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Geometry of Banach spaces and norms of ±1 matrices

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Some geometrical properties of a Banach space X are described by the behavior of norms of certain ±1 matrices between $l_r^n(X)$-spaces.

1. Definitions and notations

X, Y ; Banach spaces, $1 \leqq r, s \leqq \infty$, $1/r + 1/r' = 1/s + 1/s' = 1$,

$A = (a_{ij})$; m×n matrix, $a_{ij} = \pm 1$ (m×n ±1 matrix)

$\|A\|_{r,s;X} = \|A : l_r^n(X) \to l_s^m(X)\|$, $l_r^n(X)$; X-valued l_r^n-space

<u>Banach-Matur distance</u> : For two isomorphic Banach spaces E and F

$d(E, F) = \inf\{\|T\|\cdot\|T^{-1}\|$; T is isomorphism from E onto F}.

<u>Finite-representability</u> : Y is finitely representable (f. r.) in X if for each (some) $\lambda > 1$ and for each finite-dimensional subspace F of Y there is a finite-dimensional subspace E of X with dim E = dim F such that $d(E, F) < \lambda$.

<u>Super-reflexivity</u> : X is super-reflexive if any Banach space f. r. in X is reflexive.

Remark. Consider (P), a property of Banach spaces. (P) is said to be a super-property if X has (P) and Y is f. r. in X, then Y has (P). B-convexity and J-convexity are super-properties, but reflexivity and

Radon-Nikodym property (RNP) are not super-properties.

A finite sequence of signs ε_1, ε_2, ..., ε_n is admissible if all + signs are before all - sings (so there are n different sequences).

(1) X ; (n, ε)-convex, $\varepsilon > 0 \Leftrightarrow \forall x_1, \ldots, x_n$ with $\|x_j\| = 1$

$$\min_{\varepsilon_j = \pm 1} \|\Sigma_{j=1}^{n} \varepsilon_j x_j\| \leqq n(1 - \varepsilon).$$

(2) X ; B_n-convex $\Leftrightarrow$ X ; (n, ε)-convex for some $\varepsilon > 0$.

(3) X ; B-convex $\Leftrightarrow$ X ; B_n-convex for some n.

(4) X ; J_n-convex $\Leftrightarrow$ $\exists \delta > 0$, $\forall x_1, x_2, \ldots, x_n$ with $\|x_j\| = 1$,

$\exists \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$; admissible signs

$$\|\Sigma_{j=1}^{n} \varepsilon_j x_j\| \leqq n(1 - \delta).$$

(5) X ; J-convex $\Leftrightarrow$ X ; J_n-convex for some n.

(6) X ; uniformly non-square $\Leftrightarrow$ X ; B_2-convex ($\Leftrightarrow$ J_2-convex).

Remark. It is known that X is B-convex if and only if l_1 is not f. r. in X; and X is J-convex if and only if it is super-reflexive. It is also known that

uniformly convex $\Rightarrow$ uniformly non-square $\Rightarrow$ super-reflexive

and conversely, any super-reflexive Banach space admits an equivalent uniformly convex norm (see [1] and [2]).

2. Main results

Theorem 1 Let X be a Banach space, $1 < r \leqq \infty$ and $1 \leqq s < \infty$.

Then for any $m \times n$ ± 1 matrix $A = (a_{ij})$, the following are equivalent.

(1) $\exists \delta > 0$, $\forall x_1, \ldots\ldots, x_n$ with $\|x_i\| = 1$

$$\min_{1 \leqq i \leqq m} \|\Sigma_{j=1}^{n} a_{ij} x_j\| \leqq n(1 - \delta).$$

(2) $\|A\|_{r,s;X} = \|A : l_r^n(X) \to l_s^m(X)\| < m^{1/s} n^{1/r'}$.

Remark. For any Banach space X and for any $m \times n$ ± 1 matrix $A = (a_{ij})$, we can prove the following:

(1) $\|A\|_{r,s;X} \leqq m^{1/s} n^{1/r'}$.

(2) If l_1 is f. r. in X, then $\|A\|_{r,s;X} = m^{1/s} n^{1/r'}$.

Corollary 2 (1) X is B_n-convex if and only if $\|A\|_{r,s;X} < m^{1/s} n^{1/r'}$ for some $m \times n$ ± 1 matrix $A = (a_{ij})$.

(2) Let R_n be a $2^n \times n$ Rademacher matrix and $1 < p < \infty$. Then X is B_n-convex if and only if $\|R_n\|_{p,p;X} < 2^{n/p} n^{1/p'}$.

Remark. Rademacher matrices R_n is defined by

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \left(\begin{array}{c|c} 1 & \\ \vdots & R_n \\ 1 & \\ \hline -1 & \\ \vdots & R_n \\ -1 & \end{array}\right) \quad (n=1, 2, \ldots).$$

For the details of Rademacher matrices, see Kato, Miyazaki and Takahashi [5].

Corollary 3 (Smith and Turett [9]) Let $1<p<\infty$. Then X is B_n-convex if and only if $L_p(X)$ is. In particular, X is uniformly non-square if and only if $L_p(X)$ is, and X is B-convex if and only if $L_p(X)$ is.

Let A_2 be a 2X2 Littlewood matrix, that is, $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then nXn admissible matrices A_n ($n\geqq 2$) is defined by

$$A_{n+1} = \begin{pmatrix} 1 & & & & \\ 1 & & A_n & & \\ \vdots & & & & \\ 1 & -1 & -1 & \cdots & -1 \end{pmatrix} \qquad (n=1, 2, \ldots\ldots)$$

Corollary 4 X is J_n-convex if and only if $\|A_n\|_{p,p;X} < n$ for each (some) p with $1 < p < \infty$. In particular, X is super-reflexive if and only if $\|A_n\|_{p,p;X} < n$ for some n and each (some) p with $1 < p < \infty$.

Corollary 5 (Pisier [8]; see also Beauzamy [1]) Let $1<p<\infty$. Then X is J_n-convex if and only if $L_p(X)$ is. In particular, X is super-reflexive if and only if $L_p(X)$ is.

Theorem 6 Let X be a Banach space and $1 < r \leqq p \leqq s < \infty$. Then for any Bochner space $L_p(X)$ and for any mXn ±1 matrix $A = (a_{ij})$, it holds $\|A : l_r^n(L_p(X)) \to l_s^m(L_p(X))\| \leqq \|A : l_r^n(X) \to l_s^m(X)\|$.

Remark. (1) Corollaries 3 and 5 also follow from Theorem 6.

(2) From Theorem 5, it follows that if (p, p')-Clarkson inequality holds in X, then it also holds in $L_r(X)$, where $1 \leqq p \leqq 2$ and $p \leqq r \leqq p'$ (see Takahashi and Kato [10]).

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