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LAYER HEAT POTENTIALS FOR A BOUNDED CYLINDER WITH FRACTAL LATERAL BOUNDARY
(Problems on complex dynamical systems)

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LAYERS HEAT POTENTIALS FOR A BOUNDED CYLINDER WITH FRACTAL LATERAL BOUNDARY

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1. Introduction

Let $D$ be a bounded smooth domain in $\mathbb{R}^d$ and set

$$\Omega_D = D \times (0,T) \quad \text{and} \quad S_D = \partial D \times [0,T].$$

The double layer heat potential $\Phi f$ of $f \in L^p(S_D)$ is defined by

$$\Phi f(X) = -\int_0^T \int_{\partial D} \langle \nabla_y W(X-Y), n_y \rangle f(Y) d\sigma(y) ds$$

for $X = (x,t) \in (\mathbb{R}^d \setminus \partial D) \times \mathbb{R}$, where $\langle , \rangle$ is the inner product in $\mathbb{R}^d$, $n_y$ is the unit outer normal to $\partial D$, $\sigma$ is the surface measure on $\partial D$ and $W$ is the fundamental solution for the heat operator, i.e.,

$$W(X) = W(x,t) = \begin{cases} \frac{\exp(-\frac{|x|^2}{4t})}{(4\pi t)^{d/2}} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The double layer heat potential is important not only physically but also mathematically. For example, R. M. Brown proved that the solution to the initial-Dirichlet problem in a Lipschitz cylinder for the heat operator can be written by a double layer heat potential and the solution to the initial-Neumann problem in a Lipschitz cylinder for the heat operator is given by a single layer heat potential (cf. [B1], [B2]).

If $D$ is a bounded domain with fractal boundary, then $n_y$ and the surface measure cannot be defined. But if $D$ has a smooth boundary and $f$ is a $C^1$-function on $\mathbb{R}^{d+1}$ with compact support, then we see by the Green formula that for $X = (x,t) \in D \times \mathbb{R}$

$$\Phi f(X) = \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \langle \nabla_y f(Y), \nabla_y W(X-Y) \rangle dy$$

$$+ \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} f(Y) \Delta_y W(X-Y) dy$$

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and for $X = (x, t) \in (\mathbb{R}^d \setminus \overline{D}) \times \mathbb{R}$

\[(1.3)\]

$$
\Phi f(x) = - \int_0^T ds \int_D \langle \nabla_y f(Y), \nabla W(x - Y) \rangle dy - \int_0^T ds \int_D f(Y) \Delta_y W(x - Y) dy.
$$

So we see that, if a function $f$ defined on $S_D$ can be extended to be a function $\mathcal{E}(f)$ on $\mathbb{R}^d \times [0, T]$ such that for each $t \in [0, T]$ the function $x \mapsto \mathcal{E}(f)(x, t)$ is a $C^1$-function on $\mathbb{R}^d \setminus \partial D$ and for each $x \in \mathbb{R}^d \setminus \partial D$ and each $j$ ($j = 1, 2, \cdots, d$) the function $t \mapsto \frac{\partial \mathcal{E}(f)}{\partial x_j}(x, t)$ is measurable, then the right-hand sides of (1.2) and (1.3) may be defined.

In this paper we assume that $D$ is a bounded domain in $\mathbb{R}^d$ ($d \geq 2$) and $\partial D$ is a $\beta$-set satisfying $d - 1 \leq \beta < d$. Here, according to [JW] we say that a closed set $F$ is a $\beta$-set if there exist a positive Radon measure $\mu$ on $F$ and positive real numbers $r_0, b_1, b_2$ such that

\[(1.4)\]

$$
 b_1 r^\beta \leq \mu(B(z, r) \cap F) \leq b_2 r^\beta
$$

for all $z \in F$ and all $r \leq r_0$, where $B(z, r)$ stands for the open ball in $\mathbb{R}^d$ with center $z$ and radius $r$.

We note that, if $D$ is a bounded Lipschitz domain, then $\partial D$ is a $(d-1)$-set and the surface measure $\mu$ has the property (1.4) for $F = \partial D$ and $\beta = d - 1$. Furthermore if $\partial D$ consists of a finite number of self-similar sets, which satisfies the open set condition, and whose similarity dimensions are $\beta$, then $\partial D$ is a $\beta$-set such that the $\beta$-dimensional Hausdorff measure $\mathcal{H}^\beta$ restricted to $\partial D$ has the property (1.4) for $F = \partial D$ (cf. [Hu]).

Let $0 < \alpha \leq 1$ and $F$ be a closed set in $\mathbb{R}^d$. We denote by $\Lambda_\alpha(F \times [0, T])$ the Banach space of all continuous functions $f$ on $F \times [0, T]$ such that $f(\cdot, t)$ is $\alpha$-Hölder continuous for every $t \in [0, T]$ with norm

$$
\|f\|_{\infty, \alpha} = \sup_{X \in F \times [0, T]} |f(X)| + \sup_{x, y \in F, x \neq y, t \in [0, T]} \frac{|f(x, t) - f(y, t)|}{|x - y|^\alpha}.
$$

Further let $0 < \alpha, \lambda \leq 1$. We also denote by $\Lambda_{\alpha, \lambda}(F \times [0, T])$ the Banach space of all $f \in \Lambda_\alpha(F \times [0, T])$ such that $f$ is $\lambda$-Hölder continuous with respect to the time variable with norm

$$
\|f\|_{\infty, \alpha, \lambda} = \|f\|_{\infty, \alpha} + \sup_{x, t, s \in [0, T], t \neq s} \frac{|f(x, t) - f(x, s)|}{|t - s|^\lambda}.
$$

We will prove the following lemma in §3.
Lemma 1.1. Let $d - 1 \leq \beta < d$ and $F$ be a compact $\beta$-set in $\mathbb{R}^d$ satisfying (1.4) and $F \subset B(0, R/2)$. Then there exists a bounded operator $\mathcal{E}$ from $\Lambda_{\alpha}(F \times [0,T])$ to $\Lambda_{\alpha}(\mathbb{R}^d)$ with the following properties:

(i) $\mathcal{E}(f)(\cdot, t)$ is a $C^1$-function on $\mathbb{R}^d \setminus F$ for each $t \in [0,T]$, and both of $\mathcal{E}(f)(x, \cdot)$ and $\left( \frac{\partial \mathcal{E}(f)}{\partial x_j} \right)(x, \cdot)$ ($j = 1, \cdots, d$) are measurable for each $x \in \mathbb{R}^d$ and for each $x \in \mathbb{R}^d \setminus F$, respectively,

(ii) $\mathcal{E}(f) = f$ on $F$ and supp $\mathcal{E}(f)(\cdot, t) \subset B(0, 2R)$ for each $t \in [0,T]$.

(iii) $| \frac{\partial \mathcal{E}(f)}{\partial y_i} (y, s)| \leq c \|f\|_{\infty, \alpha} \text{dist}(y, \partial D)^{\alpha-1}$, $| \frac{\partial^2 \mathcal{E}(f)}{\partial y_i \partial y_k} (y, s)| \leq c \|f\|_{\infty, \alpha} \text{dist}(y, \partial D)^{\alpha-2}$

for every $(y, s) \in (\mathbb{R}^d \setminus F) \times [0,T]$.

(iv) If $f \in \Lambda_{\alpha, \lambda}(F \times [0,T])$, then $\mathcal{E}(f) \in \Lambda_{\alpha, \lambda}(\mathbb{R}^d \times [0,T])$.

Using Lemma 1.1 we define, for $f \in \Lambda_{\alpha}(S_{D})$,

(1.5) $\Phi f(X) = \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \langle \nabla \mathcal{E}(f)(y, s), \nabla W(X - Y) \rangle dy$

$+ \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \mathcal{E}(f)(y, s) \triangle_{y}W(X - Y) dy$

for $X = (x, t) \in D \times \mathbb{R}$ and

(1.6) $\Phi f(X) = -\int_0^T ds \int_D \langle \nabla \mathcal{E}(f)(y, s), \nabla W(X - Y) \rangle dy$

$- \int_0^T ds \int_D \mathcal{E}(f)(y, s) \triangle_{y}W(X - Y) dy$

for $X = (x, t) \in (\mathbb{R}^d \setminus \overline{D}) \times \mathbb{R}$.

Furthermore we also define the operator $K$ by

(1.7) $K f(Z) = \frac{1}{2} (I_1(Z) + I_2(Z))$,

where

$$
I_1(Z) = \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \langle \nabla \mathcal{E}(f)(y, s), \nabla W(Z - Y) \rangle dy
$$

$+ \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} (\mathcal{E}(f)(Y) - f(Z)) \triangle_{y}W(Z - Y) dy$

$+ f(Z) \int_{(\mathbb{R}^d \setminus \overline{D}) \times \{0\}} W(Z - Y) dy$

$+ f(Z) \int_{(\mathbb{R}^d \setminus \overline{D}) \times \{0\}} W(Z - Y) dy$

$+ f(Z) \int_{(\mathbb{R}^d \setminus \overline{D}) \times \{0\}} W(Z - Y) dy$. 

and

$$I_2(Z) = - \int_0^T ds \int_D \langle \nabla \mathcal{E}(f)(y, s), \nabla_y W(Z - Y) \rangle dy$$

$$- \int_0^T ds \int_D (\mathcal{E}(f)(Y) - f(Z)) \Delta_y W(Z - Y) dy - f(Z) \int_{D \times \{0\}} W(z - Y) dy$$

Under these notations we will prove the following theorem in §3.

**Theorem.** Assume that $D$ is a bounded domain in $\mathbb{R}^d$ such that $\partial D$ is a $\beta$-set. If $0 \leq \beta - (d - 1) < \alpha < 1$ and $f \in \Lambda_{\alpha, \alpha/2}(S_D)$, then, for each $Z \in \partial D \times [0, T]$

$$(1.8) \quad \lim_{X \to Z, X \in D \times (0, T)} \Phi f(X) = Kf(Z) + \frac{1}{2}f(Z)$$

and

$$(1.9) \quad \lim_{X \to Z, X \in (\mathbb{R}^d \setminus \overline{D}) \times (0, T)} \Phi f(X) = Kf(Z) - \frac{1}{2}f(Z).$$

Thus we see that our double layer heat potentials have the same boundary behavior as the usual ones for a bounded cylinder with smooth lateral boundary.

**Remark.** In this paper we shall treat the double layer heat potentials of Hölder continuous functions on $S_D$. But under a similar consideration we can also the double layer heat potentials of functions in a Besov space on $S_D$ and prove that they have the parabolically non-tangential limit at a.e. $Z \in S_D$.

### 2. Properties of $W$

In this section we recall and study properties of the function $W$. To do so, we use the parabolic metric $\delta$ defined by

$$\delta(X, Y) = (|x - y|^2 + |t - s|)^{1/2} \quad \text{for } X = (x, t) \text{ and } Y = (y, s).$$

**Lemma 2.1.** (i) $W(X) \leq c\delta(X, 0)^{-d}$,

(ii) $|\nabla_x W(X)| \leq c\delta(X, 0)^{-d-1}$ if $X \neq 0$,

(iii) $|\frac{\partial^2}{\partial x_i \partial x_j} W(X)| \leq c\delta(X, 0)^{-d-2}, \quad |\frac{\partial}{\partial t} W(X)| \leq c\delta(X, 0)^{-d-2}$ if $X \neq 0$,

(iv) $|\frac{\partial^2}{\partial x_i \partial x_j \partial x_k} W(X)| \leq c\delta(X, 0)^{-d-3}, \quad |\frac{\partial^2}{\partial x_i \partial x_j \partial x_k} W(X)| \leq c\delta(X, 0)^{-d-3}$ if $X \neq 0$,

(v) $|W(X - Y) - W(Z - Y)| \leq c\delta(X, Z)^{\epsilon} \{\delta(X, Y)^{-d-\epsilon} + \delta(Z, Y)^{-d-\epsilon}\}$

if $0 \leq \epsilon \leq 1$ and $X \neq Y$, $Z \neq Y$,

(vi) $|\nabla_y W(X - Y) - \nabla_y W(Z - Y)| \leq c\delta(X, Z)^{\epsilon} \{\delta(X, Y)^{-d-1-\epsilon} + \delta(Z, Y)^{-d-1-\epsilon}\}$

if $0 \leq \epsilon \leq 1$ and $X \neq Y$, $Z \neq Y$. 


Proof. The assertions (i), (ii), (iii) and (iv) are well known (cf. [B2, p.5]). The assertions (v) and (vi) will be shown by the same method as in the proof of Lemma 2.3 in [W2].

Let $D_0$ be a bounded piecewise smooth domain in $\mathbb{R}^d$ and $u, v$ be smooth functions on $\overline{D_0} \times [0, \rho]$. Using the divergence theorem, we obtain

\begin{equation}
\int_0^\rho \int_{D_0} (uL^*v - vLu) \, dx \, dt
= \int_0^\rho dt \int_{\partial D_0} \langle u\nabla_x v - v\nabla_x u, n_x \rangle \, d\sigma(x) - \int_{D \times \{t=0\}} uv \, dx + \int_{D \times \{t=\rho\}} uv \, dx,
\end{equation}

where \( L = \Delta - \frac{\partial}{\partial t} \) and \( L^* = \Delta + \frac{\partial}{\partial t} \).

If \( Lu = L^*v = 0 \) in $D_0 \times (0, \rho)$, then (2.1) implies

\begin{equation}
\int_0^\rho dt \int_{\partial D_0} \langle u\nabla_x v - v\nabla_x u, n_x \rangle \, d\sigma(x) - \int_{D \times \{t=0\}} uv \, dx + \int_{D \times \{t=\rho\}} uv \, dx = 0.
\end{equation}

Let $X = (x, t) (0 \leq t \leq T)$ be an exterior point of $D_0 \times (0, T)$. Then, setting $u = 1$ and $v(Y) = W(X - Y)$ and noting that $W(X - Y) = 0$ for $Y = (y, T)$, we deduce from (2.2)

\begin{equation}
\int_0^T ds \int_{\partial D_0} \langle \nabla_y W(X - Y), n_y \rangle \, d\sigma(y) - \int_{D \times \{s=0\}} W(X - Y) \, dy = 0.
\end{equation}

Hereafter we assume that $D$ is a bounded domain in $\mathbb{R}^d$ such that $\partial D$ is a $\beta$-set satisfying $\overline{D} \subset B(O, R/2)$.

Let us use the Whitney decomposition to approximate $D$ and $\mathbb{R}^d \setminus \overline{D}$ (cf. [S, p.167]). Let $\mathcal{V}(D)$ be the Whitney decomposition of $D$ and define

\[ A_n = \bigcup_{k=k_0}^{n} \bigcup_{Q \in \mathcal{V}_k(D)} Q, \]

where $\mathcal{V}_k(D) = \{ Q \in \mathcal{V}(D); Q \text{ is a } k\text{-cube} \}$ and $k_0$ is the smallest integer $k$ such that $\mathcal{V}_k(D) \neq \emptyset$. Similarly we also define

\[ B_n = \left( \bigcup_{k=-\infty}^{n} \bigcup_{Q \in \mathcal{V}_k(\mathbb{R}^d \setminus \overline{D})} Q \right). \]

Then we have the following lemma.
Lemma 2.2. Set
\[ g_n(X) = \int_0^T \int_{A_n} \Delta_y W(X - Y) dy \quad \text{and} \quad h_n(X) = \int_0^T \int_{B_n} \Delta_y W(X - Y) dy. \]

Then \( \lim_{n \to \infty} g_n(X) \) and \( \lim_{n \to \infty} h_n(X) \) exist on \( \mathbb{R}^d \times [0, T] \) and for \( X \in \mathbb{R}^d \times (0, T] \)

\[ \lim_{n \to \infty} g_n(X) = \int_{D \times \{0\}} W(X - Y) dy - \chi_D(X) \]

and

\[ \lim_{n \to \infty} h_n(X) = \int_{(\mathbb{R}^d \setminus \overline{D}) \times \{0\}} W(X - Y) dy - \chi_{\mathbb{R}^d \setminus \overline{D}}(X) \]

Proof. Let \( X = (x, t) \in \mathbb{R}^d \times (0, T] \) and \( t > \rho > 0 \). Applying (2.2) to \( A_n \times (0, \rho) \), we have

\[ \int_0^\rho ds \int_{\partial A_n} \langle \nabla_y W(X - Y), n_y \rangle d\sigma(y) - \int_{A_n \times \{0\}} W(X - Y) dy + \int_{A_n \times \{\rho\}} W(X - Y) dy = 0. \]

Using the divergence theorem for \( A_n \) in \( \mathbb{R}^d \), we have

\[ \int_0^\rho ds \int_{A_n} \Delta_y W(X - Y) dy - \int_{A_n \times \{0\}} W(X - Y) dy + \int_{A_n \times \{\rho\}} W(X - Y) dy = 0. \]

As \( \rho \to t \) and \( n \to \infty \), we obtain,

\[ \lim_{n \to \infty} g_n(X) = \int_{D \times \{0\}} W(X - Y) dy - \chi_D(X). \]

On the other hand \( g_n(X) = 0 \) for \( t = 0 \). Hence \( \lim_{n \to \infty} g_n(X) \) exists for each \( X \in \mathbb{R}^d \times [0, T] \).

Similarly we can also prove the conclusion for \( h_n \).

\[ \square \]

3. Double layer heat potentials

In this section we first prove Lemma 1.1 in §1.
Proof of Lemma 1.1 We use the extension operator $\mathcal{E}_0$ in [S, p.172] and choose a $C^\infty$-function $\phi_0$ such that

$$\phi_0 = 1 \text{ on } B(0, R), \quad \text{supp } \phi_0 \subset B(0, 2R) \quad \text{and } 0 \leq \phi_0 \leq 1.$$  

We define

$$\mathcal{E}(f)(x, t) = \mathcal{E}_0(f(\cdot, t))(x)\phi_0(x)$$

for $f \in \Lambda_\alpha(F)$ and $(x, t) \in (\mathbb{R}^d \setminus F) \times [0, T]$ and

$$\mathcal{E}(f)(x, t) = f(x, t) \quad \text{on } (x, t) \in F \times [0, T].$$

Then properties (i), (ii), (iii) follow from the definition and (13) on p.174 in [S]. Since the operator $\mathcal{E}_0$ is linear, positive and maps the constant function 1 to 1, (iv) is also valid.

In [W1] we gave the following lemma.

**Lemma A.** Let $\delta, k$ be non-negative numbers satisfying $d-\beta > \delta$ and $d-\delta-k > 0$. Then

$$\int_{B(x, r)} \text{dist}(y, \partial D)^{-\delta}|y-z|^{-k}dy \leq c r^{d-\delta-k}$$

for every $z \in \partial D$ and $r > 0$.

We next show that the double layer heat potential defined by (1.5) and (1.6) converges.

**Lemma 3.1.** Let $0 \leq \beta - (d - 1) < \alpha < 1$ and $f \in \Lambda_\alpha(S_D)$. Then $\Phi f$ is caloric in $(\mathbb{R}^d \setminus \partial D) \times \mathbb{R}$.

**Proof.** Set, for $X = (x, t) \in D \times \mathbb{R}$,

$$(3.1) \quad J_1(X) = \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y, s), \nabla_y W(X-Y) \rangle dy$$

and let $X_0 = (x_0, t_0) \in D$. Choose $\rho > 0$ satisfying $\overline{B(x_0, 2\rho)} \subset D$. If $X = (x, t) \in B(x_0, \rho) \times \mathbb{R}$, then we deduce from Lemmas 2.1 and 1.1 and Lemma A

$$|J_1(X)| \leq \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \text{dist}(y, \partial D)^{\alpha-1} \delta(X, Y)^{-1-d}dy \leq c_1 \rho^{-1-d}\|f\|_{\infty, \alpha},$$

whence $J_1$ converges locally uniformly in $D$. We denote by $g_1$ the integrand of the right-hand side on (3.1). Since

$$|\nabla_y \frac{\partial^2}{\partial x_i \partial x_j} W(X-Y)| \leq c_2 \delta(X, Y)^{-d-3} \quad \text{and} \quad |\nabla_y \frac{\partial}{\partial t} W(X-Y)| \leq c_3 \delta(X, Y)^{-d-3},$$
we see that the integral of $Lg_1$ over $(\mathbb{R}^d \setminus \overline{D}) \times [0,T]$ also converges locally uniformly on $D$. Therefore $J_1$ satisfies the heat equation in $D \times \mathbb{R}$.

Next, set

$$J_2(X) = \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \mathcal{E}(f)(y,s) \Delta W(X-Y) dy.$$

Using Lemma 1.1, (iii), we can show by the above method that $J_2$ also converges locally uniformly in $D$ and satisfies the heat equation. Thus we conclude that $\Phi f = J_1 + J_2$ has the same properties in $D \times \mathbb{R}$. We can show that $\Phi f$ also has the same properties in $(\mathbb{R}^d \setminus \overline{D}) \times \mathbb{R}$. □

Using Lemma 2.1, (iv), (v) and Lemma A, we can prove the following lemma by a similar method to that in the proof of [W1, Lemma 3.3].

**Lemma 3.2.** Let $0 \leq \beta - (d-1) < \alpha < 1$ and $f \in \Lambda_{\alpha,\alpha/2}(S_D)$. Then both of the function $J_1$ defined by (3.1) and the function $J_3$ defined by

$$J_3(X) = \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} (\mathcal{E}(f)(Y) - \mathcal{E}(f)(X)) \Delta W(X-Y) dy$$

are are continuous on $\mathbb{R}^d \times [0,T]$. Furthermore the function $J'_1$ (resp. $J'_3$) obtained by replacing $\mathbb{R}^d \setminus \overline{D}$ with $D$ in the definition of $J_1$ (resp. $J_3$) is also continuous on $\mathbb{R}^d \times [0,T]$.

**Lemma 3.3.** Let $0 \leq \beta - (d-1) < \alpha < 1$ and $g \in \Lambda_{\alpha,\alpha/2}(\mathbb{R}^d \times [0,T])$ such that $g(.t) \in C^1(\mathbb{R}^d)$, supp $g(.s) \subset B(0,r_0)$ for every $t \in [0,T]$ and $\frac{\partial g}{\partial x_j}(x,\cdot)$ is bounded for every $x \in \mathbb{R}^d$. Let $X = (x,t) \in \mathbb{R}^d \times (0,T]$ and set, for $0 < \rho \leq T$,

$$A_\rho g(X) = \int_0^\rho ds \int_{\mathbb{R}^d \setminus \overline{D}} \langle \nabla g(Y), \nabla_y W(X-Y) \rangle dy$$

$$+ \int_0^\rho ds \int_{\mathbb{R}^d \setminus \overline{D}} (g(Y) - g(X)) \Delta_y W(X-Y) dy$$

$$+ g(X) \int_{(\mathbb{R}^d \setminus \overline{D}) \times [0]} W(X-Y) dy$$

and

$$B_\rho g(X) = - \int_0^\rho ds \int_{\mathbb{R}^d \setminus \overline{D}} \langle \nabla g(Y), \nabla_y W(X-Y) \rangle dy$$

$$- \int_0^\rho ds \int_{D} (g(Y) - g(X)) \Delta_y W(X-Y) dy$$

$$- g(X) \int_{D \times [0]} W(X-Y) dy.$$
Then

\begin{equation}
A_T g(X) = B_T g(X) + g(X) \text{ for } X \in \mathbb{R}^d \times (0, T]
\end{equation}

Proof. To simplify the notation, we use $A_\rho(x)$ and $B_\rho(x)$ instead of $A_\rho g(X)$ and $B_\rho g(X)$, respectively. We first show (3.2) in case $D = D_0$ is a bounded piecewise smooth domain. Let $X = (x, t)$ and set, for $0 < \rho < t$,

$$I_\rho(X) = - \int_0^\rho ds \int_{\partial D_0} g(Y) \langle \nabla_Y W(X - Y), n_Y \rangle d\sigma(y).$$

The Green formula for $D_0$ yields

\begin{equation}
I_\rho(X) = - \int_0^\rho ds \int_{D_0} \langle \nabla g(Y), \nabla_Y W(X - Y) \rangle dy
- \int_0^\rho ds \int_{D_0} (g(Y) - g(X)) \nabla_Y W(X - Y) dy
- g(X) \int_0^\rho ds \int_{D_0} \nabla_Y W(X - Y) dy
\end{equation}

From (2.2) we deduce

$$\int_0^\rho ds \int_{D_0} \nabla_Y W(X - Y) dy
= \int_{D_0 \times \{0\}} W(X - Y) dy - \int_{D_0 \times \{\rho\}} W(X - Y) dy,$$

whence

$$\int_0^t ds \int_{D_0} \nabla_Y W(X - Y) dy = \int_{D_0 \times \{0\}} W(X - Y) dy - \chi_{D_0}(x).$$

This and (3.3) imply

\begin{equation}
I_t(X) = B_t(X) + g(X) \chi_{D_0}(x) \text{ for } X \in (\mathbb{R}^d \setminus \partial D_0) \times (0, T].
\end{equation}

Similarly, using the Green formula for $B(0, r) \setminus \overline{D_0}$ and $r \to \infty$, we obtain

$$I_t(X) = A_t(X) - g(X) \chi_{\mathbb{R}^d \setminus \overline{D_0}}(x)$$

for $X \in (\mathbb{R}^d \setminus \partial D_0) \times (0, T]$. This and (3.4) lead to

$$A_t(X) = B_t(X) + g(X) \text{ for } X \in (\mathbb{R}^d \setminus \partial D_0) \times (0, T].$$
Noting that $A_t(X) = A_T(X)$ and $B_t(X) = B_T(X)$, we obtain (3.2) for $X \in (\mathbb{R}^d \setminus \partial D) \times (0,T]$. Since $A_T$ and $B_T$ are continuous on $\mathbb{R}^d \times (0,T]$ by Lemma 3.2, (3.2) holds for a bounded piecewise smooth domain $D = D_0$.

We next show (3.2) for a bounded domain such that $\partial D$ is a $\beta$-set. We use (3.2) for $D_0 = A_n$. Since $A_T$ and $B_T$ are continuous on $\mathbb{R}^d \times (0,T]$ by Lemma 3.2, (3.2) holds for a bounded piecewise smooth domain $D = D_0$.

We next show (3.2) for a bounded domain such that $\partial D$ is a $\beta$-set. We use (3.2) for $D_0 = A_n$. Since $\int_0^T \int_{\mathbb{R}^d} |\nabla g(Y)||\nabla_y W(X - Y)| dy ds < \infty$,

$$
\int_0^T ds \int_{\mathbb{R}^d} |g(Y) - g(X)||\Delta_y W(X - Y)| dy < \infty
$$

and

$$
\int_{\mathbb{R}^d \times \{0\}} W(X - Y) dy < \infty,
$$

we see that (3.2) holds for the domain $D$ as $n \to \infty$. \hfill \Box

**Lemma 3.4.** Let $0 \leq \beta - (d - 1) < \alpha$ and $f \in \Lambda_{\alpha,\alpha/2}(S_D)$. Then (3.2) holds for $g = \mathcal{E}(f)$.

**Sketch of Proof.** Let $f \in \Lambda_{\alpha,\alpha/2}(S_D)$ and $\{v_m\}$ be a mollifier on $\mathbb{R}^d$ such that $\text{supp } v_m \subset B(0,1/m)$. We define, for $Y = (y,s) \in \mathbb{R}^d \times [0,T]$,

$$
g_m(Y) = (\mathcal{E}(f)(\cdot,s)*v_m)(y).
$$

Lemma 3.3 yields

$$
A_T g_m(X) = B_T g_m(X) + g_m(X) \text{ for } X \in \mathbb{R}^d \times (0,T].
$$

Using $g_m(X) \to \mathcal{E}(f)(X)$ uniformly as $m \to \infty$ and Lemmas A, 1.1 and 2.1, we can show that

$$
A_T g_m(X) \to A_T \mathcal{E}(f)(X)
$$

and

$$
B_T g_m(X) \to B_T \mathcal{E}(f)(X)
$$

for $X \in \mathbb{R}^d \times [0,T]$ as $m \to \infty$. \hfill \Box

We can also show the following lemma.

**Lemma 3.5.** Let $0 \leq \beta - (d - 1) < \alpha < 1$. Then the operator $K$ defined by (1.7) is a bounded operator from $\Lambda_{\alpha,\alpha/2}(S_D)$ to $\Lambda_{\alpha,\alpha/2}(S_D)$.

Let us prove our theorem.
Proof of Theorem. Let $X \in D \times (0,T]$. Using Lemma 2.2, we have $\Phi f(X) = A_T f(X)$. Since $A_T f$ is continuous on $\mathbb{R}^d \times (0,T]$ by Lemma 3.2, we have

$$\lim_{X \to Z, X \in D \times (0,T)} \Phi f(X) = A_T f(Z).$$

On the other hand Lemma 3.4 yields

$$K f(Z) = \frac{1}{2} (A_T f(Z) + B_T f(Z)) = A_T f(Z) - \frac{1}{2} f(Z).$$

Therefore we have (1.8). Similarly we can show (1.9).

References