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Kyoto University
On the Stability of Newmark's $\beta$ method

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Abstract

For the second order evolution equation in time, we consider Newmark's $\beta$ method without imposing the assumption of the Rayleigh damping for the dissipation term. We derive the trinomial recurrence relation of Newmark's method which is due to Chaix-Leleux, and give a proof of stability of the scheme for the homogeneous equation by an energy method.

1. The second order evolution equation and Newmark's method

In a finite dimensional real Hilbert space $\mathcal{H}$, we consider the following second order differential equation in time $t$:

$$\frac{d^2}{dt^2}u(t) + C \frac{d}{dt}u(t) + Ku(t) = f(t), \quad u(t) \in \mathcal{H},$$

(1)

where $C$ and $K$ are non-negative linear operators on $\mathcal{H}$ and $f$ is a given function: $f : [0, \infty) \to \mathcal{H}$.

Let $\tau$ be a time step, $U(t)$ be a difference approximation of $u(t)$, $V(t)$ be a difference approximation of $\frac{d}{dt}u(t)$, $A(t)$ be a difference approximation of $\frac{d^2}{dt^2}u(t)$, and $\beta$ and $\gamma$ be fixed real numbers. Then we can write Newmark's method[2] as follows:

$$\begin{cases}
A(t) + CV(t) + Ku(t) = f(t) \\
U(t+\tau) = U(t) + \tau V(t) + \frac{1}{2}\tau^2 A(t) + \beta \tau^2 (A(t+\tau) - A(t)) \\
V(t+\tau) = V(t) + \tau A(t) + \gamma \tau (A(t+\tau) - A(t))
\end{cases}$$

(2)

The case $\gamma = \frac{1}{2}$ is the standard Newmark's $\beta$ method.

2. The iteration scheme of Newmark's method

The iteration scheme of Newmark's method (2) for the equation (1) is written as follows:

- I. Compute $A(t)$ from initial data $U(t)$ and $V(t)$ by using (1):

$$A(t) = f(t) - (C V(t) + K U(t)).$$

- II. Compute $A(t+\tau)$ from $f(t+\tau)$, $U(t)$, $V(t)$ and $A(t)$:

$$A(t+\tau) = (I + \gamma \tau C + \beta \tau^2 K)^{-1} \times (-KU(t) - (C + \tau K)V(t) + (-\tau C + \gamma \tau C - \frac{1}{2} \tau^2 K + \beta \tau^2 K)A(t) + f(t+\tau)),$$

where $I$ is the identity operator.

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• III. Compute $V(t + \tau)$ from $V(t)$, $A(t)$ and $A(t + \tau)$:

$$V(t + \tau) = V(t) + \tau A(t) + \gamma \tau (A(t + \tau) - A(t)).$$

• IV. Compute $U(t + \tau)$ from $U(t)$, $V(t)$, $A(t)$ and $A(t + \tau)$:

$$U(t + \tau) = U(t) + \tau V(t) + \frac{1}{2} \tau^2 A(t) + \beta \tau^2 (A(t + \tau) - A(t)).$$

• V. Replace $t$ by $t + \tau$, and return to II.

3. The trinomial recurrence relation of Newmark’s method

We derive a trinomial recurrence relation for $U(t - \tau)$, $U(t)$ and $U(t + \tau)$ from the following system of equations:

$$\begin{cases} 
A(t) + CV(t) + KU(t) = f(t) \\
A(t + \tau) + CV(t + \tau) + KU(t + \tau) = f(t + \tau) \\
U(t + \tau) = U(t) + \tau V(t) + \frac{1}{2} \tau^2 A(t) + \beta \tau^2 (A(t + \tau) - A(t)) \\
V(t + \tau) = V(t) + \tau A(t) + \gamma \tau (A(t + \tau) - A(t)).
\end{cases}$$

(3)

3.1 Derivation of the trinomial recurrence relation of Newmark’s method

We eliminate $A(t)$, $A(t + \tau)$ and $V(t + \tau)$ from (3) and get an equation for $U(t)$, $U(t + \tau)$ and $V(t)$. Next we eliminate $A(t)$, $A(t + \tau)$ and $V(t)$ from (3) and substitute $t - \tau$ for $t$, and get another equation for $U(t - \tau)$, $U(t)$ and $V(t)$. Lastly we obtain the following equation eliminating $V(t)$ from these two equations:

$$(I + \gamma \tau C + \beta \tau^2 K)U(t + \tau) + \{-2I + \tau(1 - 2\gamma)C + \frac{1}{2} \tau^2 (1 - 4\beta + 2\gamma)K\}U(t) + \{I + \tau(-1 + \gamma)C + \frac{1}{2} \tau^2 (1 + 2\beta - 2\gamma)K\}U(t - \tau) = \beta \tau^2 f(t + \tau) + \tau^2 (1 - 2\beta) f(t) + \beta \tau^2 f(t - \tau).$$

(4)

In this calculation, we must take care of the non-commutativity between $C$ and $K$. In the case $\gamma = \frac{1}{2}$, we get a recurrence relation for the standard Newmark’s $\beta$ method:

$$(I + \frac{1}{2} \tau C + \beta \tau^2 K)U(t + \tau) + \{-2I + \tau^2 (1 - 2\beta)K\}U(t) + (I - \frac{1}{2} \tau C + \beta \tau^2 K)U(t - \tau) = \beta \tau^2 f(t + \tau) + \tau^2 (1 - 2\beta) f(t) + \beta \tau^2 f(t - \tau).$$

(5)

3.2 Representation by difference operators

We define difference operators with time step $\tau$ as follows:

$$\begin{align*}
D_{\tau}U(t) &\equiv \frac{1}{\tau}(U(t + \tau) - U(t)) \sim \frac{d}{dt}u(t + \tau/2), \\
D_{\tau}U(t) &\equiv \frac{1}{\tau}(U(t) - U(t - \tau)) \sim \frac{d}{dt}u(t - \tau/2), \\
D_{\tau\tau}U(t) &\equiv \frac{1}{\tau^2}(U(t + \tau) - 2U(t) + U(t - \tau)) \sim \frac{d^2}{dt^2}u(t), \\
\frac{1}{2}(D_{\tau} + D_{\tau})U(t) &\equiv \frac{1}{2\tau}(U(t + \tau) - U(t - \tau)) \sim \frac{d}{dt}u(t).
\end{align*}$$
Using these definitions, we obtain the trinomial recurrence relation for $U(t - \tau)$, $U(t)$ and $U(t + \tau)$ as follows:

$$
(I + \beta \tau^2 K)D_{\tau\overline{\tau}}U(t) + \gamma CD_{\tau}U(t) + \{(1 - \gamma)C + \tau(\gamma - \frac{1}{2})K\}D_{\overline{\tau}}U(t) + KU(t)
$$

$$
= (I + \tau(\gamma - \frac{1}{2})D_{\overline{\tau}} + \beta_{\mathcal{T}D_{\tau\overline{\tau}}})2f(t). \tag{6}
$$

Especially, in the case $\gamma = \frac{1}{2}$, we have (see [1],[3] for the case $C \equiv 0$):

$$
(I + \beta \tau^2 K)D_{\tau\overline{\tau}}U(t) + \frac{1}{2}C(D_{\tau} + D_{\overline{\tau}})U(t) + KU(t) = (I + \beta \tau^2 D_{\tau}\overline{\tau})f(t). \tag{7}
$$

4. Stability analysis by energy method

We consider Newmark's $\beta$ method for the homogeneous equation: $f(t) \equiv 0$ in (1), and derive a stability estimate for the approximate solution of (7) by means of an 'energy method'.

We take an inner-product between (7) and $\frac{1}{2}(D_{\tau} + D_{\overline{\tau}})U(t)$:

$$
((I + \beta \tau^2 K)D_{\tau\overline{\tau}}U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}})U(t))+\left(\frac{1}{2}C(D_{\tau} + D_{\overline{\tau}})U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}})U(t)\right)
$$

$$
+ (KU(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}})U(t)) = 0. \tag{8}
$$

Since $C \geq 0$, the second term in the left-hand side of (8) is non-negative. Moving this term to the right-hand side, we have

$$
((I + \beta \tau^2 K)D_{\tau\overline{\tau}}U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}})U(t)) + (KU(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}})U(t)) \leq 0.
$$

Hence, we get the inequality:

$$
((I + \beta \tau^2 K)D_{\tau\overline{\tau}}U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}})U(t)) + (KU(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}})U(t)) \leq 0. \tag{9}
$$

Multiplying both sides of (9) by $2\tau^3$, we have

$$
((I + \beta \tau^2 K)(U(t + \tau) - 2U(t) + U(t - \tau)), U(t + \tau) - U(t - \tau))
$$

$$
+ (\tau^2 KU(t), U(t + \tau) - U(t - \tau)) \leq 0.
$$

Inserting $U(t) - U(t) = 0$ in the inner-product of the first term in the left-hand side, we get

$$
((I + \beta \tau^2 K)(U(t + \tau) - U(t)), U(t + \tau) - U(t))
$$

$$
+((I + \beta \tau^2 K)(U(t + \tau) - U(t)), U(t) - U(t - \tau))
$$

$$
-((I + \beta \tau^2 K)(U(t) - U(t - \tau)), U(t + \tau) - U(t))
$$

$$
-((I + \beta \tau^2 K)(U(t) - U(t - \tau)), U(t) - U(t - \tau))
$$

$$
+ (\tau^2 KU(t), U(t + \tau) - U(t - \tau)) \leq 0.
$$

Arranging this formula, we obtain the following inequality:

$$
((I + \beta \tau^2 K)(U(t + \tau) - U(t)), U(t + \tau) - U(t)) + (\tau^2 KU(t + \tau), U(t))
$$

$$
\leq ((I + \beta \tau^2 K)(U(t) - U(t - \tau)), U(t) - U(t - \tau)) + (\tau^2 KU(t), U(t - \tau)).
$$
Dividing both sides of this inequality by $\tau^2$, we have

\[
((I + \beta\tau^2 K)D_r U(t), D_r U(t)) + (KU(t + \tau), U(t)) \\
\leq ((I + \beta\tau^2 K)D_r U(t - \tau), D_r U(t - \tau)) + (KU(t), U(t - \tau)) \\
\leq ((I + \beta\tau^2 K)D_r U(0), D_r U(0)) + (KU(\tau), U(0)).
\]

Using this inequality and the fact that

\[
(KU(t + \tau), U(t)) = (KU(t), U(t)) + \tau(KD_r U(t), U(t))
\]

and $K \geq 0$, we get

\[
||D_r U(t)||^2 + \beta\tau^2||K^{1/2}D_r U(t)||^2 + ||K^{1/2}U(t)||^2 + \tau(K^{1/2}D_r U(t), K^{1/2}U(t)) \leq C_0,
\]

where

\[
C_0 = ((I + \beta\tau^2 K)D_r U(0), D_r U(0)) + (KU(\tau), U(0)) \\
= ((I + \beta\tau^2 K)D_r U(0), D_r U(0)) + (KU(0), U(0)) + \tau(KD_r U(0), U(0)) \\
= ||D_r U(0)||^2 + \beta\tau^2||K^{1/2}D_r U(0)||^2 + ||K^{1/2}U(0)||^2 + \tau(K^{1/2}D_r U(0), K^{1/2}U(0)).
\]

If $\alpha$ is a positive real number, from Schwarz's inequality, we get

\[
|\tau(K^{1/2}D_r U(t), K^{1/2}U(t))| \leq \alpha||\tau K^{1/2}D_r U(t)|| \times \frac{1}{\alpha}||K^{1/2}U(t)|| \\
\leq \frac{1}{2}\alpha^2\tau^2||K^{1/2}D_r U(t)||^2 + \frac{1}{2\alpha^2}||K^{1/2}U(t)||^2.
\]

Moving the forth term in the left-hand side of (10) to the right-hand side and using (11), we have

\[
||D_r U(t)||^2 + \beta\tau^2||K^{1/2}D_r U(t)||^2 + ||K^{1/2}U(t)||^2 \leq C_0 - \tau(K^{1/2}D_r U(t), K^{1/2}U(t)) \\
\leq C_0 - \tau(K^{1/2}D_r U(t), K^{1/2}U(t)) \\
\leq C_0 + \frac{1}{2}\alpha^2\tau^2||K^{1/2}D_r U(t)||^2 + \frac{1}{2\alpha^2}||K^{1/2}U(t)||^2.
\]

Finally moving the second and the third terms in the last formula of (12) to the left-hand side, we obtain an energy inequality:

\[
||D_r U(t)||^2 + \tau^2(\beta - \frac{\alpha^2}{2})||K^{1/2}D_r U(t)||^2 + (1 - \frac{1}{2\alpha^2})||K^{1/2}U(t)||^2 \leq C_0.
\]

Using this inequality, we have the following results.

**Theorem 1** In the case $\beta \geq \frac{1}{4}$, we have the stability estimate, with positive constants $C_1$ and $C_2$,

\[||U(t)|| \leq C_1 + C_2t,\]

and in the case $0 \leq \beta < \frac{1}{4}$, if we choose $\tau$ such that

\[
\tau < \sqrt{\frac{1}{(\frac{1}{4} - \beta)||K^{1/2}||^2}},
\]

then we have, with positive constants $C_3$ and $C_4$,

\[||U(t)|| \leq C_3 + C_4t,\]
From now on, we show the proof of this theorem. First, we consider the case \( \beta \geq \frac{1}{4} \). If we put \( \alpha = \sqrt{2\beta} \) in (13), then we have, for \( \beta > \frac{1}{4} \), that
\[
\|D_{\tau}U(t)\|^2 + (1 - \frac{1}{4\beta})\|K^{1/2}U(t)\|^2 \leq C_0
\]
and
\[
\|D_{\tau}U(t)\|, \|K^{1/2}U(t)\| \leq C_\beta = (1 - \frac{1}{4\beta})^{-1}C_0 < \infty,
\]
where \( C_\beta \) is a constant independent of \( t \). Hence, we get
\[
\beta > \frac{1}{4} \implies \|D_{\tau}U(t)\|, \|K^{1/2}U(t)\| \leq C_\beta.
\]
And we also obtain that
\[
\beta \geq \frac{1}{4} \implies \|D_{\tau}U(t)\| \leq \sqrt{C_0}.
\]
Then recalling the definition:
\[
D_{\tau}U(t) = \frac{1}{\tau}(U(t+\tau) - U(t)),
\]
we get
\[
\|U(t+\tau) - U(t)\| \leq \sqrt{C_0\tau},
\]
and
\[
\|U(t+\tau)\| \leq \|U(t)\| + \sqrt{C_0\tau} \leq \cdots \leq \|U(0)\| + \sqrt{C_0(t+\tau)}.
\]
Putting \( C_1 = \|U(0)\| \) and \( C_2 = \sqrt{C_0} \), where \( C_1 \) is constant independent of \( \tau \), we can conclude that
\[
\beta \geq \frac{1}{4} \implies \|U(t)\| \leq C_1 + C_2 t. \tag{14}
\]
Next, we consider the case \( 0 < \beta < \frac{1}{4} \). Put \( \alpha^2 = \frac{1}{2} \) in (13). Then we have
\[
\|D_{\tau}U(t)\|^2 + \tau^2(\beta - \frac{1}{4})\|K^{1/2}D_{\tau}U(t)\|^2 \leq C_0
\]
and
\[
\|D_{\tau}U(t)\|^2 \leq C_0 + \tau^2(\frac{1}{4} - \beta)\|K^{1/2}D_{\tau}U(t)\|^2. \tag{15}
\]
Let \( y \in \mathcal{H} \) and \( \|K^{1/2}\| \) be the operator norm of \( K^{1/2} \), then we have \( \|K^{1/2}y\| \leq \|K^{1/2}\|\|y\| \). Applying this inequality to (15), we get
\[
\|D_{\tau}U(t)\|^2 \leq C_0 + \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2\|D_{\tau}U(t)\|^2
\]
and
\[
(1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2)\|D_{\tau}U(t)\|^2 \leq C_0.
\]
Noticing the fact that, for \( \tau > 0 \),
\[
0 < 1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2 \iff \tau < \sqrt{\frac{1}{(\frac{1}{4} - \beta)\|K^{1/2}\|^2}},
\]
we obtain
\[
\tau < \sqrt{\frac{1}{(\frac{1}{4} - \beta)\|K^{1/2}\|^2}} \implies \|D_{\tau}U(t)\| \leq \sqrt{\frac{C_0}{1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2}},
\]
and we obtain:
\[
\|U(t)\| \leq C_3 + C_4 t,
\]
where
\[
C_3 = \|U(0)\|, \quad C_4 = \sqrt{\frac{C_0}{1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2}}.
\]
References


