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Invariant theory of the Bergman kernel in dimension two

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Abstract. This is an elementary exposition of a joint work with Hirachi and Nakazawa [HKN2], concerning Fefferman's program [F3] on the boundary singularity of the Bergman kernel for strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth (i.e. C^∞) boundary. The main result gives, in the case $n=2$, an explicit invariant expression of the singularity of the Bergman kernel up to terms of weight ≤ 5. (A full invariant expression is discussed by Hirachi [Hi], see also his article in these proceedings.) In explaining the problem, we sometimes consider the general case $n \geq 2$, though our concern is the case $n=2$.

§1. Description of the problem. The Bergman kernel of a domain Ω in $\mathbb{C}^n$ is a real analytic function defined by $K^{\mathrm{B}}(z)=\sum |h_j(z)|^2$ for $z\in\Omega$, where $\{h_j\}_j$ is an arbitrary complete orthonormal system of the space of L^2 holomorphic functions in Ω. This is the restriction to the diagonal $w=z\in\Omega$ of a sesquiholomorphic function $K^{\mathrm{B}}(z,w)$ which is also referred to as the Bergman kernel. We assume that Ω is a strictly pseudoconvex domain with smooth boundary, and take a smooth defining function $r\in C^\infty(\overline{\Omega})$ in the sense that $\Omega=\{r>0\}$ and $dr\neq 0$ on $\partial\Omega$. Then it is well-known that $K^{\mathrm{B}}(z)\to+\infty$ as $r(z)\to+0$. Hörmander [Hö] further pointed out that

$$(1.1)\qquad r(z)^{n+1}K^{\mathrm{B}}(z)\to\frac{n!}{\pi^n}J[r](z_b)\qquad\text{as}\quad z\to z_b\in\partial\Omega,$$

where $J[\,\cdot\,]$ stands for the Levi determinant or the complex Monge-Ampère operator defined by

$$J[u]=(-1)^n\det\begin{pmatrix} u & \partial u/\partial\overline{z}_k\\ \partial u/\partial z_j & \partial^2u/\partial z_j\overline{z}_k\end{pmatrix}\qquad(j,k=1,\dots,n).$$

Here, $z = (z', z_n) = (z_1, \dots, z_n)$ is the standard coordinate system of $\mathbb{C}^n$. According to Fefferman [F1] (see also Boutet de Monvel-Sjöstrand [BS]), the singularity of K^{B} at the boundary takes the form

$$(1.2)\quad K^{\mathrm{B}}(z) = \frac{n!}{\pi^n}\left(\frac{\varphi^{\mathrm{B}}(z)}{r(z)^{n+1}} + \psi^{\mathrm{B}}(z)\log r(z)\right), \qquad \varphi^{\mathrm{B}}, \psi^{\mathrm{B}} \in C^\infty(\overline{\Omega}).$$

In particular (1.2), combined with (1.1), yields $\varphi^{\mathrm{B}} = J[r]$ on $\partial\Omega$.

REMARKS. (1°) A ball is biholomorphic to a simple model domain

$$\Omega_0 = \{r_0 > 0\} \quad \text{with} \quad r_0 = 2\,\mathrm{Re}\, z_n - |z'|^2,$$

and if $(\Omega, r) = (\Omega_0, r_0)$ then $\varphi^{\mathrm{B}} = J[r_0] = 1$ and $\psi^{\mathrm{B}} = 0$ in Ω_0. This case is exceptional and for most of the domains $\varphi^{\mathrm{B}} \neq J[r] \neq 1$ and $\psi^{\mathrm{B}} \neq 0$ in Ω.

(2°) If r is prescribed, then the singularity of $K^{\mathrm{B}}(z)$ is determined by φ^{B} modulo O^{n+1} and ψ^{B} modulo O^N for any $N \in \mathbb{N}$, where O^k stands for a general term which is smoothly divisible by r^k. The singularity of $K^{\mathrm{B}}(z)$ can be localized near a reference boundary point.

The problem in Fefferman's program [F3] is to express the singularity of K^{B} invariantly in the sense of local biholomorphic geometry:

$$(1.3)\quad \varphi^{\mathrm{B}} = \sum_{j=0}^{n} \varphi_j^{\mathrm{B}}\, r^j + O^{n+1}, \qquad \psi^{\mathrm{B}} = \sum_{j=0}^{N} \psi_j^{\mathrm{B}}\, r^j + O^{N+1} \quad (N \in \mathbb{N}).$$

We abandon $\varphi_j^{\mathrm{B}}, \psi_j^{\mathrm{B}} \in C^\infty(\partial\Omega)$ and assume $\varphi_j^{\mathrm{B}}, \psi_j^{\mathrm{B}} \in C^\infty(\overline{\Omega})$. (More precisely, we require $\varphi_j^{\mathrm{B}}, \psi_j^{\mathrm{B}}$ to be defined only near the boundary $\partial\Omega$.) To explain the reason, we need:

DEFINITION. A domain functional $K = K_\Omega$ is said to satisfy a (biholomorphic) *transformation law of weight* $w \in \mathbb{Z}$ if, for biholomorphic mappings $\Phi : \Omega_1 \to \Omega_2$,

$$(1.4)\qquad K_{\Omega_1}(z) = K_{\Omega_2}(\Phi(z))\,|\det \Phi'(z)|^{2w/(n+1)} \quad \text{for} \quad z \in \Omega_1.$$

We then write $\mathrm{w}^{\mathrm{TL}}(K) = w$.

EXAMPLES. (1°) The Bergman kernel satisfies $\mathrm{w}^{\mathrm{TL}}(K^{\mathrm{B}}) = n+1$.

(2°) Every solution of the complex Monge-Ampère equation $J[u] = 1$ satisfies $\mathrm{w}^{\mathrm{TL}}(u) = -1$. More precisely,

$$J[u_1](z) = J[u_2](\Phi(z)) \quad \text{if} \quad u_1(z) := u_2(\Phi(z))\,|\det\Phi'(z)|^{-2/(n+1)}.$$

Comparing these examples with (1.2), one might expect

$$\mathrm{w}^{\mathrm{TL}}(\varphi_j^{\mathrm{B}}) = j \quad (j \le n), \qquad \mathrm{w}^{\mathrm{TL}}(\psi_j^{\mathrm{B}}) = n+1+j \quad (j \le N)$$

for any $N \in \mathbb{N}$ by requiring r to satisfy $J[r] = 1$ near $\partial\Omega$. But then, the smoothness up to the boundary of r fails, that is, $r \notin C^\infty(\overline{\Omega})$ for most of the domains, and the program breaks down (see Section 2 below for the detail). Instead, we confine ourselves to a smooth approximate solution of $J[r] = 1$. Thus the expansion of ψ^{B} in (1.3) becomes approximate with N finite. (Hirachi [Hi] considers a complete invariant expansion of ψ^{B}, by taking account of the ambiguity of smooth approximate solutions of $J[r] = 1$, see also his article in these proceedings.)

To consider approximate invariants, we need:

DEFINITION. If a domain functional $K = K_\Omega \in C^\infty(\overline{\Omega})$ is well-defined modulo O^k and satisfies, in place of (1.4),

$$K_{\Omega_1} = (K_{\Omega_2} \circ \Phi) \cdot |\det \Phi'|^{2w/(n+1)} + O^k,$$

we write $\mathrm{w}^{\mathrm{TL}}(K) = w \mod O^k$. This notion can be localized near a reference point $z_b \in \partial\Omega$, where local biholomorphic mappings Φ are assumed to be smooth up to the boundary.

We also consider boundary invariants, and thus we need:

DEFINITION. If a boundary functional $K = K_{\partial\Omega} \in C^\infty(\partial\Omega)$ satisfies

$$K_{\partial\Omega_1} = (K_{\partial\Omega_2} \circ \Phi) \cdot |\det \Phi'|^{2w/(n+1)} \qquad \text{on} \quad \partial\Omega_1$$

for biholomorphic mappings $\Phi : \overline{\Omega}_1 \to \overline{\Omega}_2$, we write $\mathrm{w}^{\mathrm{TL}}(K) = w$ on $\partial\Omega$. This notion can be again localized near a reference point $z_b \in \partial\Omega$.

Obviously, if $\mathrm{w}^{\mathrm{TL}}(K) = w \mod O^k$ then $\mathrm{w}^{\mathrm{TL}}(K|_{\partial\Omega}) = w$ on $\partial\Omega$.

§2. The complex Monge-Ampère asymptotics. Let us begin with smooth approximate solutions due to Fefferman [F2]. Starting from an arbitrary smooth defining function of Ω, one has another defining function $r \in C^\infty(\overline{\Omega})$ such that

$$J[r] = 1 + O^{n+1}. \tag{2.1}$$

Let r^{F} denote the totality of smooth defining functions r satisfying (2.1). Abusing notation, we usually write $r = r^{\mathrm{F}}$. Fefferman's construction of $r = r^{\mathrm{F}}$ in [F2] is local, explicit and computable. Properties of r^{F} are summarized as follows:

(1^{F}) If $r_1, r_2 \in r^{\mathrm{F}}$ then $r_1 - r_2 = O^{n+2}$. If $r \in r^{\mathrm{F}}$ then $r + O^{n+2} \in r^{\mathrm{F}}$. (Consequently, the ambiguity of r^{F} is exactly O^{n+2}.)

(2^{F}) $\mathrm{w}^{\mathrm{TL}}(r^{\mathrm{F}}) = -1 \mod O^{n+2}$.

(3^{F}) r^{F} is locally defined near a boundary point.

We next state known facts on the complex Monge-Ampère boundary value problem

$$J[u] = 1 \quad (u > 0) \quad \text{in} \quad \Omega, \qquad u|_{\partial\Omega} = 0. \tag{2.2}$$

FACT 1 (unique existence, Cheng-Yau [CY]). There exists a unique solution $u = u^{\mathrm{MA}} \in C^\infty(\Omega) \cap C^{n+3/2-\varepsilon}(\overline{\Omega})$ of (2.2) for any $\varepsilon > 0$.

FACT 2 (asymptotic expansion, Lee-Melrose [LM]). For any smooth defining function r,

$$u^{\mathrm{MA}} \sim r \sum_{k=0}^{\infty} \eta_k \cdot (r^{n+1} \log r)^k, \qquad \eta_k \in C^\infty(\overline{\Omega}), \tag{2.3}$$

where each η_k is unique modulo flat functions (or as a formal power series in r). In particular, (2.3) implies $u^{\mathrm{MA}} \in C^{n+2-\varepsilon}(\overline{\Omega})$ for any $\varepsilon > 0$. This improves the regularity in Fact 1.

FACT 3 (structure of local asymptotic solutions, Graham [G1], [G2]). Let us fix $r = r^{\mathrm{F}}$ and $a \in C^\infty(\partial\Omega)$ locally near a boundary point. Then there exists a unique formal series u^{G} of the form (near the reference boundary point)

$$u^{\mathrm{G}} \sim r \sum_{k=0}^{\infty} \eta_k^{\mathrm{G}} \cdot (r^{n+1} \log r)^k, \qquad \eta_k^{\mathrm{G}} \in C^\infty(\overline{\Omega}),$$

such that $J\left[u^{\mathrm{G}}\right] \sim 1$ and $\eta_0^{\mathrm{G}} = 1 + a\, r^{n+1} + O^{n+2}$. As in ($1^{\mathrm{F}}$)–($3^{\mathrm{F}}$), properties of u^{G} are summarized as follows:

(1^{G}) Each η_k^{G} modulo O^{n+1} is independent of $r = r^{\mathrm{F}}$ and $a \in C^\infty(\partial\Omega)$.

(2^{G}) $\mathrm{w}^{\mathrm{TL}}(\eta_k^{\mathrm{G}}) = k(n+1) \mod O^{n+1}$.

(3^{G}) Each η_k^{G} modulo O^{n+1} is locally defined near a boundary point.

§3. The problem in dimension two. Now we can describe the problem and the difficulty more precisely. Let us restrict ourselves to the case $n = 2$, and thus (1.2) takes the form

$$K^{\mathrm{B}}(z) = \frac{2}{\pi^2}\left(\varphi^{\mathrm{B}}(z)\, r(z)^{-3} + \psi^{\mathrm{B}}(z) \log r(z)\right), \qquad r = r^{\mathrm{F}}.$$

Graham [G1] pointed out that $\varphi^{\mathrm{B}} = 1 + O^3$ and that

FACT 4. $\psi^{\mathrm{B}} = -3\,\eta_1^{\mathrm{G}}$ on $\partial\Omega$ locally.

Analysis of φ^{B} (for $n = 2$) is thus complete, see (1.3). To explain an implication of Fact 4, we set

$$\psi_0^{\mathrm{B}} := -3\eta_1^{\mathrm{G}}, \qquad P_4 := \left.\frac{\psi^{\mathrm{B}} - \psi_0^{\mathrm{B}}}{r}\right|_{\partial\Omega}.$$

Then $\mathrm{w}^{\mathrm{TL}}(\psi_0^{\mathrm{B}}) = 3 \mod O^3$ and $\mathrm{w}^{\mathrm{TL}}(P_4) = 4$ on $\partial\Omega$. Thus we have an approximate invariant expansion (1.3) with $N = 1$, where ψ_1^{B} is an arbitrary extension of P_4 from $\partial\Omega$ to Ω so that $\mathrm{w}^{\mathrm{TL}}(\psi_1^{\mathrm{B}}) = 4 \mod O^1$. The expansion (1.3) with $N = 1$ is completely determined in [G1] and [HKN1]. To refine this result one step further, we need to solve:

PROBLEM. Construct $\psi_1^{\mathrm{B}} \in C^\infty(\overline{\Omega})$ in such a way that

$$\psi_1^{\mathrm{B}}\big|_{\partial\Omega} = P_4, \qquad \mathrm{w}^{\mathrm{TL}}(\psi_1^{\mathrm{B}}) = 4 \mod O^2 \quad \text{locally}.$$

Assume for a moment that the Problem above is affirmatively solved. Then $\psi^{\mathrm{B}} - \psi_0^{\mathrm{B}} - \psi_1^{\mathrm{B}}\, r$ is smoothly divisible by r^2. In addition, setting

$$\widetilde{\psi} := \frac{\psi^{\mathrm{B}} - \psi_0^{\mathrm{B}} - \psi_1^{\mathrm{B}}\, r}{r^2} \in C^\infty(\overline{\Omega}), \qquad P_5 := \widetilde{\psi}\Big|_{\partial\Omega},$$

we have $\mathrm{w}^{\mathrm{TL}}(\widetilde{\psi}) = 5 \mod O^1$, and thus $\mathrm{w}^{\mathrm{TL}}(P_5) = 5$ on $\partial\Omega$. Thus we have an approximate invariant expansion (1.3) with $N = 2$, where ψ_2^{B} is an arbitrary extension of P_5. Due to the ambiguity (1^{F}) of $r = r^{\mathrm{F}}$, one cannot expect an approximate invariant expansion (1.3) with $N \geq 3$ as far as $r = r^{\mathrm{F}}$ is used. Our result is roughly stated as follows:

Result (rough statement). (1) The Problem above is affirmatively solved. Specifically, ψ_1^{B} is realized by a Weyl invariant of weight 4.

(2) P_5 is a CR invariant of weight 5, and an extension ψ_2^{B} of P_5 from $\partial\Omega$ to Ω is realized by a Weyl invariant of weight 5.

(3) ψ_1^{B} and ψ_2^{B} are given explicitly.

In the next section, we state the result more precisely in terms of Weyl invariants. Results on CR invariants are given in Section 5.

§4. Weyl invariants in the sense of Fefferman. To define Weyl invariants, it is necessary to consider a $\mathbb{C}^*$ bundle over $\overline{\Omega} \subset \mathbb{C}^n$ near the boundary $\partial\Omega$. An extra variable $z_0 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is introduced in addition to the standard coordinate system $z = (z_1, \ldots, z_n) \in \overline{\Omega} \subset \mathbb{C}^n$. Setting

$$r_\#(z_0, z) = |z_0|^2 r(z) \quad \text{with} \quad r = r^{\mathrm{F}},$$

we consider the Lorentz-Kähler metric with potential $r_\#$:

$$g = \sum_{j,k=0}^{n} (r_\#)_{j\overline{k}}\, dz_j d\overline{z}_k$$

Denoting by $R = R[g]$ the curvature tensor, we consider the covariant derivatives $R^{(p,q)} = \overline{\nabla}^{q-2} \nabla^{p-2} R$.

DEFINITION. A ***Weyl invariant*** of weight $w \in \mathbb{N}_0$ is defined to be a linear combination of complete contractions of the form

$$W_{\#} = \operatorname{contr}\Big(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}\Big), \quad w = \frac{1}{2}\sum_{j=1}^{s}(p_j+q_j) - s.$$

Then

$$W_{\#}(z_0, z) = |z_0|^{2w} W(z),$$

and the linear combination of these W is also referred to as a *Weyl invariant.* We denote by I_w^{W} the totality of Weyl invariants $W = W(z)$ of weight w.

The notion of Weyl invariants as above was introduced by Fefferman in his program [F3]. The following fact is due to Fefferman [F3] and Bailey-Eastwood-Graham [BEG].

FACT 5. For each $k = 1, \ldots, n$, there exists $W_k \in I_k^{\mathrm{W}}$ such that

$$\varphi^{\mathrm{B}} = \sum_{k=0}^{n} W_k\, r^k + O^{n+1}.$$

Properties of $W \in I_w^{\mathrm{W}}$ are summarized as follows:

(1^{W}) W modulo O^{n-w+1} is independent of $r = r^{\mathrm{F}}$.
(2^{W}) $\mathrm{w}^{\mathrm{TL}}(W) = w \mod O^{n-w+1}$.

We need to refine the ambiguity in (1^{W}) and (2^{W}) in the case $n = 2$. Our result is stated as follows.

Theorem ([HKN2]). *Assuming $n = 2$, let $W_{p,q} = \left\|R^{(p,q)}\right\|^2$ and $w = p + q - 2$, where $\|\cdot\|^2$ denotes the squared norm of a tensor with respect to the Lorentz metric g.*

(1) *If $w = 4$ or 5, then $W_{p,q}$ modulo O^{6-w} is independent of $r = r^{\mathrm{F}}$. The boundary values of $W_{p,q}$ are CR invariants of weight w.*

(2) *The boundary values of $W_{4,2}$ and $W_{3,3}$ are linearly dependent as CR invariants. The boundary values of $W_{5,2}$ and $W_{4,3}$ are linearly independent as CR invariants.*

(3) $\psi^{\mathrm{B}} = -3\eta_1^{\mathrm{G}} + \psi_1^{\mathrm{B}}\, r + \psi_2^{\mathrm{B}}\, r^2 + O^3$, *where*

$$\psi_1^{\mathrm{B}} = c_{42}\, W_{42} \quad \text{or} \quad c_{33}\, W_{33}, \qquad \psi_2^{\mathrm{B}} = c_{52}\, W_{52} + c_{43}\, W_{43}.$$

The constants $c_{42}, c_{33}, c_{52}, c_{43}$ *are explicit.* (c_{52} *and* c_{43} *depend on the choice of* ψ_1^{B}.) *Specifically,* $c_{42} = 3/1120$, $c_{33} = 1/160$, *and*

$$\text{if} \quad \psi_1^{\mathrm{B}} = c_{42}\, W_{42} \quad \text{then} \quad c_{52} = \frac{61}{141120}, \quad c_{53} = \frac{3}{7840};$$
$$\text{if} \quad \psi_1^{\mathrm{B}} = c_{33}\, W_{33} \quad \text{then} \quad c_{52} = \frac{1}{20160}, \quad c_{53} = \frac{1}{560}.$$

§5. **CR invariants.** For simplicity of the notation, we only consider the case $n = 2$. Let us begin with Moser's normal form (cf. [M], [CM]). Let $M \subset \mathbb{C}^2$ be a strictly pseudoconvex real hypersurface containing the origin as a reference point, and assume that M is real analytic. After a holomorphic change of coordinates, M is written as

$$2\,u = |z_1|^2 + F_A(z_1, \overline{z}_1, v), \qquad z_2 = u + i\,v,$$

where F_A is a power series of the form

$$F_A(z_1, \overline{z}_1, v) = \sum_{j+k+2\ell \geq 3} A^{\ell}_{j\overline{k}}\, z_1^j\, \overline{z}_1^k\, v^{\ell} = \sum_{j,k} A_{j\overline{k}}(v)\, z_1^j\, \overline{z}_1^k$$

satisfying $\overline{A_{j\overline{k}}(v)} = A_{k\overline{j}}(v)$. We then say that M is in *pre-normal form*.

Definition. M in pre-normal form is said to be in *normal form* if $A_{j\overline{k}}(v) = 0$ for $\min\{j, k\} < 2$ and $A_{2\overline{2}}(v) = A_{2\overline{3}}(v) = A_{3\overline{3}}(v) = 0$. Then z_1, z_2 are referred to as *normal coordinates*. For M in normal form, we write $M = N(A)$ and denote by $\mathcal{N}$ the totality of A giving $N(A)$.

Fact 6 ([M], [CM]). By a local biholomorphic mapping $w = \Phi(z)$, M in pre-normal form can be always put in normal form $\Phi(M)$. Φ is unique under the conditions

$$\Phi(0) = 0, \quad \Phi'(0) = \text{identity}, \quad \mathrm{Im}\,(\partial^2 w_2(0)/\partial z_2^2) = 0.$$

M has a unique normal form if and only if M is equivalent to $\partial\Omega_0$ for the model domain Ω_0 in Section 1, and the non-uniqueness is measured by the isotropy group $H = \{h \in \mathrm{Aut}(\Omega_0);\ h(0) = 0\}$. Then a group action $H \times \mathcal{N} \ni (h, A) \mapsto h.A \in \mathcal{N}$ is defined by $N(h.A) = \Phi \circ h(N(A))$ with Φ in Fact 6.

DEFINITION. A *CR invariant* of weight $w \in \mathbb{N}_0$ is a polynomial $P(A)$ in $A \in \mathcal{N}$ satisfying the transformation law

$$P(A) = |\det h'(0)|^{2w/3}\, P(h.A) \qquad (h \in H).$$

We denote by I_w^{CR} the (complexified) vector space of all CR invariants of weight w.

Even if M is not real analytic and merely smooth, $N(A)$ makes sense as a formal surface defined by a formal power series, and CR invariants are well-defined. A CR invariant $P(A)$ determines a functional $M \mapsto P_M$ defined by $P_M(p) := |\det \Phi_p'(p)|^{2w/3}\, P(A)$, where Φ_p with the reference point $p \in M$ is a formal mapping as in Fact 6 such that $\Phi_p(M) = N(A)$ and $\Phi_p(p) = 0$. Then $P_M(p)$ is independent of the choice of Φ_p, and P_M is a smooth function on M.

A list of CR invariants of weight ≤ 5 $(n = 2)$ is given as follows.

FACT 7 ([G1], [HKN2]). $I_0^{\mathrm{CR}} = \mathbb{C}$, $I_1^{\mathrm{CR}} = I_2^{\mathrm{CR}} = \{0\}$ and

$$I_3^{\mathrm{CR}} = \mathrm{span}\left(A_{4\bar{4}}^0\right), \qquad I_4^{\mathrm{CR}} = \mathrm{span}\left(|A_{2\bar{4}}^0|^2\right),$$
$$I_5^{\mathrm{CR}} = \mathrm{span}\left(F_5^{\mathrm{CR}}(1,0), F_5^{\mathrm{CR}}(0,1)\right),$$

where $F_5^{\mathrm{CR}}(a,b) := F(a, b, -2a + (10/9)b, -a + b/3)$ with

$$F(a,b,c,d) := a\,|A_{5\bar{2}}^0|^2 + b\,|A_{4\bar{3}}^0|^2 + \mathrm{Re}\left\{\left(c\,A_{3\bar{5}}^0 - i\,d\,A_{2\bar{4}}^1\right) A_{4\bar{2}}^0\right\}.$$

Assuming that $M = N(A)$ is a portion of the boundary $\partial\Omega$, let us consider the boundary values, at $0 \in M$, of η_1^{G} and W_{pq} $(p+q-2 = 4, 5)$. It was shown by Graham [G2] that $\eta_1^{\mathrm{G}} = 4\,A_{4\bar{4}}^0$ at 0. We also have:

FACT 8 ([HKN2]). For (1) of the Theorem in Section 4, the ambiguity statement holds. In addition, the following equalities hold at 0:

$$3\,W_{42} = 7\,W_{33} = 2^8 \cdot 21\,|A^0_{4\bar{2}}|^2,$$
$$W_{52} = -4 \cdot (5!)^2\, F_5^{\mathrm{CR}}(1, 18), \quad W_{43} = -4 \cdot (5!)^2\, F_5^{\mathrm{CR}}(4/3, 57/5).$$

These results imply the Theorem except for the determination of the universal constants. This determination requires expansions of η_1^{G} and W_{pq} ($p+q-2 = 4, 5$) as $t \to +0$ along the half-line $(0, t/2) \in \mathbb{C}^2$ in normal coordinates. A similar expansion of ψ^{B} is also necessary. Expansions of η_1^{G} and W_{pq}, together with the ambiguity of W_{pq}, are obtained via careful analysis of the operator $J[\,\cdot\,]$. To get an expansion of ψ^{B}, we use Boutet de Monvel's algorithm [B1], [B2], [B3] which is based on Kashiwara's microlocal characterization of the singularity of the Bergman kernel [K]. Both computations are long, see [HKN2] for the details. (Cf. also our earlier article [HKN1] for the method of computing ψ^{B}.)

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