数理解析研究所講究録
1039巻 1998年 37-52

Representation of Comonotonically Additive Functional

(共単調加法的汎関数の表現)

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1 Introduction

The Choquet integral with respect to a fuzzy measure proposed by Murofushi and Sugeno[6] is a basic tool for subjective evaluation[13] and decision analysis [4]. This integral is a functional on the class B of bounded measurable functions, which is monotone, positive homogeneous and comonotonically additive (for short c.p.m.).

Conversely, concerning the problem of whether or not a c.p.m. functional I on B can be represented by a Choquet integral with respect to a fuzzy measure, Schmeidler[9] proved that a c.p.m. functional I on B can be represented by a Choquet integral. Murofushi et al. [8] proved that the comonotonically additive functional on B can be represented by a Choquet integral with respect to a non-monotonic fuzzy measure.

Concerning the problem of whether or not a c.p.m. functional I can be represented by a Choquet integral when the domain of I is smaller than B, Greco[5] proved it when I has some continuity.

In this paper, we discuss fuzzy measures on a locally compact Hausdorff space X and the Choquet integral with respect to a fuzzy measure. We discuss the functional defined on the class K (or K^+) of (resp. positive) continuous functions with compact support on X. We generally have $K^+ \subsetneqq B^+$ on a locally compact Hausdorff space. In fact, let A be a measurable set and 1_A be the characteristic function of A, we have $1_A \in B^+$ but $1_A \notin K^+$.

In section 2, basic properties of the fuzzy measure and the Choquet integral are shown, and we define the rank and sign dependent functional I (for short r.s.d. functional), which is the difference of two Choquet integral. That is,

$$I(f) = (C)\int f^+ d\mu^+ - (C)\int f^- d\mu^-$$

where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$. This functional is used in cumulative prospect theory [14].

In section 3, we define a regular fuzzy measure, and show its properties.It is shown that if Choquet integrals of continuous functions with compact support with respect to two regular fuzzy measures are equal to each other then the regular fuzzy measures are equal to each other. This means the uniqueness of regular fuzzy measure which represents the c.m. functional.

In section 4, we show that the functional I on K is positive homogeneous if I is comonotonically additive and monotone (for short c.m.). We show that a c.m. functional I can be represented by a Choquet integral with respect to a regular fuzzy measure when the domain of I is the class K^+ of nonnegative continuous functions with compact support on the locally compact Hausdorff space.

We show that the c.m. functional on K is the r.s.d. functional in section 5. In section 6, we discuss the case of the universal set X to be a compact Hausdorff space. It is shown there that a c.m. functional can be represented by one Choquet integral.

2 Preliminaries

In this section, we define fuzzy measure, the Choquet integral and the rank and sign dependent functional, and show their basic properties. Throughout the paper we assume that $(X, \mathcal{B})$ be a measurable space.

Definition 2.1 [11] *A fuzzy measure* μ is an extended real valued set function, $\mu : \mathcal{B} \longrightarrow \overline{R^+}$ with the following properties.

(1) $\mu(\emptyset) = 0$

(2) $\mu(A) \leq \mu(B)$ whenever $A \subset B$,

where $A, B \in \mathcal{B}$ and $\overline{R^+} = [0, \infty]$ the set of extended nonnegative real numbers.

We say that μ is finite if $\mu(X) < \infty$.

When μ is finite, we define *the dual* μ^d *of* μ by

$$\mu^d(A) = \mu(X) - \mu(A^C)$$

for $A \in \mathcal{B}$.

We denote by K_0 the class of measurable functions and by K_0^+ the class of nonnegative measurable functions.

Definition 2.2 [1, 6] Let μ be a fuzzy measure on $(X, \mathcal{B})$.

(1) *The Choquet integral* of $f \in K_0^+$ with respect to a fuzzy measure μ is defined by

$$(C) \int f d\mu = \int_0^\infty \mu_f(r) dr,$$

where $\mu_f(r) = \mu(\{x | f(x) \geq r\})$.

(2) Suppose $\mu(X) < \infty$. The Choquet integral of $f \in K_0$ with respect to a fuzzy measure μ is defined by

$$(C)\int f d\mu = (C)\int f^+ d\mu - (C)\int f^- d\mu^d,$$

where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$.

Definition 2.3 [3] Let f, g be measurable nonnegative functions. We say that f and g are *comonotonic* if

$$f(x) < f(x') \Rightarrow g(x) \leq g(x')$$

for $x, x' \in X$. We denote $f \sim g$, when f and g are comonotonic.

Theorem 2.4 [2, 7] *Let $f, g \in K_0$.*

(1) *If $f \leq g$, then*

$$(C)\int f d\mu \leq (C)\int g d\mu$$

(2) *If a is a nonnegative real number,*

$$(C)\int a f d\mu = a(C)\int f d\mu.$$

(3) *If $f, g \in K_0^+$ are comonotonic, then*

$$(C)\int (f+g) d\mu = (C)\int f d\mu + (C)\int g d\mu.$$

(4) *If $\mu(X) < \infty$ and f, g are comonotonic,*

$$(C)\int (f+g) d\mu = (C)\int f d\mu + (C)\int g d\mu.$$

Definition 2.5 Let I be a real-valued functional on $K \subset K_0$. I is said to be *the rank and sign dependent functional* (for short *the r.s.d. functional*) on K, if there exist two fuzzy measures μ^+, μ^- such that for every $f \in K$

$$I(f) = (C)\int f^+ d\mu^+ - (C)\int f^- d\mu^-.$$

When $\mu^+ = \mu^-$, the r.s.d. functional is the Šipoš integral[10]. If $\mu^+(X) < \infty$ and $\mu^- = (\mu^+)^d$, the r.s.d. functional is the Choquet integral.

3 The regular fuzzy measure

In this section, we define the regular fuzzy measure and show some continuous properties of a regular fuzzy measure.

In the following we suppose that X is a topological space and K is the class of continuous functions on X with compact support. We denote $supp(f)$ the support of $f \in K$, that is,

$$supp(f) = cl\{x|f(x) \neq 0\}.$$

$\|\cdot\|$ on K means the sup norm, and $cl(\cdot)$ means the closure.

We define K^+, K^-, K_1^+ by

$$K^+ = \{f|f \in K, f \geq 0\}$$

$$K^- = \{f|f \in K, f \leq 0\}$$

$$K_1^+ = \{f|f \in K, 0 \leq f \leq 1\}.$$

Let $\mathcal{B}$ be the class of Borel subsets of X, $\mathcal{O}$ the class of open subsets of X and $\mathcal{C}$ the class of compact subsets of X.

Definition 3.1 Let μ is a fuzzy measure on the measurable space $(X, \mathcal{B})$. μ is said to be *outer regular* if

$$\mu(B) = \inf\{\mu(O) | O \in \mathcal{O}, O \supset B\}$$

for all $B \in \mathcal{B}$. An outer regular fuzzy measure μ is said to be *regular* if for all $O \in \mathcal{O}$

$$\mu(O) = \sup\{\mu(C) | C \in \mathcal{C}, C \subset O\}.$$

Definition 3.2 Let μ be a fuzzy measure on the measurable space $(X, \mathcal{B})$.

μ is said to be *o*-continuous from below if

$$O_n \uparrow O \Rightarrow \mu(O_n) \uparrow \mu(O)$$

where $n = 1, 2, 3, \cdots$ and both O_n and O are open sets.

μ is said to be *c*-continuous from above if

$$C_n \downarrow C \Rightarrow \mu(C_n) \downarrow \mu(C)$$

where $n = 1, 2, 3, \cdots$ and both C_n and C are compact sets.

The next two results follow from the definition.

Proposition 3.3 *Let μ be a regular fuzzy measure.*

(1) *μ is o-continuous from below.*

(2) *μ is c-continuous from above.*

Corollary 3.4 *Let μ be a finite regular fuzzy measure .*

(1) *μ^d is o-continuous from below.*

(2) *μ^d is c-continuous from above.*

The next result follows from Proposition 3.3.

Theorem 3.5 *Let X be a locally compact Hausdorff space, and let μ_1 and μ_2 be regular fuzzy measures. If $(C)\int f d\mu_1 = (C)\int f d\mu_2$ for all $f \in K$, then $\mu_1 = \mu_2$.*

Even if μ is regular, it is possible that μ^d is not regular. But Corollary 3.4 shows that μ^d is c-continuous from above and o-continuous from below. Therefore we can obtain the next corollary.

Corollary 3.6 *Let μ be a finite regular fuzzy measure.*
If $(C)\int f d\mu = (C)\int f d\mu^d$ for all $f \in K$ then $\mu(C) = \mu^d(C)$ where C is compact.

4 Representation of the c.m.functional on K^+

We assume that X is a locally compact Hausdorff space and $\mathcal{B}$ the class of its Borel subset. Let K be the set of continuous functions with compact support, and K^+ the set of continuous nonnegative functions with compact support.

Definition 4.1 Let I be a real valued functional on K. We say that I is *comonotonically additive* iff $f \sim g \Rightarrow I(f+g) = I(f) + I(g)$ for $f, g \in K^+$, I is *positively homogeneous* iff $I(af) = aI(f)$ for all positive real number $a > 0$, and I is *monotone* iff $f \leq g \Rightarrow I(f) \leq I(g)$ for $f, g \in K^+$. If the functional I is comonotonically additive and monotone, we say that I is *a c.m. functional.*

Let I be a c.m. functional on K. It follows from $f \sim f$ that

$$I(nf) = nI(f) \quad \text{and} \quad nI(\frac{1}{n}f) = I(f)$$

for $f \in K$ and a positive integer n. Then we have $I(qf) = qI(f)$ if q is a positive rational number. Let p be a positive real number. For every positive integer n, there exists a rational number r such that $r < p < r + \frac{1}{n}$. Since I is monotone, we have the next proposition.

Proposition 4.2 *A c.m. functional on K is positive homogeneous.*

In this section, we shall demonstrate that if I is a c.m. functional then a real-valued functional I on K^+ is represented as a Choquet integral with respect to a regular fuzzy measure.

Lemma 4.3 *Let I be a c.m. functional on K^+. We put*

$$\mu(O) = \sup\{I(f) | f \in K_1^+, supp(f) \subset O\},$$

and

$$\mu(B) = \inf\{\mu^+(O) | O \in \mathcal{O}, O \supset B\}.$$

Then μ is an outer regular fuzzy measure.

We shall say that this fuzzy measure μ is the fuzzy measure induced by a c.m. functional I.

Proposition 4.4 *Let μ is the fuzzy measure induced by a c.m. functional I.*

(1) *If $f \in K^+$,$A \subset \{x | f(x) \geq 1\}$ and $A \in \mathcal{B}$, then $\mu(A) \leq I(f)$.*

(2) *If C is a compact set in $\mathcal{B}$, then $\mu(C) < \infty$.*

(3) *If O is an open set in $\mathcal{B}$, $\mu(O) = \sup\{\mu(C) | C \in \mathcal{C}, C \subset O\}$.*

It follows from Propostion 4.4 that the fuzzy measure induced by a c.m. functional I is regular.

Theorem 4.5 *For a c.m. functional I on K^+, there exists a regular fuzzy measure μ on $\mathcal{B}$ such that for all $f \in K^+$*

$$I(f) = (C)\int f d\mu$$

proof) Let $f \in K^+$, $O_{n,k} = \{x|f(x) > (k-1)/n\}$ and $C_{n,k} = \{x|f(x) \geq k/n\}$ where $1 \leq k \leq n$. Then for all n and k, $C_{n,k}$ is a compact set, $O_{n,k}$ is an open set, and $O_{n,k+1} \subset C_{n,k} \subset O_{n,k} \subset supp(f)$. Since X is a locally compact Hausdorff space, for all n, k there exists $f_{n,k} \in K^+$ such that $0 \leq f_{n,k} \leq 1, f_{n,k}(x) = 1$ when $x \in C_{n,k}$ and $supp(f_{n,k}) \subset O_{n,k}$.

These functions $f_{n,k}$ have the following properties.

(1) For all positive integer n, k and j such that $1 \leq k \leq n$ and $1 \leq j \leq n$, $f_{n,k}$ and $f_{n,j}$ are comonotonic.

(2) For all positive integer n and k such that $1 \leq k \leq n$,

$f_{n,1} + f_{n,2} + \cdots \quad + f_{n,k}$ and $f_{n,k+1} + f_{n,k+2} + \cdots \quad + f_{n,n}$ are comonotonic.

Next define $f_n \in K^+$ by

$$f_n = \sum_{k=1}^{n} \frac{1}{n} f_{n,k}$$

for $n = 1, 2, \cdots$. If $k/n \leq f(x) < (k+1)/n$, then $k/n \leq f_n(x) \leq (k+1)/n$ since $x \in C_{n,k} \subset C_{n,k-1} \subset \cdots \subset C_{n,1}$ and $x \notin C_{n,k+1} \supset O_{n,k+2} \supset \cdots \supset O_{n,n}$. Therefore we obtain $\|f - f_n\| \leq 1/n$, where $\|\cdot\|$ is the sup norm.

Then there exists $F \in K^+$ which satisfies the following conditions.

(1) $0 \leq F \leq 1$

(2) $f_n \sim F$ for all n

(3) $x \in supp(f) \Rightarrow F(x) = 1$

(4) $0 \leq |f - f_n| \leq \frac{1}{n}F$

Therefore we have $\lim_{n\to\infty} I(f_n) = I(f)$. ∎

Since the Choquet integral with respect to every fuzzy measure is a c.p.m. functional, we can obtain the following corollary.

Corollary 4.6 *For every fuzzy measure μ, there exists an outer regular fuzzy measure μ_r such that for every $f \in K^+$*

$$(C)\int f d\mu = (C)\int f d\mu_r.$$

Example 1 Let $X = [0,1]$, $\mathcal{B}$ the family of Borel subsets of X and $J = [0, \frac{1}{2}]$.

We define a fuzzy measure μ on $\mathcal{B}$ by

$$\mu(A) = \begin{cases} 1 & \text{if } A \not\subset J \\ 0 & \text{if } A \subset J \end{cases}$$

Then this fuzzy measure μ is not outer regular. In fact, let O be an open set such that $J \subset O$. Since J is not open, $J \neq O$. Therefore we obtain $\mu(O) = 1, \inf\{\mu(O) | O \supset J\} = 1$, and $\mu(J) = 0$.

We define an outer regular fuzzy measure by

$$\mu_r(A) = \begin{cases} 1 & \text{if } A \not\subset J' \\ 0 & \text{if } A \subset J' \end{cases}$$

where $A \in \mathcal{B}$ and $J' = (0, \frac{1}{2})$. Then for all $f \in K^+$, $(C)\int f d\mu = (C)\int f d\mu_r$.

5 Representation of the c.m.functional on K

In this section, we discuss the functional defined on the class K of continuous functions with compact suport. We define the regular fuzzy measure induced by a c.m. functional on K, and show that a c.m. functional K can be represented by regular fuzzy measures.

We obtain the next lemma from Proposition4.4.

Lemma 5.1 *Let I be a c.m. functional on K.*

(1) *We put*

$$\mu^+(O) = \sup\{I(f) | f \in K_1^+, supp(f) \subset O\},$$

and

$$\mu^+(B) = \inf\{\mu^+(O) | O \in \mathcal{O}, O \supset B\}$$

for $O \in \mathcal{O}$ and $B \in \mathcal{B}$. Then μ^+ is a regular fuzzy measure.

(2) *We put*

$$\mu^-(O) = \sup\{-I(-f) | f \in K_1^+, supp(f) \subset O\},$$

and

$$\mu^-(B) = \inf\{\mu^-(O) | O \in \mathcal{O}, O \supset B\}$$

for $O \in \mathcal{O}$ and $B \in \mathcal{B}$. Then μ^- is a regular fuzzy measure.

Definition 5.2 Let I be a c.m. functional on K. We say that μ^+ defined in Lemma 5.1 is *the regular fuzzy measure induced by* I^+, and μ^- *the regular fuzzy measure induced by* I^-.

It follows from Proposition 3.3 that the induced regular fuzzy measures μ^+ and μ^- are o-continuous from below and c-continuous from above.

We obtain the next lemma from Theorem 4.5.

Lemma 5.3 *Let I be a c.m. functional on K.*

(1) *If μ^+ is the regular fuzzy measure induced by I^+, then we have*

$$I(f) = (C)\int f d\mu^+$$

for all $f \in K^+$.

(2) *If μ^- is the regular fuzzy measure induced by I^-, then we have*

$$I(f) = -(C)\int (-f) d\mu^-$$

for all $f \in K^-$.

Let $f \in K$ and I be a c.m. functional on K. Since $f \vee 0 \sim f \wedge 0$ and $f = (f \vee 0) + (f \wedge 0)$, we have

$$I(f) = I(f \vee 0) + I(f \wedge 0).$$

Since $f \vee 0 \in K^+$ and $f \wedge 0 \in K^-$, the next theorem follows from Lemma 5.1 and Lemma 5.3.

This is the main result in this section.

Theorem 5.4 *A c.m. functional on K is a r.s.d. functional. That is, let μ^+ and μ^- be the regular fuzzy measures induced by I^+ and I^- respectively. We have*

$$I(f) = (C)\int (f \vee 0) d\mu^+ - (C)\int -(f \wedge 0) d\mu^-.$$

for every $f \in K$.

The next corollary gives the nesessary and sufficient condition that a c.m. functional I is the Šipoš integral.

Corollary 5.5 *Let I be a c.m. functional on K. If $I(f) = -I(-f)$ for every $f \in K$, then I is the* Šipoš *integral.*

It seems that $\mu^+ = \mu^-$ and $\overline{\mu^+} = \mu^-$ if the functional I is a c.m. functional. But it is not always true. The next example shows that $\mu^+ \neq \mu^-$ and $(\mu^+)^d \neq \mu^-$.

Example 2 Let $X = R$ and $J = [0, 1]$. Define a fuzzy measure $\mu : \mathcal{B} \longrightarrow \{0, 1\}$ by

$$\mu(A) = \begin{cases} 1 & \text{if } J \subset A \text{ or } A \text{ is not bounded} \\ 0 & \text{if } A \text{ is bounded and } J \cap A^c \neq \emptyset \end{cases}$$

for $A \in \mathcal{B}$. And define a functional on K by

$$I(f) = (C)\int f d\mu.$$

Then I is a c.m. functional. Therefore there exist two regular fuzzy measure μ^+ and μ^- induced by I^+ and I^- respectively. Indeed for $A \in \mathcal{B}$

$$\mu^+(A) = \begin{cases} 1 & \text{if } J \subset A \\ 0 & \text{if } J \cap A^c \neq \emptyset \end{cases}$$

and

$$\mu^-(A) = 0 \quad \text{whenever} \quad A \in \mathcal{B}.$$

Then we have

$$(\mu^+)^d(A) = \begin{cases} 1 & \text{if } J \cap A \neq \emptyset \\ 0 & \text{if } A \subset J^c \end{cases}$$

Thus $\mu^+ \neq \mu^-$ and $(\mu^+)^d \neq \mu^-$.

6 The case of compact Hausdorff space

If X is a locally compact Hausdorff space, $\mu^+(X) = \mu^-(X)$ is not always true and $I(f)$ is not always equal to the Choquet integral of f with respect to μ^+. Throughout this section, we suppose that X is a compact Hausdorff space.

If X is compact, then the class K of continuous functions on X with compact support is the class of continuous functions on X. Since we have $1_X \in K$, we have $\mu^+(X) = I(1_X)$ and $\mu^-(X) = -I(-1_X)$. The next result follows immediately from this fact.

Proposition 6.1 *Let I be a c.m. functional on K and μ^+ and μ^- the regular fuzzy measure induced by I^+ and I^- respectively. Then we have $\mu^+(X) = \mu^-(X)$.*

Let I be a c.m. functional on K, μ^+ the regular fuzzy measure induced by I^+ and $f \in K$. There exists $a > 0$ such that $\|f\| < a$. It follows from Theorem 5.4 that

$$I(f + a1_X) = (C)\int (f + a1_X)d\mu^+.$$

Since $f \sim a1_X$, we have the next theorem.

Theorem 6.2 *Let I be a c.m. functional on K and μ^+ the regular fuzzy measure induced by I^+. Then I can be represented by the Choquet integral with respect to μ^+. That is,*

$$I(f) = (C)\int f d\mu^+$$

for $f \in K$.

The proof of the next corollary is much the same.

Corollary 6.3 *Let I be a c.m. functional on K, and μ^- the regular fuzzy measure induced by I^-. Then we have*

$$I(f) = -(C)\int (-f)d\mu^-.$$

for $f \in K$.

The next result follows immediately from Theorem 5.4, Theorem 6.2 and Corollary 3.6.

Corollary 6.4 *Let I be a c.m. functional on K and μ^+ and μ^- the regular fuzzy measure induced by I^+ and I^- respectively.*

(1) $(C)\int f d\mu^- = (C)\int f d(\mu^+)^d$ *for all* $f \in K^+$.

(2) $\mu^- = (\mu^+)^d$ *and* $\mu^+ = (\mu^-)^d$.

Corollary 6.4 (2) means that both μ^+ and $(\mu^+)^d$ (μ^- and $(\mu^-)^d$) are regular when X is compact.

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