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# 最節約復元順序集合の極値問題について

– On extremal problems of MPR-posets –

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We first review several definitions and theorems used latter. The set $\{1, 2, \cdots, n\}$ of $n$ elements is denoted by $[n]$. Let $a_i$ $(i \in [2n])$ be any elements in $\Omega$, and be sorted in ascending order as follows:

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots \leq x_{2n}.$$

Then we call $x_n$ and $x_{n+1}$ the *median two points* of the numbers $a_i$ $(i \in [2n])$, and denote $\langle x_n, x_{n+1}\rangle$ by

$$\mathrm{med2}\langle a_1, a_2, \cdots, a_{2n}\rangle \text{ or } \mathrm{med2}\langle a_i : i \in [2n]\rangle.$$

We also call $x_{n-1}, x_n, x_{n+1}$ and $x_{n+2}$ the *median four points* of the numbers $a_i$ $(i \in [2n])$, and denote $\langle x_{n-1}, x_n, x_{n+1}, x_{n+2}\rangle$ by

$$\mathrm{med4}\langle a_1, a_2, \cdots, a_{2n}\rangle \text{ or } \mathrm{med4}\langle a_i : i \in [2n]\rangle.$$

Let $I_i = [a_i, b_i]$ $(i \in [m])$ be any family of closed intervals in $\Omega$. Then we denote the median two points $\mathrm{med2}\langle a_i : i \in [m], b_i : i \in [m]\rangle$ of all the endpoints $a_i$ and $b_i$ of $I_i$ $(i \in [m])$ by

$$\mathrm{med2}\langle I_1, I_2, \cdots, I_m\rangle \text{ or } \mathrm{med2}\langle I_i : i \in [m]\rangle.$$

We also denote the median four points $\mathrm{med4}\langle a_i : i \in [m], b_i : i \in [m]\rangle$ of all the endpoints $a_i$ and $b_i$ of $I_i$ $(i \in [m])$ by

$$\mathrm{med4}\langle I_1, I_2, \cdots, I_m\rangle \text{ or } \mathrm{med4}\langle I_i : i \in [m]\rangle.$$

Let $\mathrm{med2}\langle I_i : i \in [m]\rangle = \langle x_m, x_{m+1}\rangle$. Then we call the closed interval $[x_m, x_{m+1}]$ in $\Omega$ the *median interval* of the closed intervals $I_i$ $(i \in [m])$, which is the key concept in a series of our papers, and denote it by

$$\mathrm{med}\langle I_1, I_2, \cdots, I_m\rangle \text{ or } \mathrm{med}\langle I_i : i \in [m]\rangle.$$

The following is Lemma 1 in [3] (lemma B in [5]). It is very useful to investigate characteristics of each MPR.

**Lemma A.** *Let $a$ and $b_i$ $(i \in [2m])$ be any elements in $\Omega$. Then*

$$\mathrm{med2}\langle a, a, b_i : i \in [2m]\rangle = \mathrm{med2}\langle a, a, \mathrm{med4}\langle b_i : i \in [2m]\rangle\rangle. \quad \square$$

From Theorem 1 in [1], we see that $\mathrm{med}\langle[\lambda(p(u)), \lambda(p(u))], I(v) : u \to v\rangle$ is the MPR-set of node $u$ under the condition that an element $\lambda(p(u))$ in $S_{p(u)}$ has been assigned to $u$'s parent $p(u)$. This subset of the MPR-set $S_u$ is denoted by $S_u \,|\, x$. That is,

$$S_u \,|\, x = \mathrm{med}\langle[x, x], I(v) : u \to v\rangle,$$

where $x$ is an element in $S_{p(u)}$. The following is Theorem 1 in [3].

**Theorem B.** *Let $T$ be a rooted el-tree $(T_s, r)$. Then each MPR-set $S_u$ for each internal node $u$ of $T$ is recursively decided by*

$$S_u = [\min(S_u \,|\, \min(S_{p(u)})), \max(S_u \,|\, \max(S_{p(u)}))]. \quad \square$$

We now show a sufficient condition for a $\sigma(r)$-version MPR-poset to have both the greatest element and the least element.

**Proposition 1.** *Let $T$ be an el-tree, and $r$ be any element in $V_O$. If*

$$\sigma(r) \;\leq\; \min\{\ \min(S_u) | \forall u \in V_H, S_u \textit{is a non-singleton}\ \}$$

*or*

$$\sigma(r) \;\geq\; \max\{\ \max(S_u) | \forall u \in V_H, S_u \textit{is a non-singleton}\ \},$$

*then* $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$ *has both the greatest element and the least element.* $\square$

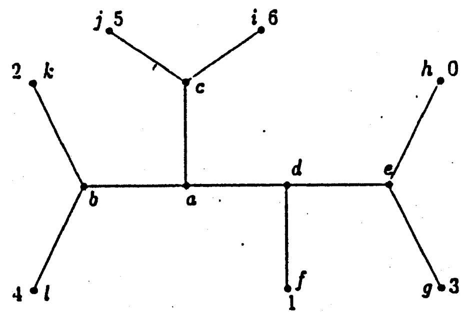



Figure 1: An el-tree $T_1$

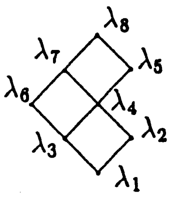
We here give some examples to illustrate proposition 1. Let $T_1$ be an el-tree shown in Fig 1. The set $\mathbf{Rmp}(T_1)$ of MPRs is also given in Table 1. Then $f, g, i, j$ and $l$ in $V_O(T_1)$ satisfy the conditions in proposition 1. Therefore we see that $(\mathbf{Rmp}(T_1), \leq_{\sigma(f)})$, $(\mathbf{Rmp}(T_1), \leq_{\sigma(g)})$, $(\mathbf{Rmp}(T_1), \leq_{\sigma(i)})$, $(\mathbf{Rmp}(T_1), \leq_{\sigma(j)})$ and $(\mathbf{Rmp}(T_1), \leq_{\sigma(l)})$ have both the greatest element and the least element (Fig 2 (a) $\sim$ (e)).

We get easily the following remark from proposition 1.

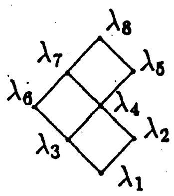
**Remark 1.** *Let $T$ be an el-tree rooted at $r$ such that $\sigma(r) = \min(\sigma(V_O))$. Then $\sigma(r)$-version MPR-poset,* $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$ *has both the greatest element and the least element.* $\square$

| $\lambda \backslash u$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ | $j$ | $k$ | $l$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | 2 | 2 | 5 | 1 | 1 | 1 | 3 | 0 | 6 | 5 | 2 | 4 |
| $\lambda_2$ | 2 | 2 | 5 | 2 | 2 | 1 | 3 | 0 | 6 | 5 | 2 | 4 |
| $\lambda_3$ | 3 | 3 | 5 | 1 | 1 | 1 | 3 | 0 | 6 | 5 | 2 | 4 |
| $\lambda_4$ | 3 | 3 | 5 | 2 | 2 | 1 | 3 | 0 | 6 | 5 | 2 | 4 |
| $\lambda_5$ | 3 | 3 | 5 | 3 | 3 | 1 | 3 | 0 | 6 | 5 | 2 | 4 |
| $\lambda_6$ | 4 | 4 | 5 | 1 | 1 | 1 | 3 | 0 | 6 | 5 | 2 | 4 |
| $\lambda_7$ | 4 | 4 | 5 | 2 | 2 | 1 | 3 | 0 | 6 | 5 | 2 | 4 |
| $\lambda_8$ | 4 | 4 | 5 | 3 | 3 | 1 | 3 | 0 | 6 | 5 | 2 | 4 |

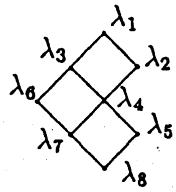
Table 1: $\mathbf{Rmp}(T_1)$



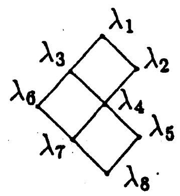

(a) $(\mathbf{Rmp}(T_1), \leq_{\sigma(f)})$



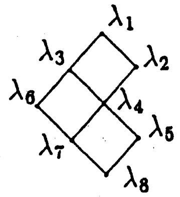

(b) $(\mathbf{Rmp}(T_1), \leq_{\sigma(g)})$




(c) $(\mathbf{Rmp}(T_1), \leq_{\sigma(i)})$




(d) $(\mathbf{Rmp}(T_1), \leq_{\sigma(j)})$




(e) $(\mathbf{Rmp}(T_1), \leq_{\sigma(l)})$

Figure 2: $\sigma(r)$-version MPR-posets

We now have the main theorem in this paper, which answers for whether there exists the least element in a $\sigma(r)$-version MPR-poset or not.

Let $T$ be a rooted el-tree $(T_s, r)$. We define a reconstruction $\lambda$ on $T$ as follows. We define $\lambda$ by $\lambda(u) = x$ in $S_u$ satisfying $x \leq_{\sigma(r)} y$ for any $y$ in $S_u$, that is, $x$ is the least element of a subposet $(S_u, \leq_{\sigma(r)})$ in the poset $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$. This reconstruction $\lambda$ is particularly written as $\lambda_{\min}^{<\sigma(r)>}$.

We can get the following implicitly from propostion 1.

**Remark 2.** *Let $T$ be a rooted el-tree $(T_s, r)$. If*

$$\sigma(r) \leq \min\{\ \min(S_u) | \forall u \in V_H, S_u \text{is a non-singleton}\ \},$$

*then $\lambda_{\min}^{<\sigma(r)>} = \lambda_{\min}$. The dual case also holds.* □

**Lemma 1.** *Let $T$ be a rooted el-tree $(T_s, r)$. For each $u$ in $V_H$, we have*

$$\lambda_{\min}^{<\sigma(r)>}(u) = \begin{cases} \min(S_u) & (\ \sigma(r) \leq \min(S_u)\ ) \\ \sigma(r) & (\ \min(S_u) < \sigma(r) < \max(S_u)\ ) \\ \max(S_u) & (\ \sigma(r) \geq \max(S_u)\ ) \end{cases}$$ □

**Theorem 1.** *Let $T$ be a rooted el-tree $(T_s, r)$. Then the reconstruction $\lambda_{\min}^{<\sigma(r)>}$ is the least element of* $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$. □

We here show some examples of the MPR $\lambda_{\min}^{<\sigma(r)>}$. Let $T_1 = (T_c, j)$ be the tree $T_1$ rooted at node $j$. Then for each $u$ in $V$ we can decide $\lambda_{\min}^{<\sigma(j)>}(u)$, which is shown in Fig 3. We also can see that $\lambda_{\min}^{<\sigma(j)>}$ is equal to $\lambda_8$ in $\mathbf{Rmp}(T_1)$, i.e, the least element of $(\mathbf{Rmp}(T_1), \leq_{\sigma(j)})$ (Fig 4).

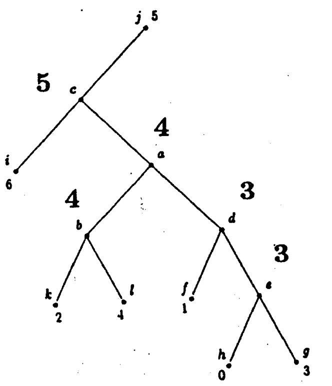



Figure 3: $\lambda_{\min}^{<\sigma(j)>}$ on $T_1 = (T_c, j)$

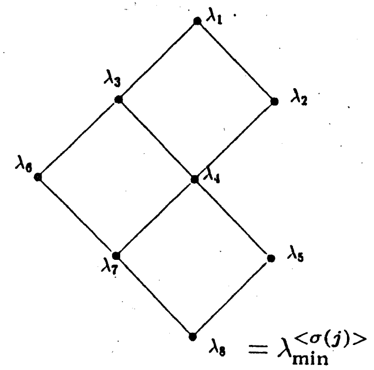



Figure 4: $(\mathbf{Rmp}(T_1), \leq_{\sigma(j)})$

It is known that $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$ dosen't always have the greatest element. So, we show one of the requirements for a reconstruction $\lambda$ in $\mathbf{Rmp}(T)$ to be a maximal element in the $\sigma(r)$-version MPR-poset.

**Lemma 2.** *Let $T$ be a rooted el-tree $(T_s, r)$. For each $u$ in $V$, we have* $\max S_{p(u)} \leq \min I(u)$, $\max I(u) \leq \min S_{p(u)}$ *or* $S_{p(u)} \subseteq I(u)$ *hold.* □

Let $T$ be a rooted el-tree $(T_s, r)$. We define two reconstructions $\alpha^{<\sigma(r)>}, \beta^{<\sigma(r)>}$ on $T$ as follows. We define $\alpha^{<\sigma(r)>}$ and $\beta^{<\sigma(r)>}$ by $\alpha^{<\sigma(r)>}(u)$ = the smallest element $x$ under the usual ordering $\leq$ of maximal elements in the subposet $(S_u, \leq_{\sigma(r)})$ and $\beta^{<\sigma(r)>}(u)$ = the greatest element $x$ under the usual ordering $\leq$ of maximal elements in the subposet $(S_u, \leq_{\sigma(r)})$.

**Lemma 3.** *Let $T$ be a rooted el-tree $(T_s, r)$. For each $u$ in $V_H$, we have*

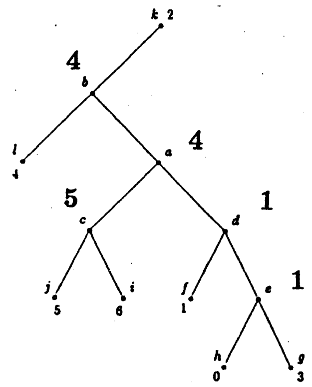
$$\alpha^{<\sigma(r)>}(u) = \begin{cases} \min(S_u) & (\ \sigma(r) > \min(S_u)\ ) \\ \max(S_u) & (\ \sigma(r) \leq \min(S_u)\ ) \end{cases}$$

$$\beta^{<\sigma(r)>}(u) = \begin{cases} \min(S_u) & (\ \sigma(r) \geq \max(S_u)\ ) \\ \max(S_u) & (\ \sigma(r) < \max(S_u)\ ) \end{cases} \quad \square$$

Then, we get the following proposition.

**Proposition 2.** *Let $T$ be a rooted el-tree $(T_s, r)$. Then, both $\alpha^{<\sigma(r)>}$ and $\beta^{<\sigma(r)>}$ are maximal elements of* $(\mathbf{Rmp}(T), \leq_{\sigma(r)})$. □

We here show some examples of the MPR $\alpha^{<\sigma(r)>}$ and $\beta^{<\sigma(r)>}$. Let $T_1 = (T_b, k)$ be the tree $T_1$ rooted at node $k$. Then for each $u$ in $V$ we can decide $\alpha^{<\sigma(k)>}(u)$ and $\beta^{<\sigma(k)>}(u)$, which are shown in Fig 5(a) and (b) respectively. We also see that $\alpha^{<\sigma(k)>}$ and $\beta^{<\sigma(k)>}$ are equal to $\lambda_6$ and $\lambda_8$ in $\mathbf{Rmp}(T_1)$, respectively, which are maximal elements of $(\mathbf{Rmp}(T_1), \leq_{\sigma(k)})$ (Fig 6).



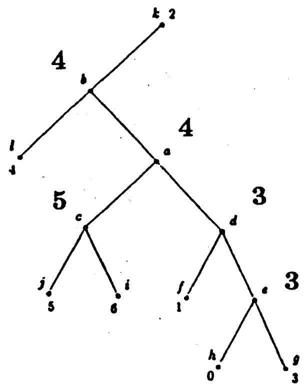




Figure 5(a): $\alpha^{<\sigma(k)>}$ on $T_1 = (T_b, k)$ (b): $\beta^{<\sigma(k)>}$ on $T_1 = (T_b, k)$

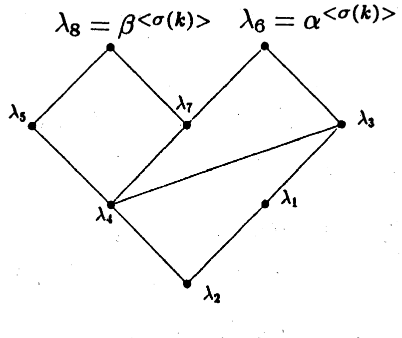



Figure 6: $(\mathbf{Rmp}(T_1), \leq_{\sigma(k)})$

Finally, we show interesting examples on the number of maximal elements of a $\sigma(r)$-version MPR-poset. Let $T_2$ be an el-tree shown in Fig 7. When $T_2$ is rooted at $p$ in $V_O(T_2)$, we see that $(\mathbf{Rmp}(T_2), \leq_{\sigma(p)})$ has three maximal elements $\lambda_1, \lambda_3$ and $\lambda_6$ shown in Fig 8. In other words, it shows that the number of maximal elements of a $\sigma(r)$-version MPR-poset is not necessarily at most two.

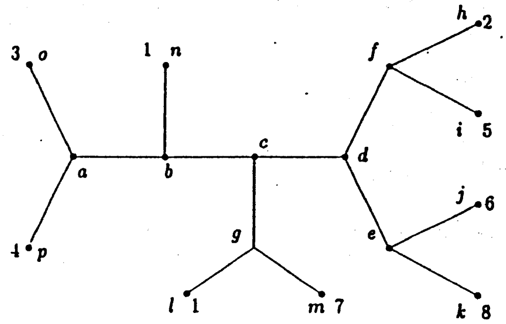



Figure 7: An el-tree $T_2$

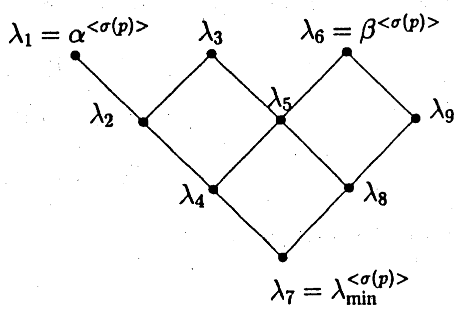



Figure 8: $(\mathbf{Rmp}(T_2), \leq_{\sigma(p)})$

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