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On the Stability of Newmark’s $\beta$ method

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Abstract

For the second order evolution equation in time, we consider Newmark’s $\beta$ method without imposing the assumption of the Rayleigh damping for the dissipation term. We derive the trinomial recurrence relation of Newmark’s method which is due to Chaix-Leleux, and give a proof of stability of the scheme for the homogeneous equation by an energy method.

1. The second order evolution equation and Newmark’s method

In a finite dimensional real Hilbert space $\mathcal{H}$, we consider the following second order differential equation in time $t$:

$$\frac{d^2}{dt^2}u(t) + C\frac{d}{dt}u(t) + Ku(t) = f(t), \quad u(t) \in \mathcal{H},$$  \hspace{1cm} (1)

where $C$ and $K$ are non-negative linear operators on $\mathcal{H}$ and $f$ is a given function: $f : [0, \infty) \rightarrow \mathcal{H}$.

Let $\tau$ be a time step, $U(t)$ be a difference approximation of $u(t)$, $V(t)$ be a difference approximation of $\frac{d}{dt}u(t)$, $A(t)$ be a difference approximation of $\frac{d^2}{dt^2}u(t)$, and $\beta$ and $\gamma$ be fixed real numbers. Then we can write Newmark’s method[2] as follows:

$$\begin{align*}
A(t) + CV(t) + KU(t) &= f(t) \\
U(t + \tau) &= U(t) + \tau V(t) + \frac{1}{2}\tau^2 A(t) + \beta \tau^2 (A(t + \tau) - A(t)) \\
V(t + \tau) &= V(t) + \tau A(t) + \gamma \tau (A(t + \tau) - A(t)).
\end{align*}$$  \hspace{1cm} (2)

The case $\gamma = \frac{1}{2}$ is the standard Newmark’s $\beta$ method.

2. The iteration scheme of Newmark’s method

The iteration scheme of Newmark’s method (2) for the equation (1) is written as follows:

- I. Compute $A(t)$ from initial data $U(t)$ and $V(t)$ by using (1):
  $$A(t) = f(t) - (C V(t) + K U(t)).$$

- II. Compute $A(t + \tau)$ from $f(t + \tau)$, $U(t)$, $V(t)$ and $A(t)$:
  $$A(t + \tau) = \left( I + \gamma \tau C + \beta \tau^2 K \right)^{-1} \times \{ -KU(t) - (C + \tau K)V(t) \\
+ (\gamma \tau C + \gamma \tau^2 C - \frac{1}{2}\tau^2 K + \beta \tau^2 K)A(t) + f(t + \tau) \},$$

where $I$ is the identity operator.

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III. Compute $V(t + \tau)$ from $V(t)$, $A(t)$ and $A(t + \tau)$:

$$V(t + \tau) = V(t) + \tau A(t) + \gamma \tau (A(t + \tau) - A(t)).$$

IV. Compute $U(t + \tau)$ from $U(t)$, $V(t)$, $A(t)$ and $A(t + \tau)$:

$$U(t + \tau) = U(t) + \tau V(t) + \frac{1}{2} \tau^2 A(t) + \beta \tau^2 (A(t + \tau) - A(t)).$$

V. Replace $t$ by $t + \tau$, and return to II.

3. The trinomial recurrence relation of Newmark’s method

We derive a trinomial recurrence relation for $U(t - \tau), U(t)$ and $U(t + \tau)$ from the following system of equations:

$$\begin{align*}
A(t) + CV(t) + KU(t) &= f(t) \\
A(t + \tau) + CV(t + \tau) + KU(t + \tau) &= f(t + \tau) \\
U(t + \tau) &= U(t) + \tau V(t) + \frac{1}{2} \tau^2 A(t) + \beta \tau^2 (A(t + \tau) - A(t)) \\
V(t + \tau) &= V(t) + \tau A(t) + \gamma \tau (A(t + \tau) - A(t)).
\end{align*}$$

(3)

3.1 Derivation of the trinomial recurrence relation of Newmark’s method

We eliminate $A(t), A(t + \tau)$ and $V(t + \tau)$ from (3) and get an equation for $U(t), U(t + \tau)$ and $V(t)$. Next we eliminate $A(t), A(t + \tau)$ and $V(t)$ from (3) and substitute $t - \tau$ for $t$, and get another equation for $U(t - \tau), U(t)$ and $V(t)$. Lastly we obtain the following equation eliminating $V(t)$ from these two equations:

$$(I + \gamma \tau C + \beta \tau^2 K)U(t + \tau) + \{-2I + \tau (.1 - 2\gamma) C + \frac{1}{2} \tau^2 (1 - 4\beta + 2\gamma) K\}U(t) + \{I + \tau (-1 + \gamma) C + \frac{1}{2} \tau^2 (1 + 2\beta - 2\gamma) K\}U(t - \tau) = \beta \tau^2 f(t + \tau) + \frac{1}{2} \tau^2 (1 - 4\beta + 2\gamma) f(t) + \frac{1}{2} \tau^2 (1 + 2\beta - 2\gamma) f(t - \tau).$$

(4)

In this calculation, we must take care of the non-commutativity between $C$ and $K$. In the case $\gamma = \frac{1}{2}$, we get a recurrence relation for the standard Newmark’s $\beta$ method:

$$\begin{align*}
(I + \frac{1}{2} \tau C + \beta \tau^2 K)U(t + \tau) + \{-2I + \tau^2 (1 - 2\beta)K\}U(t) + (I - \frac{1}{2} \tau C + \beta \tau^2 K)U(t - \tau) &= \beta \tau^2 f(t + \tau) + \tau^2 (1 - 2\beta) f(t) + \beta \tau^2 f(t - \tau).
\end{align*}$$

(5)

3.2 Representation by difference operators

We define difference operators with time step $\tau$ as follows:

$$\begin{align*}
D_{\tau}U(t) &\equiv \frac{1}{\tau}(U(t + \tau) - U(t)) \sim \frac{d}{dt}u(t + \tau/2), \\
D_{\tau}U(t) &\equiv \frac{1}{\tau}(U(t) - U(t - \tau)) \sim \frac{d}{dt}u(t - \tau/2), \\
D_{\tau^2}U(t) &\equiv \frac{1}{\tau^2}(U(t + \tau) - 2U(t) + U(t - \tau)) \sim \frac{d^2}{dt^2}u(t), \\
\frac{1}{2}(D_{\tau} + D_{\tau})U(t) &\equiv \frac{1}{2\tau}(U(t + \tau) - U(t - \tau)) \sim \frac{d}{dt}u(t).
\end{align*}$$
Using these definitions, we obtain the trinomial recurrence relation for $U(t-\tau)$, $U(t)$ and $U(t+\tau)$ as follows:

\[
(I + \beta\tau^2 K) D_{\tau\overline{\tau}} U(t) + \gamma CD_{\tau} U(t) + \{(1-\gamma)C + \tau(\gamma - \frac{1}{2})K\} D_{\overline{\tau}} U(t) + K U(t)
\]

\[
= \{I + \gamma\tau(\gamma - \frac{1}{2})D_{\overline{\tau}} + \beta\tau^2 D_{\tau\overline{\tau}}\} f(t).
\]

Especially, in the case $\gamma = \frac{1}{2}$, we have (see [1],[3] for the case $C \equiv 0$):

\[
(I + \beta\tau^2 K) D_{\tau\overline{\tau}} U(t) + \frac{1}{2} C(D_{\tau} + D_{\overline{\tau}}) U(t) + K U(t) = \{I + \beta\tau^2 D_{\tau\overline{\tau}}\} f(t).
\]

4. Stability analysis by energy method

We consider Newmark's $\beta$ method for the homogeneous equation: $f(t) \equiv 0$ in (1), and derive a stability estimate for the approximate solution of (7) by means of an 'energy method'.

We take an inner-product between (7) and $\frac{1}{2}(D_{\tau} + D_{\overline{\tau}}) U(t)$:

\[
((I + \beta\tau^2 K) D_{\tau\overline{\tau}} U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}}) U(t)) + \frac{1}{2} C(D_{\tau} + D_{\overline{\tau}}) U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}}) U(t))
\]

\[
+ (K U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}}) U(t)) = 0.
\]

Since $C \geq 0$, the second term in the left-hand side of (8) is non-negative. Moving this term to the right-hand side, we have

\[
((I + \beta\tau^2 K) D_{\tau\overline{\tau}} U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}}) U(t)) + (K U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}}) U(t)) \leq 0.
\]

Hence, we get the inequality:

\[
((I + \beta\tau^2 K) D_{\tau\overline{\tau}} U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}}) U(t)) + (K U(t), \frac{1}{2}(D_{\tau} + D_{\overline{\tau}}) U(t)) \leq 0.
\]

Multiplying both sides of (9) by $2\tau^3$, we have

\[
((I + \beta\tau^2 K)(U(t+\tau) - U(t+\tau) - U(t) + U(t-\tau)), \tau^3 U(t)) + \tau^2 KU(t), \tau^2 KU(t, \tau^2 KU(t, \tau^2 KU(t)) \leq 0.
\]

Inserting $U(t) - U(t) = 0$ in the inner-product of the first term in the left-hand side, we get

\[
((I + \beta\tau^2 K)(U(t+\tau) - U(t)), \tau^3 U(t)) + ((I + \beta\tau^2 K)(U(t+\tau) - U(t)), \tau^3 U(t)) \leq 0.
\]

Arranging this formula, we obtain the following inequality:

\[
((I + \beta\tau^2 K)(U(t+\tau) - U(t)), \tau^3 U(t)) + ((I + \beta\tau^2 K)(U(t+\tau) - U(t)), \tau^3 U(t)) \leq 0.
\]
Dividing both sides of this inequality by $\tau^2$, we have

\[
(I + \beta \tau^2 K)D_\tau U(t), D_\tau U(t) + (K U(t + \tau), U(t)) \leq (I + \beta \tau^2 K)D_\tau U(t - \tau), D_\tau U(t - \tau) + (K U(t), U(t))
\]

Using this inequality and the fact that

\[
(KU(t + \tau), U(t)) = (KU(t), U(t)) + \tau(KD_\tau U(t), U(t))
\]

and $K \geq 0$, we get

\[
\|D_\tau U(t)\|^2 + \beta \tau^2 \|K^{1/2}D_\tau U(t)\|^2 + \|K^{1/2}U(t)\|^2 + \tau(K^{1/2}D_\tau U(t), K^{1/2}U(t)) \leq C_0,
\]

where

\[
C_0 = ((I + \beta \tau^2 K)D_\tau U(0), D_\tau U(0)) + (K U(\tau), U(\tau))
\]

\[
= ((I + \beta \tau^2 K)D_\tau U(0), D_\tau U(0)) + (K U(0), U(0)) + \tau(KD_\tau U(0), U(0))
\]

\[
= \|D_\tau U(0)\|^2 + \beta \tau^2 \|K^{1/2}D_\tau U(0)\|^2 + \|K^{1/2}U(0)\|^2 + \tau(K^{1/2}D_\tau U(0), K^{1/2}U(0)).
\]

If $\alpha$ is a positive real number, from Schwarz's inequality, we get

\[
|\tau(K^{1/2}D_\tau U(t), K^{1/2}U(t))| \leq \alpha \|\tau K^{1/2}D_\tau U(t)\| \times \frac{1}{\alpha} \|K^{1/2}U(t)\| \leq \frac{1}{2} \alpha^2 \tau^2 \|K^{1/2}D_\tau U(t)\|^2 + \frac{1}{2 \alpha^2} \|K^{1/2}U(t)\|^2.
\]

Moving the forth term in the left-hand side of (10) to the right-hand side and using (11), we have

\[
\|D_\tau U(t)\|^2 + \beta \tau^2 \|K^{1/2}D_\tau U(t)\|^2 + \|K^{1/2}U(t)\|^2 \leq C_0 - \tau(K^{1/2}D_\tau U(t), K^{1/2}U(t)) \leq C_0 + \frac{1}{2} \alpha^2 \tau^2 \|K^{1/2}D_\tau U(t)\|^2 + \frac{1}{2 \alpha^2} \|K^{1/2}U(t)\|^2.
\]

Finally moving the second and the third terms in the last formula of (12) to the left-hand side, we obtain an energy inequality:

\[
\|D_\tau U(t)\|^2 + 2\tau(\beta - \frac{\alpha^2}{2})\|K^{1/2}D_\tau U(t)\|^2 + (1 - \frac{1}{2\alpha^2})\|K^{1/2}U(t)\|^2 \leq C_0.
\]

Using this inequality, we have the following results.

**Theorem 1** In the case $\beta \geq \frac{1}{4}$, we have the stability estimate, with positive constants $C_1$ and $C_2$,

\[
\|U(t)\| \leq C_1 + C_2 t,
\]

and in the case $0 \leq \beta < \frac{1}{4}$, if we choose $\tau$ such that

\[
\tau < \sqrt{\frac{1}{\beta\alpha^2}}.
\]

then we have, with positive constants $C_3$ and $C_4$,

\[
\|U(t)\| \leq C_3 + C_4 t,
\]
From now on, we show the proof of this theorem. First, we consider the case $\beta \geq \frac{1}{4}$. If we put $\alpha = \sqrt{2\beta}$ in (13), then we have, for $\beta > \frac{1}{4}$, that

$$\|D_\tau U(t)\|^2 + (1 - \frac{1}{4\beta})\|K^{1/2}U(t)\|^2 \leq C_0$$

and

$$\|D_\tau U(t)\|, \|K^{1/2}U(t)\| \leq C_\beta = (1 - \frac{1}{4\beta})^{-1}C_0 < \infty,$$

where $C_\beta$ is a constant independent of $t$. Hence, we get

$$\beta > \frac{1}{4} \implies \|D_\tau U(t)\|, \|K^{1/2}U(t)\| \leq C_\beta.$$  

And we also obtain that

$$\beta \geq \frac{1}{4} \implies \|D_\tau U(t)\| \leq \sqrt{C_0}.$$  

Then recalling the definition:

$$D_\tau U(t) = \frac{1}{\tau}(U(t+\tau) - U(t)),$$

we get

$$\|U(t+\tau) - U(t)\| \leq \sqrt{C_0}\tau,$$

and

$$\|U(t+\tau)\| \leq \|U(t)\| + \sqrt{C_0}\tau \leq \cdots \cdots \leq \|U(0)\| + \sqrt{C_0}(t+\tau).$$

Putting $C_1 = \|U(0)\|$ and $C_2 = \sqrt{C_0}$, where $C_1$ is constant independent of $\tau$, we can conclude that

$$\beta \geq \frac{1}{4} \implies \|U(t)\| \leq C_1 + C_2t.$$  

(14)

Next, we consider the case $0 \leq \beta < \frac{1}{4}$. Put $\alpha^2 = \frac{1}{2}$ in (13). Then we have

$$\|D_\tau U(t)\|^2 + \tau^2(\beta - \frac{1}{4})\|K^{1/2}D_\tau U(t)\|^2 \leq C_0$$

and

$$\|D_\tau U(t)\|^2 \leq C_0 + \tau^2(\beta - \frac{1}{4})\|K^{1/2}D_\tau U(t)\|^2.$$  

(15)

Let $y \in \mathcal{H}$ and $\|K^{1/2}\|$ be the operator norm of $K^{1/2}$, then we have $\|K^{1/2}y\| \leq \|K^{1/2}\||y||$. Applying this inequality to (15), we get

$$\|D_\tau U(t)\|^2 \leq C_0 + \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2\|D_\tau U(t)\|^2$$

and

$$\|D_\tau U(t)\|^2 \leq C_0 + \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2\|D_\tau U(t)\|^2 \leq C_0.$$  

Noticing the fact that, for $\tau > 0$,

$$0 < 1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2 \iff \tau < \sqrt{\frac{1}{(\frac{1}{4} - \beta)\|K^{1/2}\|^2}},$$

we obtain

$$\tau < \sqrt{\frac{1}{(\frac{1}{4} - \beta)\|K^{1/2}\|^2}} \implies \|D_\tau U(t)\| \leq \sqrt{\frac{C_0}{1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2}},$$

and we obtain:

$$\|U(t)\| \leq C_3 + C_4t,$$

where

$$C_3 = \|U(0)\|, \quad C_4 = \sqrt{\frac{C_0}{1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2}}.$$
References


