Structures of Lyapunov Regular Sets with Non-Zero Exponents

Masahiro KURATA
Nagoya Institute of Technology

1 Introduction

Let $M$ be a closed manifold with a Riemannian metric and $f: M \to M$ be a $C^2$-diffeomorphism. For $\lambda, \mu > 0$ and $1 \leq k \leq \dim M - 1$, we denote by $\Lambda = \Lambda(-\mu, \lambda, k)$ the set consisting of Lyapunov regular points $x$ such that

1. Lyapunov exponents of $x$ are less than $-\mu$ or greater than $\lambda$,
2. the dimension of the fiber of stable bundle $E^s(x)$ equals $k$.

Without loss of generality, we may assume that $\Lambda$ has a dense orbit. In this note we construct a "symbolic dynamics" that represents $\Lambda$, and study the structure of $\Lambda$ in the case when $\Lambda$ is a fractal set.

2 symbolic dynamics of $\Lambda$

The set $\Lambda$ is represented by a "symbolic dynamics" as follows.

I Let $\mathcal{W} = \{W_n\}_{n \geq 0}$ be a family of sets $W_n$ of words of length $n + 1$ such that

1. There are symbols $\{B_1, \cdots, B_p, C_1, \cdots, C_q\}$, and for any $n \geq 0$

   \[ W_n \subset \{B_1, \cdots, B_p, C_1, \cdots, C_q\}^{n+1}, \]

2. $(\alpha_0, \cdots, \alpha_n) \in W_n$ implies $\alpha_0, \alpha_n \in \{B_1, \cdots, B_p\}$,
3. If $(\alpha_0, \cdots, \alpha_n) \in W_n, (\beta_0, \cdots, \beta_m) \in W_m$ and $\alpha_n = \beta_0$, then

   \[ (\alpha_0, \cdots, \alpha_n, \beta_1, \cdots, \beta_m) \in W_{n+m}, \]

4. for any $B_i, B_j$ $(1 \leq i, j \leq p)$, there are $n \geq 1$ and $(\alpha_0, \cdots, \alpha_n) \in W_n$ with

   \[ \alpha_0 = B_i, \alpha_n = B_j. \]

II For a family of sets of words $\mathcal{W} = \{W_n\}_{n \geq 0}$ as above, we define a subset of shift (not necessarily subshift) $\Sigma = \Sigma(\mathcal{W})$ as follows:

1. $\Sigma = \Sigma(\mathcal{W}) \subset \{B_1, \cdots, B_p, C_1, \cdots, C_q\}^\mathbb{Z}$
2. $\Sigma$ is generated by $\mathcal{W} = \{W_n\}_{n \geq 0}$, that is,

   \[ (a) \; \alpha = (\alpha_n)_{n \in \mathbb{Z}} \in \Sigma \text{ if and only if for any } N > 0 \text{ there are } m, n \geq N \text{ such that } (\alpha_{-m}, \alpha_{-m+1}, \cdots, \alpha_0, \cdots, \alpha_n) \in W_{m+n} \]
(b) $\alpha_0 \in \{B_0, \cdots, B_p\}$

III We denote by $\Sigma$ a collection of $\Sigma = \Sigma(W)$, where $W = \{W_n\}_{n \geq 0}$, is defined in I and II. And an equivalence relation $\sim$ is defined in the disjoint union $\cup \Sigma(W)$ as follows:

if $a \sim b$ for $a \in \Sigma, b \in \Sigma'$ and $\sigma^n(a) \in \Sigma, \sigma^n(b) \in \Sigma'$

then $\sigma^n(a) \sim \sigma^n(b)$, where $\sigma$ is the shift map.

IV For the quotient space $\overline{\Sigma} = \cup \Sigma(W)/\sim$, a shift map $\tilde{\sigma} : \overline{\Sigma} \to \overline{\Sigma}$ is defined as follows:

for $\alpha = (\alpha_n)_n \in \Sigma$ with $\alpha_0, \alpha_k \in \{B_1, \cdots, B_p\}$, we have $\tilde{\alpha}^k[\alpha] = [\beta]$, where $\beta = (\beta_n)_n \in \Sigma$ is given by $\beta_n = \alpha_{n+k}$.

With the notations as above, we have the following.

**Theorem 2.1.** For a Lyapunov regular set $\Lambda = \Lambda(-\mu, \lambda, k)$, there are

1. a collection $\Sigma = \{\Sigma(W)\}$ of countable subsets of shifts $\Sigma(W)$,

2. an equivalence relation $\sim$ on $\cup \Sigma(W)$ and the shift map $\tilde{\sigma} : \overline{\Sigma} = \cup \Sigma(W)/\sim \to \overline{\Sigma}$,

3. a collection of maps $\Psi = \Psi_\Sigma : \Sigma \to \Lambda$ (for $\Sigma \in \Sigma$) which is compatible with the equivalence relation $\sim$,

such that the map

$$\tilde{\Psi} : \overline{\Sigma} \to \Lambda$$

induced from $\{\Psi = \Psi_\Sigma \mid \Sigma \in \Sigma\}$ is surjective and the diagram

$$\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\tilde{\sigma}} & \tilde{\Sigma} \\
\downarrow & & \downarrow \tilde{\Psi} \\
\Lambda & \overset{f|\Lambda}{\longrightarrow} & \Lambda
\end{array}$$

is commutative.

**Remark 2.1.** Let $\varepsilon$ be an arbitrary positive number. In Theorem 2.1. we may choose the symbols $\{B_1, \cdots, B_p, C_1, \cdots, C_q\}$ of any $W$ and maps $\Psi = \Psi_\Sigma$ such that

1. $p = 1$,

2. $\text{diam} \Psi(\Sigma) < \varepsilon$ for $\Sigma = \Sigma(W) \in \Sigma$,

3. for $x = \Psi((\alpha_n)_n) \in \Psi(W)$

$$\|Tf^n | E^s(x)| \| < \exp(-\mu n) \quad \text{if } \alpha_0 = \alpha_n = B_1,$$

$$\|Tf^{-n} | E^u(x)| \| < \exp(-\lambda n) \quad \text{if } \alpha_{-n} = \alpha_0 = B_1.$$
3 Locally self-similarity with countable contractions

Let $\Lambda = \Lambda(-\mu, \lambda, k)$ be a Lyapunov regular set. In the sequel we assume that $\Sigma = \{\Sigma\}$, where $\Sigma = \Sigma(W)$, and maps $\Psi = \Psi_\Sigma : \Sigma \to \Lambda$ ($\Sigma \in \Sigma$) are given as in section 2 and satisfy Remark 2.1.

Then $\Lambda$ is a countable union of closed sets:

$$\Lambda = \bigcup_{\Sigma \in \Sigma} \Psi(\Sigma).$$

In this section we consider the structure of $\Psi(\Sigma)$.

Let $G_k(W)$ be the set of generators of $W$, that is,

$$G_k(W) = \{(a_0, \cdots, a_k) \in W_k | a_0 = a_k = B_1, a_i \in \{C_1, \cdots, C_q\} 1 \leq i \leq k - 1\},$$

$$G(W) = \bigcup_k G_k(W).$$

For $a = (a_0, \cdots, a_k) \in G(W)$, the right contraction

$$R(a) : \Psi(\Sigma) \to \Psi(\Sigma)$$

is defined by

$$R(a)(\Psi((\alpha_n)_n) = \Psi((\beta_n)_n) \quad \text{for} \quad (\alpha_n)_n \in \Sigma,$$

where

$$\beta_n = \begin{cases} 
\alpha_{n-k}, & k \leq n, \\
\alpha_n, & 0 \leq n \leq k, \\
\alpha_n, & n \leq 0.
\end{cases}$$

Similarly the left contraction

$$L(a) : \Psi(\Sigma) \to \Psi(\Sigma)$$

is defined by

$$L(a)(\Psi((\alpha_n)_n) = \Psi((\beta_n)_n) \quad \text{for} \quad (\alpha_n)_n \in \Sigma,$$

where

$$\beta_n = \begin{cases} 
\alpha_n, & 0 \leq n, \\
\alpha_{n+k}, & -k \leq n \leq 0, \\
\alpha_{n+k}, & n \leq -k.
\end{cases}$$

Then we have the following.

**Proposition 3.1.** The set $\Psi(\Sigma)$ is a countable union of images of $\Psi(\Sigma)$ by maps $R(a)L(b)$;

$$\Psi(\Sigma) = \bigcup_{a,b \in G(W)} R(a)L(b)(\Psi(\Sigma)).$$

If the map $\Psi = \Psi_\Sigma : \Sigma \to \Lambda$ is injective, then for any $a, b \in G(W)$ the map

$$L(b)R(a) = R(a)L(b) : \Psi(\Sigma) \to \Sigma(\Sigma)$$

is a contraction. And $\Psi(\Sigma)$ is self-similar by countable contractions.
4 Hausdorff dimension of local stable manifolds

Form the propositions in the previous section, we have
\[
\Psi(\Sigma) = \bigcup_{a_1,b_1 \in G(W)} R(a_1)L(b_1) (\Psi(\Sigma))
\]
\[
= \bigcup_{a_2,b_2 \in G(W)} \bigcup_{a_1,b_1 \in G(W)} R(a_2)L(b_2)R(a_1)L(b_1) (\Psi(\Sigma))
\]
\[
= \bigcup_{a_1,a_2 \in G(W)} \bigcup_{b_1,b_2 \in G(W)} R(a_2)R(a_1)L(b_2)L(b_1) (\Psi(\Sigma))
\]
\[
= \bigcup_{a_1,b_1 \in G(W)^N} R(a)L(b) (\Psi(\Sigma)),
\]
where
\[
a = (a_1, a_2, \cdots), \ b = (b_1, b_2, \cdots) \in G(W)^N,
\]
and
\[
L(b) (\Psi(\Sigma)) = \bigcap_{N \geq 1} L(b_N) \cdots L(b_1) (\Psi(\Sigma)).
\]

Besides the set \(L(b) (\Psi(\Sigma))\) coincides with an intersection of a local unstable manifold and \(\Psi(\Sigma)\).

Let \(\text{Lip} (R(a) | L(b) (\Psi(\Sigma)))\) be the Lipschitz constant of the map
\[
R(a): L(b) (\Psi(\Sigma)) \longrightarrow L(b) (\Psi(\Sigma)).
\]

By choosing \(\varepsilon > 0\) in Remark 2.1 sufficiently small, we have
\[
\text{Lip} (R(a) | L(b) (\Psi(\Sigma))) < \exp(-\lambda n).
\]

Because the number of the elements of \(G_k(W)\) is less than or equals \(q^{k-1}\), this implies the following.

**Proposition 4.1.** For \(b \in G(W)^N\), there is \(c(b) > 0\) such that
\[
\sum_{a \in G(W)} \text{Lip} (R(a) | L(b) (\Psi(\Sigma)))^{c(b)} = 1.
\]

The number \(c(b)\) dominates the Hausdorff dimension of the intersection of the local unstable manifold and \(\Psi(\Sigma)\):

**Proposition 4.2.** The Hausdorff dimension of \(L(b) (\Psi(\Sigma))\) is less than or equals \(c(b)\).

References
