<table>
<thead>
<tr>
<th>Title</th>
<th>Structures of Lyapunov Regular Sets with Non-Zero Exponents (Problems on complex dynamical systems)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kurata, Masahiro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1998), 1042: 143-146</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1998-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62089">http://hdl.handle.net/2433/62089</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
Structures of Lyapunov Regular Sets with Non-Zero Exponents

Masahiro KURATA
Nagoya Institute of Technology

1 Introduction

Let $M$ be a closed manifold with a Riemannian metric and $f; M \to M$ be a $C^2$-diffeomorphism. For $\lambda, \mu > 0$ and $1 \leq k \leq \dim M - 1$, we denote by $\Lambda = \Lambda(-\mu, \lambda, k)$ the set consisting of Lyapunov regular points $x$ such that

1. Lyapunov exponents of $x$ are less than $-\mu$ or greater than $\lambda$,
2. the dimension of the fiber of stable bundle $E^s(x)$ equals $k$.

Without loss of generality, we may assume that $\Lambda$ has a dense orbit. In this note we construct a "symbolic dynamics" that represents $\Lambda$, and study the structure of $\Lambda$ in the case when $\Lambda$ is a fractal set.

2 symbolic dynamics of $\Lambda$

The set $\Lambda$ is represented by a "symbolic dynamics" as follows.

I Let $\mathcal{W} = \{W_n\}_{n \geq 0}$ be a family of sets $W_n$ of words of length $n + 1$ such that

1. There are symbols $\{B_1, \cdots, B_p, C_1, \cdots, C_q\}$, and for any $n \geq 0$
   \[ W_n \subset \{B_1, \cdots, B_p, C_1, \cdots, C_q\}^{n+1}, \]
2. $(\alpha_0, \cdots, \alpha_n) \in W_n$ implies $\alpha_0, \alpha_n \in \{B_1, \cdots, B_p\}$,
3. If $(\alpha_0, \cdots, \alpha_n) \in W_n, (\beta_0, \cdots, \beta_m) \in W_m$ and $\alpha_n = \beta_0$, then $(\alpha_0, \cdots, \alpha_n, \beta_1, \cdots, \beta_m) \in W_{n+m}$,
4. for any $B_i, B_j$ (1 $\leq i, j \leq p$), there are $n \geq 1$ and $(\alpha_0, \cdots, \alpha_n) \in W_n$ with $\alpha_0 = B_i, \alpha_n = B_j$.

II For a family of sets of words $\mathcal{W} = \{W_n\}_{n \geq 0}$ as above, we define a subset of shift (not necessarily subshift) $\Sigma = \Sigma(\mathcal{W})$ as follows:

1. $\Sigma = \Sigma(\mathcal{W}) \subset \{B_1, \cdots, B_p, C_1, \cdots, C_q\}^\mathbb{Z}$
2. $\Sigma$ is generated by $\mathcal{W} = \{W_n\}_{n \geq 0}$, that is,
   \[ (a) \; \alpha = (\alpha_n)_{n \in \mathbb{Z}} \in \Sigma \text{ if and only if for any } N > 0 \text{ there are } m, n \geq N \text{ such that } (\alpha_{-m}, \alpha_{-m+1}, \cdots, \alpha_0, \cdots, \alpha_n) \in W_{m+n} \]
We denote by $\Sigma$ a collection of $\Sigma = \Sigma(\mathcal{W})$, where $\mathcal{W} = \{W_n\}_{n\geq 0}$, is defined in I and II. And an equivalence relation $\sim$ is defined in the disjoint union $\cup \Sigma(\mathcal{W})$ as follows:

if $a \sim b$ for $a \in \Sigma, b \in \Sigma'$ and $\sigma^n(a) \in \Sigma, \sigma^n(b) \in \Sigma'$
then $\sigma^n(a) \sim \sigma^n(b)$, where $\sigma$ is the shift map.

For the quotient space $\bar{\Sigma} = \cup \Sigma(\mathcal{W})/\sim$, a shift map $\tilde{\sigma}: \bar{\Sigma} \to \bar{\Sigma}$ is defined as follows:

for $\underline{\alpha} = (\alpha_n)_n \in \Sigma$ with $\alpha_0, \alpha_k \in \{B_1, \cdots, B_p\}$, we have $\tilde{\sigma}^k[\underline{\alpha}] = [\underline{\beta}]$, where $\underline{\beta} = (\beta_n)_n \in \Sigma$ is given by $\beta_n = \alpha_{n+k}$.

With the notations as above, we have the following.

**Theorem 2.1.** For a Lyapunov regular set $\Lambda = \Lambda(-\mu, \lambda, k)$, there are

1. a collection $\Sigma = \{\Sigma(\mathcal{W})\}$ of countable subsets of shifts $\Sigma(\mathcal{W})$,
2. an equivalence relation $\sim$ on $\cup \Sigma(\mathcal{W})$ and the shift map $\tilde{\sigma}: \bar{\Sigma} = \cup \Sigma(\mathcal{W})/\sim \to \bar{\Sigma}$,
3. a collection of maps $\Psi = \Psi_\Sigma: \Sigma \to \Lambda$ (for $\Sigma \in \Sigma$) which is compatible with the equivalence relation $\sim$,

such that the map

$$\tilde{\Psi}: \bar{\Sigma} \to \Lambda$$

induced from $\{\Psi = \Psi_\Sigma \mid \Sigma \in \Sigma\}$ is surjective and the diagram

$$\begin{array}{ccc}
\bar{\Sigma} & \xrightarrow{\tilde{\sigma}} & \bar{\Sigma} \\
\downarrow \psi & & \downarrow \tilde{\psi} \\
\Lambda & \xrightarrow{f|\Lambda} & \Lambda
\end{array}$$

is commutative.

**Remark 2.1.** Let $\epsilon$ be an arbitrary positive number. In Theorem 2.1, we may choose the symbols $\{B_1, \cdots, B_p, C_1, \cdots, C_q\}$ of any $\mathcal{W}$ and maps $\Psi = \Psi_\Sigma$ such that

1. $p = 1$,
2. $\text{diam } \Psi(\Sigma) < \epsilon$ for $\Sigma = \Sigma(\mathcal{W}) \in \Sigma$,
3. for $x = \Psi((\alpha_n)_n) \in \Psi(\mathcal{W})$

$||Tf^n| E^s(x)|| < \exp(-\mu n)$ if $\alpha_0 = \alpha_n = B_1$,
$||Tf^{-n}| E^u(x)|| < \exp(-\lambda n)$ if $\alpha_{-n} = \alpha_0 = B_1$. 

\[144\]
3 Locally self-similarity with countable contractions

Let $\Lambda = \Lambda(-\mu, \lambda, k)$ be a Lyapunov regular set. In the sequel we assume that $\Sigma = \{\Sigma\},$ where $\Sigma = \Sigma(W),$ and maps $\Psi = \Psi_\Sigma : \Sigma \to \Lambda$ $(\Sigma \in \Sigma)$ are given as in section 2 and satisfy Remark 2.1.

Then $\Lambda$ is a countable union of closed sets:

$$\Lambda = \bigcup_{\Sigma \in \Sigma} \Psi(\Sigma).$$

In this section we consider the structure of $\Psi(\Sigma)$.

Let $G(W)$ be the set of generators of $W,$ that is,

$$G_k(W) = \{(a_0, \cdots, a_k) \in W_k \mid a_0 = a_k = B_1, a_i \in \{C_1, \cdots, C_q\} \ 1 \leq i \leq k - 1\},$$

$$G(W) = \bigcup_k G_k(W).$$

For $a = (a_0, \cdots, a_k) \in G(W),$ the right contraction

$$R(a) : \Psi(\Sigma) \to \Psi(\Sigma)$$

is defined by

$$R(a)(\Psi((\alpha_n)_n) = \Psi((\beta_n)_n) \ \text{for } (\alpha_n)_n \in \Sigma,$$

where

$$\beta_n = \begin{cases} 
\alpha_{n-k}, & k \leq n, \\
\alpha_n, & 0 \leq n \leq k, \\
\alpha_n, & n \leq 0.
\end{cases}$$

Similarly the left contraction

$$L(a) : \Psi(\Sigma) \to \Psi(\Sigma)$$

is defined by

$$L(a)(\Psi((\alpha_n)_n) = \Psi((\beta_n)_n) \ \text{for } (\alpha_n)_n \in \Sigma,$$

where

$$\beta_n = \begin{cases} 
\alpha_n, & 0 \leq n, \\
\alpha_{n+k}, & -k \leq n \leq 0, \\
\alpha_n, & n \leq -k.
\end{cases}$$

Then we have the following.

**Proposition 3.1.** The set $\Psi(\Sigma)$ is a countable union of images of $\Psi(\Sigma)$ by maps $R(a)L(b);$ 

$$\Psi(\Sigma) = \bigcup_{a, b \in G(W)} R(a)L(b)(\Psi(\Sigma)).$$

If the map $\Psi = \Psi_\Sigma : \Sigma \to \Lambda$ is injective, then for any $a, b \in G(W)$ the map

$$L(b)R(a) = R(a)L(b) : \Psi(\Sigma) \to \Sigma(\Sigma)$$

is a contraction. And $\Psi(\Sigma)$ is self-similar by countable contractions.
4 Hausdorff dimension of local stable manifolds

Form the propositions in the previous section, we have

\[ \Psi(\Sigma) = \bigcup_{a_1,b_1 \in G(\mathcal{W})} R(a_1)L(b_1) (\Psi(\Sigma)) \]

\[ = \bigcup_{a_2,b_2 \in G(\mathcal{W})} \bigcup_{a_1,b_1 \in G(\mathcal{W})} R(a_2)L(b_2)R(a_1)L(b_1) (\Psi(\Sigma)) \]

\[ = \bigcup_{a_1,a_2 \in G(\mathcal{W})} \bigcup_{b_1,b_2 \in G(\mathcal{W})} R(a_2)R(a_1)L(b_2)L(b_1) (\Psi(\Sigma)) \]

\[ = \bigcup_{a,b \in G(\mathcal{W})^N} R(a)L(b) (\Psi(\Sigma)), \]

where

\[ a = (a_1,a_2,\ldots), \quad b = (b_1,b_2,\ldots) \in G(\mathcal{W})^N, \]

and

\[ L(b) (\Psi(\Sigma)) = \bigcap_{N \geq 1} L(b_N) \cdots L(b_1) (\Psi(\Sigma)). \]

Besides the set \( L(b) (\Psi(\Sigma)) \) coincides with an intersection of a local unstable manifold and \( \Psi(\Sigma) \).

Let \( \text{Lip} (R(a) | L(b) (\Psi(\Sigma))) \) be the Lipshitz constant of the map

\[ R(a): L(b) (\Psi(\Sigma)) \longrightarrow L(b) (\Psi(\Sigma)). \]

By choosing \( \varepsilon > 0 \) in Remark2.1 sufficiently small, we have

\[ \text{Lip} (R(a) | L(b) (\Psi(\Sigma))) < \exp(-\lambda n). \]

Because the number of the elements of \( G_k(\mathcal{W}) \) is less than or equals \( q^{k-1} \), this implies the following.

**Proposition 4.1.** For \( b \in G(\mathcal{W})^N \), there is \( c(b) > 0 \) such that

\[ \sum_{a \in G(\mathcal{W})} \text{Lip} (R(a) | L(b) (\Psi(\Sigma)))^{c(b)} = 1. \]

The number \( c(b) \) dominates the Hausdorff dimension of the intersection of the local unstable manifold and \( \Psi(\Sigma) \):

**Proposition 4.2.** The Hausdorff dimension of \( L(b) (\Psi(\Sigma)) \) is less than or equals \( c(b) \).

**References**
