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Kneading sequences for symmetric PL bimodal maps

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1 PL bimodal map

PL bimodal maps are the piecewise linear maps on the interval with three monotone segments. We consider the family of symmetric bimodal maps on the interval $[-1, 1]$, which map $-1$ and $1$ to $-1$ and $1$ respectively. A bimodal map has two turning points $t_1$ and $t_2$ on $(-1, 1)$. The map is strictly increasing on $[-1, t_1]$ and on $[t_2, 1]$, and strictly decreasing on $[t_1, t_2]$. Such maps are called $\{+ - +\}$ type bimodal maps. Let $\lambda$ be the slope on the first and third segments, and $-\mu$ be that on the second segment $(\lambda, \mu > 0)$. The map is given by the formula

$$F_{\lambda, \mu}(x) = \begin{cases} 
\lambda x + \lambda - 1 & (x \leq t_1) \\
-\mu x & (t_1 \leq x \leq t_2) \\
\lambda x - (\lambda - 1) & (x \geq t_2)
\end{cases}$$

where $t_1 = -t_2 = \frac{1 - \lambda}{\lambda + \mu}$. The parameters $\lambda$ and $\mu$ are chosen from the set

$$D = \{ (\lambda, \mu) : \lambda > 1, \mu > 1, \frac{2}{\lambda} + \frac{1}{\mu} \geq 1 \}.$$  

Since the maps are bimodal and symmetric at the origin, $t_1 < 0$. Therefore $\lambda > 1$. If $\mu < 1$ then there exists an attracting fixed point. If $\mu = 1$ then $x \in [t_1, t_2] \setminus \{0\}$ is 2-periodic. Therefore the condition $\mu > 1$ is necessary for topological entropy to be positive. The last condition $\frac{2}{\lambda} + \frac{1}{\mu} \geq 1$ is required for the existence of an $F$-invariant interval.

**Lemma 1.1** Assume that $\lambda, \mu > 1$. There exists an interval whose interior is invariant under $F$, if and only if $\frac{2}{\lambda} + \frac{1}{\mu} \geq 1$. 
If $\frac{2}{\lambda} + \frac{1}{\mu} < 1$, then the invariant interval is a Cantor set. In this case, topological entropy is log 3.

Now we consider another set of the parameters $\lambda$ and $\mu$

$$D_0 = \{ (\lambda, \mu) \in D : \frac{1}{\lambda} + \frac{1}{\mu} < 1 \}.$$ 

For $(\lambda, \mu) \in D \setminus D_0$ there exists an $F$-invariant interval, such that $F$ maps the negative points and the positive points to positive points and negative points respectively.

**Lemma 1.2** For $(\lambda, \mu) \in D$, we have $F^{2}_{\lambda, \mu}(t_1) > 0$, if and only if $\frac{1}{\lambda} + \frac{1}{\mu} < 1$.

2 Symbolic dynamics for the bimodal maps

2.1 Kneading sequences

We consider kneading sequences adapted to a special case of bimodal maps that are symmetric at the origin. Let $f$ be a $\{+ - +\}$ type bimodal map with two turning points $t_1 < 0$ and $t_2 > 0$. Let $f^n$ is the $n$th iterate of $f$. The orbit of $x_0 \in [f(t_2), f(t_1)]$ is

$$O(x_0) = (x_0, f(x_0), f^2(x_0), \cdots)$$

$$= (x_0, x_1, x_2, \cdots).$$

Its itinerary is

$$I(x_0) = (A_0 A_1 A_2 \cdots),$$

where

$$A_i = \begin{cases} 
L & (f(t_2) \leq x_i < t_1) \\
C_L & (x_i = t_1) \\
M & (t_1 < x_i t_2) \\
C_R & (x_i = t_2) \\
R & t_2 < (x_i \leq f(t_1)).
\end{cases}$$
The itineraries $K_L(f), K_R(f)$ for the critical values $f(t_1), f(t_2)$ are called the **kneading sequences** of the map:

$$K_L(f) = I(f(t_1)), K_R(f) = I(f(t_2)).$$

For the itinerary $I(x_0) = (A_0A_1A_2\cdots)$, we denote the number of $i$'s such that $A_i = M$ (for $i < n$) by $\theta_n$, and let $\epsilon_n = (-1)^{\theta_n}$. We denote a symbol by a capital letter without underline and a sequence by a capital letter with underline.

We define an order on the symbols and the sequences as follows.

(i) $L < C_L < M < C_R < R$.

(ii) let $S = (A_0A_1A_2\cdots)$ and $T = (B_0B_1B_2\cdots)$ be two different sequences. Let $k$ be the smallest non-negative integer with $A_k \neq B_k$. We say $S < T$ if $A_k < B_k$ and $\epsilon_k = 1$ or if $A_k > B_k$ and $\epsilon_k = -1$ for the above $k$.

For $x, y \in [f(t_2), f(t_1)]$, it follows that

(i) if $I(x) < I(y)$ then $x < y$.

(ii) if $x < y$ then $I(x) \leq I(y)$.

$|A|$ denote the cardinality of $A$. When $|A| = 0$, we write $A = \phi$. If $|A| > 0$, then $A > \phi$.

For $A = (A_0A_1A_2A_3\cdots)$, define the **shift operator** $\sigma$ by

$$\sigma(A) = \begin{cases} 
\phi & \text{if } A = C_L, C_R \text{ or } \phi \\
(A_1A_2A_3\cdots) & \text{otherwise}
\end{cases}$$

We call a sequence $A$ **maximal** if $\sigma^k(A) \leq A$ for $k = 1, 2, \cdots$. The kneading sequence $K_L(f)$ is maximal.
2.2 The products of the sequences

We say a sequence $A$ is **even** or **odd** according to the parity of the number of $M$'s it contains. We shall write $AB$ for the concatenation of $A$ and $B$, and $A^n = A \cdots A$ ($n$ times) and $A^\infty = AA \cdots$. Let $A \neq \phi$ and $B \neq C_L$ or $C_R$. We define $\bar{A}$ as follows:

$$\bar{A}_i = \begin{cases} 
R & (A_i = L) \\
C_R & (A_i = C_L) \\
M & (A_i = M) \\
C_L & (A_i = C_R) \\
L & (A_i = R)
\end{cases}$$

We define ***-product and ****-product as follows:

(i) if $A$ is even

$$A * B_0 B_1 \cdots = AB_0 AB_1 \cdots$$

(ii) if $A$ is odd

$$A * B_0 B_1 \cdots = \bar{A}_0 \bar{B}_1 B_1 \cdots$$

where

$$\bar{B}_i = \begin{cases} 
M & (B_i = L) \\
L & (B_i = M) \\
R & (B_i = R)
\end{cases}$$

(iii) if $A$ is even

$$A ** B_0 B_1 \cdots = AB_0 \bar{A} B_1 \cdots$$

(iv) if $A$ is odd
\[ A**B_0B_1\cdots = A\check{B}_0\underline{A}\hat{B}_1\cdots \]

where

\[ B_i = \begin{cases} 
L & (B_i = L) \\
R & (B_i = M) \\
M & (B_i = R) 
\end{cases} \]

\[ \check{B}_i = \begin{cases} 
M & (B_i = L) \\
L & (B_i = M) \\
R & (B_i = R) 
\end{cases} \]

\[ \hat{B}_i = \begin{cases} 
M & (B_i = L) \\
L & (B_i = M) \\
R & (B_i = R) \end{cases}. \]

Let \( A \) be maximal. We say \( A \) is \textbf{primary} if it cannot be written as \( B*D \) or \( B**D \) with \( B \neq \phi \) and \( D \neq \phi \).

There are some kneading sequences \( K_L(f) \) that contain the symbol \( C_L \). These sequences can be written \((AC_L)^\infty \) or \((AC_R\overline{A}C_L)^\infty \) by using a sequence \( A \neq \phi \). These sequences satisfy the following inequalities:

\[ A*L^\infty < AC_L < A*ML^\infty \]

\[ A**M^\infty < AC_R\overline{A}C_L < A**RM^\infty \]

**Proposition 2.1** Let \( AC_L \) be maximal. If \( A*L^\infty \leq K_L(f) \leq A*ML^\infty \) then there is a \( B \) such that \( K_L(f) = A*B \). This \( B \) is maximal.

**Proof.** Put \( n = |AC_L| \). Assume that \( A \) is even. We first show that

\[ A*L^\infty \leq \sigma^n(K_L(f)) \leq A*ML^\infty. \]  \hspace{1cm} (1)

Our assumptions implies

\[ (AL)^\infty \leq K_L(f) \leq AM(AL)^\infty. \]
Then we have $K_L(f) = AB_0 \cdots$, where $B_0 = L, C_L$, or $M$. If $B_0 = L$ then $\sigma^n((AL)^\infty) \leq \sigma^n(K_L(f))$. If $B_0 = M$ then $\sigma^n(AM(AL)^\infty) \leq \sigma^n(K_L(f))$. In both of the cases we have $(AL)^\infty \leq \sigma^n(K_L(f))$. We get $\sigma^n(K_L(f)) \leq K_L(f)$ since $K_L(f)$ is maximal. Therefore we obtain the inequality (1). If $B_0 = C_L$, the inequality (1) holds since $\sigma^n(K_L(f)) = K_L(f)$.

By induction, for all $p \geq 1$

$$A* L^\infty \leq \sigma^{np}(I\iota_L'(f)) \leq A* ML^\infty.$$  

Thus $K_L(f)$ is of the form $AB_0X$. From the reasoning in the preceding paragraph, it follows that $X$ must be again of the same form. Hence

$$K_L(f) = AB_0AB_1 \cdots = A*B.$$  

In the case that $A$ is odd, we also have $K_L(f) = A*B$.

Next, we show that $B = B_0B_1 \cdots$ is maximal. For that purpose it is enough to prove that $\sigma^k(B) \leq B$ for any $k$. Since $K_L(f)$ is maximal, it follows that for any $k$

$$\sigma^{kn}(A*B) \leq A*B.$$  

(2)

We assume that for some $\tilde{k}$

$$B_k B_{k+1} \cdots B_{k+\tilde{k}-1} = B_0B_1 \cdots B_{\tilde{k}-1}$$  

(3)

and

$$B_{k+\tilde{k}} \neq B_k.$$  

(4)

If $A$ is even, then from (2)

$$AB_k AB_{k+1} \cdots AB_{k+\tilde{k}} < AB_0 AB_1 \cdots AB_{\tilde{k}}.$$  

Since $AB_k AB_{k+1} \cdots B_{k+\tilde{k}-1}A = AB_0 AB_1 \cdots B_{\tilde{k}-1}A$, we obtain $\sigma^k(B) < B$. We also have the same inequality in the case $A$ is odd. If the assumption (3) (4) does not hold for any $\tilde{k}$, then $B$ is periodic with period $k$, i.e. $\sigma^k(B) = B$.  

\[ \square \]
Lemma 2.2 Assume $A \ast L^\infty \leq K_L(f) \leq A \ast ML^\infty$. If $K_L(f) = A \ast B$ then $L^\infty \leq B \leq ML^\infty$ and $\sigma(B) \leq \sigma^k(B)$ for any $k \geq 1$.

Lemma 2.3 Assume $A \ast L^\infty \leq K_L(f) \leq A \ast ML^\infty$. If $\sigma^n(K_L(f)) \leq I(x) \leq K_L(f)$, then $\sigma^n(K_L(f)) \leq I(f^n(x)) \leq K_L(f)$, where $n = |AC_L|$.

Proposition 2.4 Let $AC_R \overline{AC}_L$ be maximal. If $A \ast M^\infty \leq K_L(f) \leq A \ast RM^\infty$ then there is a $B$ such that $K_L(f) = A \ast B$. This $B$ is maximal.

Lemma 2.5 Assume $A \ast M^\infty \leq K_L(f) \leq A \ast RM^\infty$ If $K_L(f) = A \ast B$ then $M^\infty \leq B \leq RM^\infty$ and $\sigma(B) \leq \sigma^k(B)$ for any $k \geq 1$.

Lemma 2.6 Assume $A \ast M^\infty \leq K_L(f) \leq A \ast RM^\infty$. If $\sigma^n(K_L(f)) \leq I(x) \leq K_L(f)$, then $\sigma^n(K_L(f)) \leq I(f^n(x)) \leq K_L(f)$, where $n = |AC_R|$.

2.3 The properties of kneading sequence for PL bimodal maps

Now we remember Lemma 1.2 that for $(\lambda, \mu) \in D_0$ we have $F_{\lambda,\mu}^2(t_1) > 0$. The map $F_{\lambda,\mu}(x)$ has a fixed point $x = 0$. The itinerary of this point is $M^\infty$. Therefore for $(\lambda, \mu) \in D_0$ $K_L(F_{\lambda,\mu}) > RM^\infty$.

Proposition 2.7 If $K_L(F_{\lambda,\mu}) > RM^\infty$, then $K_L(F_{\lambda,\mu})$ is primary.

Proof. Assume that $K_L(F_{\lambda,\mu}) = AB$. Lemma 2.3 implies that there is an interval $J$ such that $F_{\lambda,\mu}^n(J) = J$ for $n = |AC_L|$. We can take $\{x | \sigma^n(K_L(F_{\lambda,\mu})) \leq I(x) \leq K_L(F_{\lambda,\mu})\}$ for the above $J$. Then we find $F_{\lambda,\mu}^n$ on the interval $J$ is unimodal. Let $\kappa$ and $-\nu$ ($\kappa, \nu > 0$) be the slopes of $F_{\lambda,\mu}^n(J)$. Let $k$ be the total number of $L$'s and $R$'s in $A$, so the number of $M$'s is $n - 1 - k$. If $A$ is even, then we get the slopes

$$\kappa = (-\mu)^{n-1-k}\lambda^k \lambda \geq \lambda^2$$
$$(-\nu) = (-\mu)^{n-1-k}\lambda^k(-\mu) \leq -\lambda \mu$$
respectively. If $A$ is odd, then

\[
\kappa = (-\mu)^{n-1-k}\lambda^{k}(-\mu) \geq \mu^{2}
\]

\[
(-\nu) = (-\mu)^{n-1-k}\lambda^{k}\lambda \leq -\lambda\mu.
\]

In both of the cases, we get $\frac{1}{\kappa} + \frac{1}{\nu} < 1$, since $K_{L}(F_{\lambda,\mu}) > RM^\infty$ implies $\frac{1}{\lambda} + \frac{1}{\mu} < 1$. This contradicts the result in Misiurewicz-Visinescu [2] that $\frac{1}{\kappa} + \frac{1}{\nu} > 1$ is necessary for the existence of an $F$-invariant interval for unimodal maps.

**Proposition 2.8** If $K_{L}(F_{\lambda,\mu}) > RM^\infty$, then $K_{L}(F_{\lambda,\mu})$ is primary.

**Theorem 2.9** Let $K_{L}(F_{\lambda,\mu})$ be a maximal and primary sequence such that $K_{L}(F_{\lambda,\mu}) > RM^\infty$. There is $\nu$ such that $K(g_{\nu}) = K_{L}(F_{\lambda,\mu})$, where $g_{\nu}$ is the PL bimodal map with the slopes alternately $\nu$, $-\nu$, $\nu(\nu > 1)$.

**Proof.** From the maximality and the primarity of $K_{L}(F_{\lambda,\mu})$ as well as Proposition 2.1, we have one of the inequalities $K_{L}(F_{\lambda,\mu}) < A* L^\infty$ or $K_{L}(F_{\lambda,\mu}) > A* ML^\infty$. We set

\[
M_{F_{\lambda,\mu}} = \{\nu : K_{L}(g_{\nu}) < K_{L}(F_{\lambda,\mu})\}
\]

and

\[
P_{F_{\lambda,\mu}} = \{\nu : K_{L}(g_{\nu}) > K_{L}(F_{\lambda,\mu})\},
\]

and we claim these are open. We show only that $M_{F_{\lambda,\mu}}$ is open, and we can prove that $P_{F_{\lambda,\mu}}$ is open in the same way. We put $n = |AC_{L}|$, and assume that $A$ is even. We take $\nu \in M_{F_{\lambda,\mu}}$ with $K_{L}(g_{\nu}) = AD_{n}D_{n+1} \cdots < K_{L}(F_{\lambda,\mu})$ such that $D_{n}$ is not equal to the $n$th symbol of $K_{L}(F_{\lambda,\mu})$. If $D_{n} \neq C_{L}$ then it is obvious that $M_{K(F_{\lambda,\mu})}$ is open. If $D_{n} = C_{L}$ then $K_{L}(F_{\lambda,\mu}) > A* ML^\infty$. In this case there exists $s_{0}$ such that $K_{L}(F_{\lambda,\mu}) > AMLs_{0} \cdots$. Thus the sets $\{\tilde{D} = AM(\tilde{AL})^{s} : s > s_{0}\}$ and $\{\tilde{D} = AL(\tilde{AL})^{s} : s > s_{0}\}$ are included in $M_{F_{\lambda,\mu}}$, and $M_{F_{\lambda,\mu}}$ is open. If $A$ is odd, we can also prove that $M_{F_{\lambda,\mu}}$ is open in the same way.
3 Monotonicity of topological entropy

In this section we consider the monotonicity of kneading sequences and that of topological entropy.

We define an order on a pair of the parameters.

(i) We say $(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2)$ if $\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2$.

(ii) We say $(\lambda_1, \mu_1) < (\lambda_2, \mu_2)$ if $(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2)$ and either $\lambda_1 \neq \lambda_2$ or $\mu_1 \neq \mu_2$.

We denote the pair of $K_L(F_{\lambda,\mu})$ and $K_R(F_{\lambda,\mu})$ by $K(\lambda, \mu)$. We say $K(\lambda_1, \mu_1) < K(\lambda_2, \mu_2)$ if and only if $K_L(F_{\lambda_1,\mu_1}) < K_L(F_{\lambda_2,\mu_2})$ and $K_R(F_{\lambda_1,\mu_1}) > K_R(F_{\lambda_2,\mu_2})$.

Let $h(\lambda, \mu)$ be the topological entropy of $F_{\lambda,\mu}$.

**Proposition 3.1** Let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in D \setminus D_0$. If $(\lambda_1, \mu_1) < (\lambda_2, \mu_2)$ then $h(\lambda_1, \mu_1) < h(\lambda_2, \mu_2)$.

We can show this proposition applying the result of [2] that proved the monotonicity of the topological entropy for PL unimodal maps.

Let

$$D_1 = \left\{ (\lambda, \mu) \in D_0 : R C_R L C_L < K_L(F_{\lambda,\mu}) < R(RL)^\infty \right\}.$$  

We obtain a proposition about the monotonicity of kneading sequences as follows:

**Proposition 3.2** Let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in D_0 \setminus D_1$. If $(\lambda_1, \mu_1) < (\lambda_2, \mu_2)$ then $K(\lambda_1, \mu_1) < K(\lambda_2, \mu_2)$.

The proof of this proposition is given by an analytical estimation.

**Theorem 3.3** Let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in D_0 \setminus D_1$. If $(\lambda_1, \mu_1) < (\lambda_2, \mu_2)$ then $h(\lambda_1, \mu_1) < h(\lambda_2, \mu_2)$.

We can prove this theorem from Theorem 2.9 and Proposition 3.2.
**Theorem 3.4** For a constant $c$ with $0 < c < \log(3)$, the iso-entropy curve given by $h(\lambda, \mu) = c$ is connected.

We prove this theorem in [3].

**References**

