyzw + x^{5+p} + y^{5+q} + z^{5+r} + w^{5+s} = 0
\]
in $\mathbb{C}^4$. Blow up the point $O = (0,0,0,0)$, let $F$ be the exceptional set, and let $Y$ be the strict transform of $X$. In this case the morphism $\pi : Y \to X$ is the canonical resolution of $X$. The exceptional set $E$ consists of four 2-dimensional projective spaces in $F$, forming a tetrahedron.

**Example.** Let $(X, x)$ be the hypersurface purely elliptic singularity

$$x^2 + y^3 + z^7 + w^{43+s} + xyzw = 0$$

in $\mathbb{C}^4$. Blow up the point $O = (0,0,0,0)$ with weight $(21,14,6,1)$, let $F$ be the exceptional set, and let $Y$ be the strict transform of $X$. In this case the morphism $\pi : Y \to X$ is the canonical resolution of $X$. The exceptional set $E$ is a rational surface with a singularity $T_{2,3,7}$ in a weighted projective space $F$, i.e., $\mathbb{P}(21,14,6,1)$.

**Example.** Let $(X, x)$ be the hypersurface singularity defined by the equation

$$x^2 + y^3 + z^7 + z^6w^6 + w^{43+s} + xyzw = 0$$

in $\mathbb{C}^4$. Then the singularity $(X, x)$ is a purely elliptic singularity of $(0,1)$-type.

**Example.** Let $(X, x)$ be the hypersurface singularity defined by the equation

$$x^2 + y^3 + z^7 + \lambda z^6w^6 + \mu w^{42} + w^{43+s} + xyzw = 0$$

in $\mathbb{C}^4$. Then the we obtain:

1. $\mu \neq 0 \iff (0,2)$-type.
2. $\mu = 0, \lambda \neq 0 \iff (0,1)$-type.
3. $\mu = 0, \lambda = 0 \iff (0,0)$-type.

**References**


