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Kyoto University
Order preserving operator function via the inequality

\[ A \geq B \geq 0 \text{ ensures } (A^\frac{r}{2} A^p A^\frac{r}{2})^{\frac{1+r}{p+r}} \geq (A^\frac{r}{2} B^p A^\frac{r}{2})^{\frac{1+r}{p+r}} \text{ for } p \geq 1 \text{ and } r \geq 0 \]

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space \( H \). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( (Tx, x) \geq 0 \) for all \( x \in H \) and also an operator \( T \) is strictly positive (denoted by \( T > 0 \)) if \( T \) is positive and invertible. The following Theorem F is an extension of the celebrated Löwner-Heinz theorem [12][10].

**Theorem F** (Furuta inequality) [4].

If \( A \geq B \geq 0 \), then for each \( r \geq 0 \)

(i) \[ (B^\frac{r}{2} A^p B^\frac{r}{2})^{\frac{1}{q}} \geq (B^\frac{r}{2} B^p B^\frac{r}{2})^{\frac{1}{q}} \]

and

(ii) \[ (A^\frac{r}{2} A^p A^\frac{r}{2})^{\frac{1}{q}} \geq (A^\frac{r}{2} B^p A^\frac{r}{2})^{\frac{1}{q}} \]

hold for \( p \geq 0 \) and \( q \geq 1 \) with \( (1+r)q \geq p+r \).

We remark that Theorem F is essentially the same as the inequality made in its title and Theorem F yields the Löwner-Heinz theorem when we put \( r = 0 \) in (i) or (ii) stated above: \( A \geq B \geq 0 \) ensures \( A^\alpha \geq B^\alpha \) for any \( \alpha \in [0, 1] \). Alternative proofs of Theorem F are given in [2] [5] and [11] and also elementary one page proof in [6]. It was shown in [13] that the domain surrounded by \( p, q \) and \( r \) in the Figure is the best possible one for Theorem F. In [8] we established the following Theorem G as extensions of Theorem F.

**Theorem G** (Generalized Furuta inequality) [8]. If \( A \geq B \geq 0 \) with \( A > 0 \), then for each \( t \in [0, 1] \) and \( p \geq 1 \),

\[ F_{p,t}(A, B, r, s) = A^\frac{-r}{2} \{A^\frac{r}{2} (A^\frac{-t}{2} B^p A^\frac{-t}{2}) A^\frac{r}{2}\}^{\frac{1-t+r}{(p-t)s+r}} A^\frac{-r}{2} \]

is decreasing for \( r \geq t \) and \( s \geq 1 \) and \( F_{p,t}(A, A, r, s) \geq F_{p,t}(A, B, r, s) \), that is, for each \( t \in [0, 1] \) and \( p \geq 1 \),

\[ A^{1-t+r} \geq \{A^\frac{r}{2} (A^\frac{-t}{2} B^p A^\frac{-t}{2}) A^\frac{r}{2}\}^{\frac{1-t+r}{(p-t)s+r}} \]

holds for any \( s \geq 1 \) and \( r \geq t \).
Recently a nice mean theoretic proof of Theorem G is shown in [3]. Ando-Hiai [1] established excellent log majorization results and proved the useful inequality equivalent to the main log majorization theorem as follows; If $A \geq B \geq 0$ with $A > 0$, then
\[
A^r \geq \{A^{\frac{r}{2}} (A^\frac{-1}{2} B^p A^\frac{1}{2})^r A^{\frac{1}{2}}\}^\frac{1}{r}
\]
holds for any $p \geq 1$ and $r \geq 1$. Theorem G interpolates the inequality stated above by Ando-Hiai and Theorem F itself and also extends results of [7].

Since now, many applications of Theorem F and Theorem G have been developed in the following branches by many authors.

**APPLICATIONS OF THEOREM F**

(A) OPERATOR INEQUALITIES

(1) Characterizations of operators satisfying $\log A \geq \log B$
(2) Generalizations of Ando's theorem
(3) Other order preserving operator inequalities
(4) Applications to the relative operator entropy
(5) Applications to Ando-Hiai log majorization
(6) Generalized Aluthge transformation

(B) NORM INEQUALITIES

(1) Several generalizations of Heinz-Kato theorem
(2) Generalizations of some theorems on norms
(3) An extension of Kosaki trace inequality and parallel results

(C) OPERATOR EQUATIONS

(1) Generalizations of Pedersen-Takesaki theorem and related results

Very recently the following result is obtained as an extension of Theorem G.

**Theorem H** [9]. If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$, $q \geq 0$ and $p \geq \max\{q,t\}$,
\[
G_{p,q,t}(A, B, r, s) = A^{\frac{r}{2}} \{A^{\frac{r}{2}} (A^\frac{-t}{2} B^p A^\frac{1}{2})^r A^{\frac{1}{2}}\} \geq_{t+r} A^\frac{r}{2}
\]
is decreasing for $r \geq t$ and $s \geq 1$. Moreover for each $t \in [0,1]$, $q \in [t,1]$ and $p \geq q$, $G_{p,q,t}(A, A, r, s) \geq G_{p,q,t}(A, B, r, s)$, that is,
\[
A^{q-t+r} \geq \{A^{\frac{r}{2}} (A^\frac{-t}{2} B^p A^\frac{1}{2})^r A^{\frac{1}{2}}\} \geq_{t+r} A^\frac{r}{2}
\]
holds for any $s \geq 1$ and $r \geq t$. 
The proof in [8] of Theorem G is complicated and technical and also the proof in [3] is based on mean theoretic one. Here we show a simplified proof of Theorem H which is an extension form of Theorem G only using Theorem F and the following Lemma F.

**Lemma F (Furuta lemma) [8].** Let $A > 0$ and $B$ be an invertible operator. Then

$$(BAB^*)^\lambda = BA^\frac{1}{2}(A^\frac{1}{2}B^*BA^\frac{1}{2})^{\lambda-1}A^\frac{1}{2}B^*$$

holds for any real number $\lambda$.

Firstly we show a short proof of the inequality (1.2) of Theorem H. Secondly we show a proof of the monotonicity of the function $G_{p,q,t}(A, B, r, S)$ of Theorem H. Lastly we give three counterexamples and a conjecture related to Theorem G and Theorem H.

## 2 Results on inequalities

**Theorem H-i** [9]. If $A \geq B \geq 0$ with $A > 0$, then for each $1 \geq q \geq t \geq 0$ and $p \geq q$,

$$(1.2) \quad A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}B^{p}A^{\frac{-t}{2}}})SA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}$$

holds for $s \geq 1$ and $r \geq t$.

Theorem H-i is proved as an immediate consequence of the following Theorem 1.

**Theorem 1.** Let $S$ and $T$ be positive invertible operators on a Hilbert space such that $S^{\alpha_{0}} \geq (S^\frac{\beta}{2}T^{\alpha_{0}}S^\frac{\beta}{2})_{\frac{\beta_{0}}{\alpha_{0}+\beta_{0}}} \alpha_{0}^{\alpha_{0}+\beta_{0}}$ holds for fixed $\alpha_{0} > 0$ and $\beta_{0} > 0$. Then

$$(2.1) \quad S^{\beta}T^{\alpha_{0}}S^{\frac{\beta}{2}} \geq (S^\frac{\beta}{2}T^{\alpha_{S}}S^\frac{\beta}{2})_{\frac{\beta_{0}}{\alpha+\beta}}$$

holds for any $\alpha \geq \alpha_{0}$ and $\beta \geq \beta_{0}$.

**Proof of Theorem 1.** Applying (ii) of Theorem F to the hypothesis $S^{\alpha_{0}} \geq (S^\frac{\beta}{2}T^{\alpha_{0}}S^\frac{\beta}{2})_{\frac{\beta_{0}}{\alpha_{0}+\beta_{0}}}$, we have

$$(2.2) \quad S^{(1+r_{1})\beta_{0}} \geq \{S^{\frac{\beta_{0}}{2}}(S^\frac{\beta}{2}T^{\alpha_{0}}S^\frac{\beta}{2})_{\frac{\beta_{0}}{\alpha_{0}+\beta_{0}}} S^{\frac{\beta_{0}}{2}}\}^{\frac{1+r_{1}}{p_{1}+r_{1}}}$$

for any $p_{1} \geq 1$ and $r_{1} \geq 0$.

Putting $p_{1} = \frac{\alpha_{0}+\beta_{0}}{\beta_{0}} \geq 1$ in (2.2), we have

$$(2.3) \quad S^{(1+r_{1})\beta_{0}} \geq (S^\frac{\beta_{0}}{2}T^{\alpha_{0}}S^{\frac{\beta}{2}})^{(1+r_{1})\beta_{0}}_{\alpha_{0}+(1+r_{1})\beta_{0}}.$$

Put $\beta = (1 + r_{1})\beta_{0} \geq \beta_{0}$ in (2.3). Then we have

$$(2.4) \quad S^{\beta} \geq (S^\frac{\beta}{2}T^{\alpha_{0}}S^\frac{\beta}{2})_{\frac{\beta_{0}}{\alpha_{0}+\beta}} \alpha_{0}^{\frac{\beta_{0}}{\alpha_{0}+\beta}}$$

for $\beta \geq \beta_{0}$.
(2.4) is equivalent to the following (2.5) by Lemma F

\[(2.5) \quad T^{\alpha_0} \leq (T^{\frac{\alpha_0}{2}} S^\beta T^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha + \beta}} \quad \text{for } \beta \geq \beta_0.\]

Again applying (i) of Theorem F to (2.5), we have

\[(2.6) \quad T^{(1 + r_2)\alpha_0} \leq (T^{\frac{\alpha_0}{2}} (T^{\frac{\alpha_0}{2}} S^\beta T^{\frac{\alpha_0}{2}})^{\frac{\alpha_0 + \beta}{\alpha + \beta}} T^{\frac{\alpha_0}{2}})^{\frac{1 + r_2}{p_2 + r_2}} \quad \text{for any } p_2 \geq 1 \text{ and } r_2 \geq 0.

Putting \(p_2 = \frac{\alpha_0 + \beta}{\alpha_0} \geq 1\) in (2.6), we have

\[(2.7) \quad T^{(1 + r_2)\alpha_0} \leq (T^{\frac{(1 + r_2)\alpha_0}{2}} S^\beta T^{\frac{(1 + r_2)\alpha_0}{2}})^{\frac{(1 + r_2)\alpha_0}{(1 + r_2)\alpha_0 + \beta}} \quad \text{by Lemma F.}\]

Refining (2.7) and taking inverses of both sides, we obtain (2.1).

**Proof of Theorem H-i.** If \(A \geq B \geq 0\), then the following (2.10) holds

\[(2.10) \quad A^{q \geq r} \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{q + r}{q + r}} \quad \text{for } p \geq q, q \in [0, 1] \text{ and } r \geq 0\]

by (ii) of Theorem F since \((1 + r)^{\frac{q + r}{q + r}} \geq p + r \text{ and } \frac{q + r}{q + r} \geq 1\) in this case.

In the case \(t = 0, (1.2)\) is valid by (2.10) in this case.

In the case \(p = q = t \in [0, 1]\). Let \(C = A^{\frac{t}{2}} B^p A^{\frac{t}{2}}\). As \(I \geq C \geq 0\) holds by Löwner-Heinz theorem, \(A^r \geq A^{\frac{r}{2}} C^s A^{\frac{r}{2}}\) for \(s \geq 1\), that is, (1.2) holds in this case.

In the case \(p > t > 0\). Put \(X = (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1}{p - t}}\). Then we have \(A^{\frac{t}{2}} X^{p - t} A^{\frac{t}{2}} = B^p\) and \(A \geq (A^{\frac{t}{2}} X^{p - t} A^{\frac{t}{2}})^{\frac{1}{2}}\) by the hypothesis \(A \geq B \geq 0\). Put \(\beta_0 = t \in (0, 1]\) and \(\alpha_0 = p - t > 0\). Then \(A \geq (A^{\frac{t}{2}} X^{\alpha_0} A^{\frac{t}{2}})^{\frac{\alpha_0}{\alpha_0 + \beta_0}}\), and

\[A^{\beta_0} \geq (A^{\frac{t}{2}} X^{\alpha_0} A^{\frac{t}{2}})^{\frac{\alpha_0}{\alpha_0 + \beta_0}}\]

holds by Löwner-Heinz theorem. Put \(\alpha = (p - t)s\) and \(\beta = r\). Then \(\alpha \geq \alpha_0\) and \(\beta \geq \beta_0\) hold since \(s \geq 1\) and \(r \geq t\) hold, so that Theorem 1 ensures the following inequality

\[(A^{\frac{t}{2}} X^{\alpha} A^{\frac{t}{2}})^{\frac{\alpha_0 + \beta}{\alpha_0 + \beta}} \leq A^{\frac{t}{2}} X^\alpha A^{\frac{t}{2}},\]

that is, we have

\[(2.11) \quad \{A^{\frac{t}{2}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s A^{\frac{t}{2}}\}^{\frac{p - t + t}{p - t + r}} \leq A^{\frac{t}{2}} A^{\frac{t}{2}} B^p A^{\frac{t}{2}} A^{\frac{t}{2}}.\]
Raising each side of (2.11) to the power \(\frac{q-t+r}{p-t+r} \in [0,1]\) by Löwner-Heinz theorem, we have the first inequality of the following (2.12).

\[
\{A^\frac{r}{2}(A^\frac{-t}{2}B^p A^\frac{-t}{2})^s A^\frac{r}{2}\}^{\frac{q-t+r}{p-t+r}} \\
\leq (A^\frac{r}{2} B^p A^\frac{-t}{2})^{\frac{q-t+r}{p-t+r}} \\
\leq A^{q-t+r}
\]

and the last inequality holds by replacing \(r\) by \(r-t \geq 0\) in (2.10), so the proof of Theorem H-i is complete.

### 3 Results on functions

**Theorem H-f** [9]. Let \(A \geq B \geq 0\) with \(A > 0\). For each \(t \in [0,1], q \geq 0\) and \(p \geq \max\{q, t\}\),

\[
G_{p,q,t}(A, B, r, s) = A^\frac{r}{2} \{A^\frac{-r}{2} B^p A^\frac{-t}{2}\}^s A^\frac{r}{2} \}
^{\frac{q-t+r}{(p-t)_s+r}} A^\frac{r}{2}
\]

is decreasing for \(r \geq t\) and \(s \geq 1\).

Theorem H-f is proved as an immediate consequence of the following Theorem 2.

**Theorem 2.** Let \(S\) and \(T\) be positive invertible operators on a Hilbert space such that \(S^{\alpha_0} \geq (S^\frac{\alpha}{2} T^\alpha S^\frac{\alpha}{2})^{\frac{\alpha_0}{\alpha+\beta}}\) holds for fixed \(\alpha_0 > 0\) and \(\beta_0 > 0\). Then for fixed \(\delta \geq -\beta_0\),

\[
f(\alpha, \beta) = S^{-\frac{\delta}{2}} (S^\frac{\delta}{2} T^\alpha S^\frac{\delta}{2})^{\frac{\alpha+\beta}{\alpha+\beta}} S^{-\frac{\delta}{2}}
\]

is a decreasing function of both \(\alpha\) and \(\beta\) for \(\alpha \geq \max\{\delta, \alpha_0\}\) and \(\beta \geq \beta_0\).

**Proof of Theorem 2.**

(a) *Proof of the result that \(f(\alpha, \beta)\) is a decreasing function of \(\alpha\) for \(\alpha \geq \max\{\delta, \alpha_0\}\).*

The hypothesis in Theorem 2 ensures (3.1) in the same way as the proof of Theorem 1

\[
(T^\frac{\alpha}{2} S^\theta T^\frac{\beta}{2})^{\frac{\alpha+\beta}{\alpha+\beta}} \geq T^\alpha
\]

for all \(\alpha \geq \alpha_0\) and \(\beta \geq \beta_0\).

(3.1) yields the following (3.2) by Löwner-Heinz theorem

\[
(T^\frac{\alpha}{2} S^\theta T^\frac{\beta}{2})^{\frac{\alpha+\beta}{\alpha+\beta}} \geq T^u
\]

for all \(\alpha \geq \alpha_0, \beta \geq \beta_0\) and any \(u\) such that \(\alpha \geq u \geq 0\).

Then we have

\[
g(\alpha) = \{S^\frac{\delta}{2} T^\alpha S^\frac{\delta}{2}\}^{\frac{\alpha+\beta}{\alpha+\beta}}
\]

\[
= \{S^\frac{\delta}{2} T^\alpha S^\frac{\delta}{2}\}^{\frac{\alpha+\beta}{\alpha+\beta}}
\]

\[
= \{S^\frac{\delta}{2} T^\alpha S^\frac{\delta}{2}\}^{\frac{\alpha+\beta}{\alpha+\beta}}
\]

by Lemma F

\[
\geq \{S^\frac{\delta}{2} T^\alpha S^\frac{\delta}{2}\}^{\frac{\alpha+\beta}{\alpha+\beta}}
\]

\[
= (S^\frac{\delta}{2} T^\alpha S^\frac{\delta}{2})^{\frac{\alpha+\beta}{\alpha+\beta}} = g(\alpha + u)
\]
and the last inequality holds by (3.2) and Löwner-Heinz theorem since $\frac{\delta + \beta}{\alpha + \beta + \delta} \in [0, 1]$ holds by the hypothesis on $\alpha, \beta$ and $\delta$. Hence $f(\alpha, \beta) = S^{-\frac{\beta}{2}} g(\alpha) S^{-\frac{\beta}{2}}$ is a decreasing function of $\alpha$ for $\alpha \geq \max\{\delta, \alpha_0\}$.

(b) Proof of the result that $f(\alpha, \beta)$ is a decreasing function of $\beta$ for $\beta \geq \beta_0$.

By Lemma F,

$$f(\alpha, \beta) = S^{-\frac{\beta}{2}} (S^{\frac{\beta}{2}} T^\alpha S^{\frac{\beta}{2}})^{\frac{\delta + \beta}{\alpha + \beta}} S^{-\frac{\beta}{2}}$$

and (3.1) is equivalent to the following (3.3) by Lemma F

$$(3.3) \quad S^\alpha \geq (S^{\frac{\beta}{2}} T^\alpha S^{\frac{\beta}{2}})^{\frac{\delta + \beta}{\alpha + \beta}} \quad \text{for all } \alpha \geq \alpha_0 \text{ and } \beta \geq \beta_0.$$ (3.3) yields the following (3.4) by Löwner-Heinz theorem

$$(3.4) \quad S^\beta \geq (S^{\frac{\beta}{2}} T^\alpha S^{\frac{\beta}{2}})^{\frac{\delta + \beta}{\alpha + \beta}} \quad \text{for all } \alpha \geq \alpha_0, \beta \geq \beta_0 \text{ and any } v \text{ such that } \beta \geq v \geq 0.$$ Then we have

$$h(\beta) = (T^\beta S^\beta T^\beta)^{\frac{\delta + \beta}{\alpha + \beta}}$$

$$(3.3) \quad = \left\{ \left( T^\beta S^\beta T^\beta \right)^{\frac{\delta + \beta}{\alpha + \beta}} \right\}^{\frac{\delta + \beta}{\alpha + \beta}}$$

$$= \left\{ T^\beta S^\beta S^\beta (S^{\frac{\beta}{2}} T^\alpha S^{\frac{\beta}{2}}) S^\beta T^\beta \right\}^{\frac{\delta + \beta}{\alpha + \beta}}$$

$$\geq \left\{ T^\beta S^\beta S^\beta (S^{\frac{\beta}{2}} T^\alpha S^{\frac{\beta}{2}}) S^\beta T^\beta \right\}^{\frac{\delta + \beta}{\alpha + \beta}}$$

$$= (T^\beta S^\beta T^\beta)^{\frac{\delta + \beta}{\alpha + \beta}} = h(\beta + v)$$

and the last inequality holds by (3.4) and Löwner-Heinz theorem since $\frac{\delta - \alpha}{\alpha + \beta + \delta} \in [-1, 0]$ and taking inverses. Hence $f(\alpha, \beta) = T^\beta h(\beta) T^\beta$ is a decreasing function of $\beta$ for $\beta \geq \beta_0$.

Consequently we have finished a proof of Theorem 2 by (a) and (b).

Proof of Theorem H-f. We consider the case $p > t > 0$. Put $X = (\frac{\alpha}{\beta} B^p A^{\beta})^{\frac{1}{p-t}}$. Then we have $A^\frac{\beta}{2} X^{p-t} A^\frac{\beta}{2} = B^p$ and $A \geq (A^\frac{\beta}{2} X^{p-t} A^\frac{\beta}{2})^p$ by the hypothesis $A \geq B \geq 0$. Put $\beta_0 = t \in (0, 1]$ and $\alpha_0 = p - t > 0$. Then $A \geq (A^\frac{\beta}{2} X^{\alpha_0} A^\frac{\beta}{2})^{\frac{\beta_0}{\alpha_0 + \beta_0}}$, so that

$$A^\beta \geq (A^\frac{\beta}{2} X^{\alpha_0} A^\frac{\beta}{2})^{\frac{\beta_0}{\alpha_0 + \beta_0}}$$

holds by Löwner-Heinz theorem. Put $\alpha = (p - t)s, \beta = r$ and $\delta = q - t$. The hypothesis $t \in (0, 1], q \geq 0$ and $p \geq \max\{q, t\}$ in Theorem H-f satisfy the conditions required on $\alpha, \beta$ and $\delta$ in Theorem 2, that is, $\delta \geq -\beta_0, \alpha \geq \max\{\alpha_0, \delta\}$ and $\beta \geq \beta_0$. Applying Theorem 2,

$$f(\alpha, \beta) = A^\frac{\beta}{2} (A^\frac{\beta}{2} X^\alpha A^\frac{\beta}{2})^{\frac{\delta + \beta}{\alpha + \beta}} A^\frac{\beta}{2}$$

$$= A^\frac{\beta}{2} \left\{ A^\frac{\beta}{2} (A^\frac{\beta}{2} B^p A^{\beta})^s A^\frac{\beta}{2} \right\}^{\frac{\delta + \beta}{\alpha + \beta}} A^\frac{\beta}{2}$$

$$= G_{p,q,t}(A, B, r, s)$$
is decreasing for $r \geq t$ and $s \geq 1$, so the proof in the case $p > t > 0$ is complete.

In the case $t = 0$, Theorem H-f easily follows by [7, Theorem 3].

In the case $p = t \geq q \geq 0$. Let $C = A^\frac{p}{2}B^tA^\frac{-p}{2}$. Then $I \geq C \geq 0$ by Löwner-Heinz theorem, so that $A^r \geq A^\frac{r}{2}C^sA^\frac{r}{2}$ holds since $I \geq C \geq 0$ and $s \geq 1$, and again by Löwner-Heinz theorem

\[(3.5) \quad A^u \geq (A^\frac{r}{2}C^sA^\frac{r}{2})^\frac{s}{r} \quad \text{for } r \geq u \geq 0.\]

Then we obtain

\[
G_{t,q,t}(A, B, r, s) = A^\frac{r}{2}(A^\frac{r}{2}C^sA^\frac{r}{2})^{\frac{q-t+r}{r}A^\frac{r}{2}}
\]

\[= C^\frac{r}{2}(C^\frac{r}{2}A^\frac{r}{2}C^\frac{r}{2})^{\frac{q-t+r}{r}A^\frac{r}{2}} \quad \text{by Lemma F}
\]

\[= C^\frac{r}{2}\{(C^\frac{r}{2}A^\frac{r}{2}C^\frac{r}{2})^{\frac{q-t+r}{r}A^\frac{r}{2}}\}^{\frac{q-t+r}{r}A^\frac{r}{2}}
\]

\[= C^\frac{r}{2}\{(C^\frac{r}{2}A^\frac{r}{2}C^\frac{r}{2})^{\frac{q-t+r}{r}A^\frac{r}{2}}\}^{\frac{q-t+r}{r}A^\frac{r}{2}}
\]

\[= C^\frac{r}{2}A^\frac{r-\frac{q-t+r}{r}A^\frac{r}{2}}{2}\{A^\frac{r-\frac{q-t+r}{r}A^\frac{r}{2}}{2}\}^{\frac{q-t+r}{r}A^\frac{r}{2}} = G_{t,q,t}(A, B, r + u, s)
\]

and the last inequality holds by (3.5) and Löwner-Heinz theorem since $q-t+r \in [-1, 0]$ and taking inverses. Consequently $G_{t,q,t}(A, B, r, s)$ is a decreasing function of both $r \geq t$ and $s \geq 1$ because $G_{t,q,t}(A, B, r, s)$ is decreasing of $s \geq 1$ by (3.6) since $I \geq C \geq 0$.

Whence the proof of Theorem H-f is complete.

4 Best possibility and counterexamples

We discuss best possibility of (1.1) in Theorem G and also we cite counterexamples related to Theorem G.

Counterexample 1. There exists a counterexample to (1.1) of Theorem G if we replace $A \geq B$ in Theorem G by $\log A \geq \log B$. Let $p = 2, t = 1, r = 2$ and $s = 2$. Then $p, t, r$ and $s$ satisfy the condition in Theorem G. Take $A$ and $B$ as

\[
A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}^2, \quad B = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}^2
\]

Then it turns out that $\log A \geq \log B$ holds since $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \geq \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$ and $\log t$ is operator monotone, but $A \not\geq B$ holds and

\[
A^{1-t+r} - \{A^\frac{r}{2}(A^\frac{r}{2}B^pA^\frac{-r}{2})sA^\frac{r}{2}\}^{\frac{1-t+r}{p-t-s+r}} = \begin{pmatrix} 50.1594 \cdots & 61.8403 \cdots \\ 61.8403 \cdots & 74.8485 \cdots \end{pmatrix},
\]
so that the eigenvalues of $A^{1-t+r} - \{A^\frac{s}{2}(A^\frac{-t}{2}B^pA^\frac{-t}{2})^{s}A^\frac{s}{2}\}^\frac{1-t+r}{(p-t)s+r}$ are $-0.5563 \cdots$ and $125.5643 \cdots$; therefore $A^{1-t+r} \not\geq \{A^\frac{s}{2}(A^\frac{-t}{2}B^pA^\frac{-t}{2})^{s}A^\frac{s}{2}\}^\frac{1-t+r}{(p-t)s+r}$.

Hence we can’t replace $A \geq B$ in Theorem G by $\log A \geq \log B$, which is weaker than $A \geq B \geq 0$.

**Counterexample 2.** There exists a counterexample to (1.1) of Theorem G if $r$ and $t$ don’t satisfy the condition $r \geq t$. Let $p = 2, s = 2, t = 1 \in [0, 1]$ and $r = \frac{1}{2}$. Then $r \not\geq t$. Take $A$ and $B$ as

$$A = \begin{pmatrix} 28 & 44 \\ 44 & 73 \end{pmatrix}, \quad B = \begin{pmatrix} 20 & 36 \\ 36 & 65 \end{pmatrix}.$$  

Then $A \geq B \geq 0$ and

$$A^{1-t+r} - \{A^\frac{s}{2}(A^\frac{-t}{2}B^pA^\frac{-t}{2})^{s}A^\frac{s}{2}\}^\frac{1-t+r}{(p-t)s+r} = \begin{pmatrix} 1.9229 \cdots & 0.6555 \cdots \\ 0.6555 \cdots & -0.0547 \cdots \end{pmatrix},$$

so that the eigenvalues of $A^{1-t+r} - \{A^\frac{s}{2}(A^\frac{-t}{2}B^pA^\frac{-t}{2})^{s}A^\frac{s}{2}\}^\frac{1-t+r}{(p-t)s+r}$ are $-0.2523 \cdots$ and $2.1205 \cdots$, therefore $A^{1-t+r} \not\geq \{A^\frac{s}{2}(A^\frac{-t}{2}B^pA^\frac{-t}{2})^{s}A^\frac{s}{2}\}^\frac{1-t+r}{(p-t)s+r}$.

**Counterexample 3.** There exists a counterexample to (1.1) of Theorem G if $t$ don’t satisfy the condition $t \in [0, 1]$. Let $t = 1.2 \not\in [0, 1], p = 2, r = 2, s = 2$. Then $r \geq t$. Take $A$ and $B$ as

$$A = \begin{pmatrix} 125 & 90 \\ 90 & 69 \end{pmatrix}, \quad B = \begin{pmatrix} 125 & 90 \\ 90 & 65 \end{pmatrix}.$$  

Then $A \geq B \geq 0$ and

$$A^{1-t+r} - \{A^\frac{s}{2}(A^\frac{-t}{2}B^pA^\frac{-t}{2})^{s}A^\frac{s}{2}\}^\frac{1-t+r}{(p-t)s+r} = \begin{pmatrix} 33.3128 \cdots & 43.4624 \cdots \\ 43.4624 \cdots & 55.3433 \cdots \end{pmatrix},$$

so that the eigenvalues of $A^{1-t+r} - \{A^\frac{s}{2}(A^\frac{-t}{2}B^pA^\frac{-t}{2})^{s}A^\frac{s}{2}\}^\frac{1-t+r}{(p-t)s+r}$ are $-0.5084 \cdots$ and $89.1646 \cdots$, therefore $A^{1-t+r} \not\geq \{A^\frac{s}{2}(A^\frac{-t}{2}B^pA^\frac{-t}{2})^{s}A^\frac{s}{2}\}^\frac{1-t+r}{(p-t)s+r}$.

**Remark.** We remark the following result. By using his skillful and excellent technique as almost same as one in [13], K. Tanahashi [14] asserts that $\frac{1-t+r}{(p-t)s+r}$ of the right hand side of (1.1) of Theorem G is best possible in the sense of the following: $A^{(1-t+r)\alpha} \geq \{A^\frac{s}{2}(A^\frac{-t}{2}B^pA^\frac{-t}{2})^{s}A^\frac{s}{2}\}^\frac{1-t+r}{(p-t)s+r}$ does not hold for any $\alpha > 1$ in Theorem G.

At the end of this section, we cite the following conjecture related to Theorem H and Theorem G.

**Conjecture.** There exists a counterexample to Theorem G in general for any $r < t$.

If $t = 0$ and $r < 0$ in Theorem G, we have already obtained a counterexample.
参考文献


[4] T. Furuta, *A ≥ B ≥ 0 assures \((B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}\) for \(r \geq 0, p \geq 0, q \geq 1\) with \((1 + 2r)q \geq p + 2r\)*, Proc. Amer. Math. Soc. **101** (1987), 85–88.


