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ニューマークのベータ法の安定性について

(On the Stability of Newmark's β method)

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Abstract

For the second order evolution equation in time, we consider Newmark's β method without imposing the assumption of the Rayleigh damping for the dissipation term. We derive the trinomial recurrence relation of Newmark's method which is due to Chaix-Leleux, and give a proof of stability of the scheme for the homogeneous equation by an energy method.

1. The second order evolution equation and Newmark's method

In a finite dimensional real Hilbert space $\mathcal{H}$, we consider the following second order differential equation in time t:

$$\frac{d^2}{dt^2}u(t) + C\frac{d}{dt}u(t) + Ku(t) = f(t), \quad u(t) \in \mathcal{H}, \tag{1}$$

where C and K are non-negative linear operators on $\mathcal{H}$ and f is a given function: $f : [0, \infty) \to \mathcal{H}$.

Let τ be a time step, $U(t)$ be a difference approximation of $u(t)$, $V(t)$ be a difference approximation of $\frac{d}{dt}u(t)$, $A(t)$ be a difference approximation of $\frac{d^2}{dt^2}u(t)$, and β and γ be fixed real numbers. Then we can write Newmark's method[2] as follows:

$$\begin{cases} A(t) + CV(t) + KU(t) = f(t) \\ U(t+\tau) = U(t) + \tau V(t) + \frac{1}{2}\tau^2 A(t) + \beta\tau^2(A(t+\tau) - A(t)) \\ V(t+\tau) = V(t) + \tau A(t) + \gamma\tau(A(t+\tau) - A(t)). \end{cases} \tag{2}$$

The case $\gamma = \frac{1}{2}$ is the standard Newmark's β method.

2. The iteration scheme of Newmark's method

The iteration scheme of Newmark's method (2) for the equation (1) is written as follows:

- I. Compute $A(t)$ from initial data $U(t)$ and $V(t)$ by using (1):

$$A(t) = f(t) - (C\,V(t) + K\,U(t)).$$

- II. Compute $A(t+\tau)$ from $f(t+\tau)$, $U(t)$, $V(t)$ and $A(t)$:

$$\begin{aligned} A(t+\tau) &= (I + \gamma\tau C + \beta\tau^2 K)^{-1} \\ &\quad \times\{-KU(t) - (C + \tau K)V(t) \\ &\quad +(-\tau C + \gamma\tau C - \tfrac{1}{2}\tau^2 K + \beta\tau^2 K)A(t) + f(t+\tau)\}, \end{aligned}$$

where I is the identity operator.

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- III. Compute $V(t+\tau)$ from $V(t)$, $A(t)$ and $A(t+\tau)$:

$$V(t+\tau) = V(t) + \tau A(t) + \gamma\tau(A(t+\tau) - A(t)).$$

- IV. Compute $U(t+\tau)$ from $U(t)$, $V(t)$, $A(t)$ and $A(t+\tau)$:

$$U(t+\tau) = U(t) + \tau V(t) + \frac{1}{2}\tau^2 A(t) + \beta\tau^2(A(t+\tau) - A(t)).$$

- V. Replace t by $t+\tau$, and return to II.

3. The trinomial recurrence relation of Newmark's method

We derive a trinomial recurrence relation for $U(t-\tau)$, $U(t)$ and $U(t+\tau)$ from the following system of equations:

$$\left\{\begin{array}{l} A(t) + CV(t) + KU(t) = f(t) \\ A(t+\tau) + CV(t+\tau) + KU(t+\tau) = f(t+\tau) \\ U(t+\tau) = U(t) + \tau V(t) + \frac{1}{2}\tau^2 A(t) + \beta\tau^2(A(t+\tau) - A(t)) \\ V(t+\tau) = V(t) + \tau A(t) + \gamma\tau(A(t+\tau) - A(t)). \end{array}\right. \tag{3}$$

3.1 Derivation of the trinomial recurrence relation of Newmark's method

We eliminate $A(t)$, $A(t+\tau)$ *and* $V(t+\tau)$ from (3) and get an equation for $U(t)$, $U(t+\tau)$ and $V(t)$. Next we eliminate $A(t)$, $A(t+\tau)$ *and* $V(t)$ from (3) and substitute $t-\tau$ for t, and get another equation for $U(t-\tau)$, $U(t)$ and $V(t)$. Lastly we obtain the following equation eliminating $V(t)$ from these two equations:

$$\begin{aligned} & (I + \gamma\tau C + \beta\tau^2 K)U(t+\tau) + \{-2I + \tau(1-2\gamma)C + \frac{1}{2}\tau^2(1 - 4\beta + 2\gamma)K\}U(t) \\ & +\{I + \tau(-1+\gamma)C + \frac{1}{2}\tau^2(1 + 2\beta - 2\gamma)K\}U(t-\tau) \\ = \; & \beta\tau^2 f(t+\tau) + \frac{1}{2}\tau^2(1 - 4\beta + 2\gamma)f(t) + \frac{1}{2}\tau^2(1 + 2\beta - 2\gamma)f(t-\tau). \end{aligned} \tag{4}$$

In this calculation, we must take care of the non-commutativity between C and K. In the case $\gamma = \frac{1}{2}$, we get a recurrence relation for the standard Newmark's β method:

$$\begin{aligned} & (I + \tfrac{1}{2}\tau C + \beta\tau^2 K)U(t+\tau) + \{-2I + \tau^2(1-2\beta)K\}U(t) + (I - \tfrac{1}{2}\tau C + \beta\tau^2 K)U(t-\tau) \\ = \; & \beta\tau^2 f(t+\tau) + \tau^2(1-2\beta)f(t) + \beta\tau^2 f(t-\tau). \end{aligned} \tag{5}$$

3.2 Representation by difference operators

We define difference operators with time step τ as follows:

$$\begin{aligned} D_\tau U(t) &\equiv \frac{1}{\tau}(U(t+\tau) - U(t)) \sim \frac{d}{dt}u(t+\tau/2), \\ D_{\bar\tau} U(t) &\equiv \frac{1}{\tau}(U(t) - U(t-\tau)) \sim \frac{d}{dt}u(t-\tau/2), \\ D_{\tau\bar\tau} U(t) &\equiv \frac{1}{\tau^2}(U(t+\tau) - 2U(t) + U(t-\tau)) \sim \frac{d^2}{dt^2}u(t), \\ \frac{1}{2}(D_\tau + D_{\bar\tau})U(t) &\equiv \frac{1}{2\tau}(U(t+\tau) - U(t-\tau)) \sim \frac{d}{dt}u(t). \end{aligned}$$

Using these definitions, we obtain the trinomial recurrence relation for $U(t-\tau)$, $U(t)$ and $U(t+\tau)$ as follows:

$$\begin{aligned} & (I+\beta\tau^2K)D_{\tau\bar{\tau}}U(t)+\gamma CD_\tau U(t)+\{(1-\gamma)C+\tau(\gamma-\tfrac{1}{2})K\}D_{\bar{\tau}}U(t)+KU(t) \\ = \; & \{I+\tau(\gamma-\tfrac{1}{2})D_{\bar{\tau}}+\beta\tau^2D_{\tau\bar{\tau}}\}f(t). \end{aligned} \tag{6}$$

Especially, in the case $\gamma=\frac{1}{2}$, we have (see [1],[3] for the case $C\equiv 0$):

$$(I+\beta\tau^2K)D_{\tau\bar{\tau}}U(t)+\frac{1}{2}C(D_\tau+D_{\bar{\tau}})U(t)+KU(t)=(I+\beta\tau^2D_{\tau\bar{\tau}})f(t). \tag{7}$$

4. Stability analysis by energy method

We consider Newmark's β method for the homogeneous equation: $f(t)\equiv 0$ in (1), and derive a stability estimate for the approximate solution of (7) by means of an 'energy method'.

We take an inner-product between (7) and $\frac{1}{2}(D_\tau+D_{\bar{\tau}})U(t)$:

$$\begin{aligned} ((I+\beta\tau^2K)D_{\tau\bar{\tau}}U(t),\ \frac{1}{2}(D_\tau+D_{\bar{\tau}})U(t))+(\frac{1}{2}C(D_\tau+D_{\bar{\tau}})U(t),\ \frac{1}{2}(D_\tau+D_{\bar{\tau}})U(t)) & \\ +(KU(t),\ \frac{1}{2}(D_\tau+D_{\bar{\tau}})U(t))=0. & \end{aligned} \tag{8}$$

Since $C\geq 0$, the second term in the left-hand side of (8) is non-negative. Moving this term to the right-hand side, we have

$$\begin{aligned} ((I+\beta\tau^2K)D_{\tau\bar{\tau}}U(t),\ \frac{1}{2}(D_\tau+D_{\bar{\tau}})U(t))+(KU(t),\ \frac{1}{2}(D_\tau+D_{\bar{\tau}})U(t)) & \\ =-(\frac{1}{2}C(D_\tau+D_{\bar{\tau}})U(t),\ \frac{1}{2}(D_\tau+D_{\bar{\tau}})U(t))\leq 0. & \end{aligned}$$

Hence, we get the inequality:

$$((I+\beta\tau^2K)D_{\tau\bar{\tau}}U(t),\ \frac{1}{2}(D_\tau+D_{\bar{\tau}})U(t))+(KU(t),\ \frac{1}{2}(D_\tau+D_{\bar{\tau}})U(t))\leq 0. \tag{9}$$

Multiplying both sides of (9) by $2\tau^3$, we have

$$\begin{aligned} ((I+\beta\tau^2K)(U(t+\tau)-2U(t)+U(t-\tau)),\ U(t+\tau)-U(t-\tau)) & \\ +(\tau^2KU(t),\ U(t+\tau)-U(t-\tau))\leq 0. & \end{aligned}$$

Inserting $U(t)-U(t)=0$ in the inner-product of the first term in the left-hand side, we get

$$\begin{aligned} & ((I+\beta\tau^2K)(U(t+\tau)-U(t)),\ U(t+\tau)-U(t)) \\ & +((I+\beta\tau^2K)(U(t+\tau)-U(t)),\ U(t)-U(t-\tau)) \\ & -((I+\beta\tau^2K)(U(t)-U(t-\tau)),\ U(t+\tau)-U(t)) \\ & -((I+\beta\tau^2K)(U(t)-U(t-\tau)),\ U(t)-U(t-\tau)) \\ & \qquad +(\tau^2KU(t),\ U(t+\tau)-U(t-\tau)) \;\leq\; 0. \end{aligned}$$

Arranging this formula, we obtain the following inequality:

$$\begin{aligned} & ((I+\beta\tau^2K)(U(t+\tau)-U(t)),\ U(t+\tau)-U(t))+(\tau^2KU(t+\tau),\ U(t)) \\ \leq\; & ((I+\beta\tau^2K)(U(t)-U(t-\tau)),\ U(t)-U(t-\tau))+(\tau^2KU(t),\ U(t-\tau)). \end{aligned}$$

Dividing both sides of this inequality by τ^2, we have

$$\begin{aligned} & ((I+\beta\tau^2K)D_\tau U(t),\ D_\tau U(t)) + (KU(t+\tau),\ U(t)) \\ \le\ & ((I+\beta\tau^2K)D_\tau U(t-\tau),\ D_\tau U(t-\tau)) + (KU(t),\ U(t-\tau)) \\ \le\ & ((I+\beta\tau^2K)D_\tau U(0),\ D_\tau U(0)) + (KU(\tau),\ U(0)). \end{aligned}$$

Using this inequality and the fact that

$$(KU(t+\tau),\ U(t)) = (KU(t),\ U(t)) + \tau(KD_\tau U(t),\ U(t))$$

and $K \ge 0$, we get

$$\|D_\tau U(t)\|^2 + \beta\tau^2\|K^{1/2}D_\tau U(t)\|^2 + \|K^{1/2}U(t)\|^2 + \tau(K^{1/2}D_\tau U(t),\ K^{1/2}U(t)) \le C_0, \tag{10}$$

where

$$\begin{aligned} C_0 &= ((I+\beta\tau^2K)D_\tau U(0),\ D_\tau U(0)) + (KU(\tau),\ U(0)) \\ &= ((I+\beta\tau^2K)D_\tau U(0),\ D_\tau U(0)) + (KU(0),\ U(0)) + \tau(KD_\tau U(0),\ U(0)) \\ &= \|D_\tau U(0)\|^2 + \beta\tau^2\|K^{1/2}D_\tau U(0)\|^2 + \|K^{1/2}U(0)\|^2 + \tau(K^{1/2}D_\tau U(0),\ K^{1/2}U(0)). \end{aligned}$$

If α is a positive real number, from Schwarz's inequality, we get

$$\begin{aligned} |\tau(K^{1/2}D_\tau U(t),\ K^{1/2}U(t))| &\le \|\tau K^{1/2}D_\tau U(t)\|\|K^{1/2}U(t)\| \\ &= \alpha\|\tau K^{1/2}D_\tau U(t)\| \times \tfrac{1}{\alpha}\|K^{1/2}U(t)\| \\ &\le \tfrac{1}{2}\alpha^2\tau^2\|K^{1/2}D_\tau U(t)\|^2 + \tfrac{1}{2\alpha^2}\|K^{1/2}U(t)\|^2. \end{aligned} \tag{11}$$

Moving the forth term in the left-hand side of (10) to the right-hand side and using (11), we have

$$\begin{aligned} \|D_\tau U(t)\|^2 + \beta\tau^2\|K^{1/2}D_\tau U(t)\|^2 &+ \|K^{1/2}U(t)\|^2 \\ &\le C_0 - \tau(K^{1/2}D_\tau U(t),\ K^{1/2}U(t)) \\ &\le C_0 + |\tau(K^{1/2}D_\tau U(t),\ K^{1/2}U(t))| \\ &\le C_0 + \tfrac{1}{2}\alpha^2\tau^2\|K^{1/2}D_\tau U(t)\|^2 + \tfrac{1}{2\alpha^2}\|K^{1/2}U(t)\|^2. \end{aligned} \tag{12}$$

Finally moving the second and the third terms in the last formula of (12) to the left-hand side, we obtain an energy inequality:

$$\|D_\tau U(t)\|^2 + \tau^2(\beta - \frac{\alpha^2}{2})\|K^{1/2}D_\tau U(t)\|^2 + (1 - \frac{1}{2\alpha^2})\|K^{1/2}U(t)\|^2 \le C_0. \tag{13}$$

Using this inequality, we have the following results.

Theorem 1 *In the case $\beta \ge \frac{1}{4}$, we have the stability estimate, with positive constants C_1 and C_2,*

$$\|U(t)\| \le C_1 + C_2 t,$$

and in the case $0 \le \beta < \frac{1}{4}$, if we choose τ such that

$$\tau < \sqrt{\frac{1}{(\frac{1}{4}-\beta)\|K^{1/2}\|^2}},$$

then we have, with positive constants C_3 and C_4,

$$\|U(t)\| \le C_3 + C_4 t,$$

From now on, we show the proof of this theorem. First, we consider the case $\beta \geq \frac{1}{4}$. If we put $\alpha = \sqrt{2\beta}$ in (13), then we have, for $\beta > \frac{1}{4}$, that

$$\|D_\tau U(t)\|^2 + (1 - \frac{1}{4\beta})\|K^{1/2}U(t)\|^2 \leq C_0$$

and

$$\|D_\tau U(t)\|,\ \|K^{1/2}U(t)\| \leq C_\beta = (1 - \frac{1}{4\beta})^{-1}C_0 \ < \infty,$$

where C_β is a constant independent of t. Hence, we get

$$\beta > \frac{1}{4} \implies \|D_\tau U(t)\|,\ \|K^{1/2}U(t)\| \leq C_\beta.$$

And we also obtain that

$$\beta \geq \frac{1}{4} \implies \|D_\tau U(t)\| \leq \sqrt{C_0}.$$

Then recalling the definition:

$$D_\tau U(t) = \frac{1}{\tau}(U(t+\tau) - U(t)),$$

we get

$$\|U(t+\tau) - U(t)\| \leq \sqrt{C_0}\tau,$$

and

$$\|U(t+\tau)\| \leq \|U(t)\| + \sqrt{C_0}\tau \leq \cdots\cdots \leq \|U(0)\| + \sqrt{C_0}(t+\tau).$$

Putting $C_1 = \|U(0)\|$ and $C_2 = \sqrt{C_0}$, where C_1 is constant independent of τ, we can conclude that

$$\beta \geq \frac{1}{4} \Longrightarrow \|U(t)\| \leq C_1 + C_2 t. \tag{14}$$

Next, we consider the case $0 \leq \beta < \frac{1}{4}$. Put $\alpha^2 = \frac{1}{2}$ in (13). Then we have

$$\|D_\tau U(t)\|^2 + \tau^2(\beta - \frac{1}{4})\|K^{1/2}D_\tau U(t)\|^2 \leq C_0$$

and

$$\|D_\tau U(t)\|^2 \leq C_0 + \tau^2(\frac{1}{4} - \beta)\|K^{1/2}D_\tau U(t)\|^2. \tag{15}$$

Let $y \in \mathcal{H}$ and $\|K^{1/2}\|$ be the operator norm of $K^{1/2}$, then we have $\|K^{1/2}y\| \leq \|K^{1/2}\|\|y\|$. Applying this inequality to (15), we get

$$\|D_\tau U(t)\|^2 \leq C_0 + \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2\|D_\tau U(t)\|^2$$

and

$$(1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2)\|D_\tau U(t)\|^2 \leq C_0.$$

Noticing the fact that, for $\tau > 0$,

$$0 < 1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2 \Longleftrightarrow \tau < \sqrt{\frac{1}{(\frac{1}{4} - \beta)\|K^{1/2}\|^2}},$$

we obtain

$$\tau < \sqrt{\frac{1}{(\frac{1}{4} - \beta)\|K^{1/2}\|^2}} \implies \|D_\tau U(t)\| \leq \sqrt{\frac{C_0}{1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2}},$$

and we obtain:

$$\|U(t)\| \leq C_3 + C_4 t,$$

where

$$C_3 = \|U(0)\|,\ C_4 = \sqrt{\frac{C_0}{1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2}}.$$

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