<table>
<thead>
<tr>
<th>Title</th>
<th>LAYER HEAT POTENTIALS FOR A BOUNDED CYLINDER WITH FRACTAL LATERAL BOUNDARY (Problems on complex dynamical systems)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Watanabe, Hisako</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1998), 1042: 112-122</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1998-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62092">http://hdl.handle.net/2433/62092</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
1. Introduction

Let $D$ be a bounded smooth domain in $\mathbb{R}^d$ and set

$$\Omega_D = D \times (0,T) \quad \text{and} \quad S_D = \partial D \times [0,T].$$

The double layer heat potential $\Phi f$ of $f \in L^p(S_D)$ is defined by

$$\Phi f(X) = -\int_0^T \int_{\partial D} \langle \nabla_y W(X - Y), n_y \rangle f(Y) d\sigma(y) ds$$

for $X = (x, t) \in (R^d \setminus \partial D) \times \mathbb{R}$, where $\langle , \rangle$ is the inner product in $\mathbb{R}^d$, $n_y$ is the unit outer normal to $\partial D$, $\sigma$ is the surface measure on $\partial D$ and $W$ is the fundamental solution for the heat operator, i.e.,

$$W(X) = W(x, t) = \begin{cases} \exp \left( -\frac{t|x|^2}{4t} \right) & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The double layer heat potential is important not only physically but also mathematically. For example, R. M. Brown proved that the solution to the initial-Dirichlet problem in a Lipschitz cylinder for the heat operator can be written by a double layer heat potential and the solution to the initial-Neumann problem in a Lipschitz cylinder for the heat operator is given by a single layer heat potential (cf. $[B_1], [B_2]$).

If $D$ is a bounded domain with fractal boundary, then $n_y$ and the surface measure can not be defined. But if $D$ has a smooth boundary and $f$ is a $C^1$-function on $\mathbb{R}^{d+1}$ with compact support, then we see by the Green formula that for $X = (x, t) \in D \times \mathbb{R}$

$$\Phi f(X) = \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \langle \nabla_y f(Y), \nabla_y W(X - Y) \rangle dy$$

$$+ \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} f(Y) \Delta_y W(X - Y) dy$$

Typeset by \LaTeX
and for \( X = (x, t) \in (\mathbb{R}^d \setminus \overline{D}) \times \mathbb{R} \)

\[
\Phi f(x) = -\int_0^T ds \int_D (\nabla_y f(Y), \nabla W(x - Y)) dy \\
- \int_0^T ds \int_D f(Y) \Delta_y W(x - Y) dy
\]

(1.3)

So we see that, if a function \( f \) defined on \( S_D \) can be extended to be a function \( \mathcal{E}(f) \) on \( \mathbb{R}^d \times [0, T] \) such that for each \( t \in [0, T] \) the function \( x \mapsto \mathcal{E}(f)(x, t) \) is a \( C^1 \)-function on \( \mathbb{R}^d \setminus \partial D \) and for each \( x \in \mathbb{R}^d \setminus \partial D \) and each \( j \) \((j = 1, 2, \cdots, d)\) the function \( t \mapsto \frac{\partial \mathcal{E}(f)}{\partial x_j}(x, t) \) is measurable, then the right-hand sides of (1.2) and (1.3) may be defined.

In this paper we assume that \( D \) is a bounded domain in \( \mathbb{R}^d \) \((d \geq 2)\) and \( \partial D \) is a \( \beta \)-set satisfying \( d - 1 \leq \beta < d \). Here, according to [JW] we say that a closed set \( F \) is a \( \beta \)-set if there exist a positive Radon measure \( \mu \) on \( F \) and positive real numbers \( r_0, b_1, b_2 \) such that

\[
b_1 r^\beta \leq \mu(B(z, r) \cap F) \leq b_2 r^\beta
\]

for all \( z \in F \) and all \( r \leq r_0 \), where \( B(z, r) \) stands for the open ball in \( \mathbb{R}^d \) with center \( z \) and radius \( r \).

We note that, if \( D \) is a bounded Lipschitz domain, then \( \partial D \) is a \((d - 1)\)-set and the surface measure \( \mu \) has the property (1.4) for \( F = \partial D \) and \( \beta = d - 1 \). Furthermore if \( \partial D \) consists of a finite number of self-similar sets, which satisfies the open set condition, and whose similarity dimensions are \( \beta \), then \( \partial D \) is a \( \beta \)-set such that the \( \beta \)-dimensional Hausdorff measure \( \mathcal{H}^\beta \) restricted to \( \partial D \) has the property (1.4) for \( F = \partial D \) (cf. [Hu]).

Let \( 0 < \alpha \leq 1 \) and \( F \) be a closed set in \( \mathbb{R}^d \). We denote by \( \Lambda_\alpha(F \times [0, T]) \) the Banach space of all continuous functions \( f \) on \( F \times [0, T] \) such that \( f(\cdot, t) \) is \( \alpha \)-Hölder continuous for every \( t \in [0, T] \) with norm

\[
\|f\|_{\infty, \alpha} = \sup_{X \in F \times [0, T]} |f(X)| + \sup_{x, y \in F, x \neq y, t \in [0, T]} \frac{|f(x, t) - f(y, t)|}{|x - y|^\alpha}.
\]

Further let \( 0 < \alpha, \lambda \leq 1 \). We also denote by \( \Lambda_{\alpha, \lambda}(F \times [0, T]) \) the Banach space of all \( f \in \Lambda_\alpha(F \times [0, T]) \) such that \( f \) is \( \lambda \)-Hölder continuous with respect to the time variable with norm

\[
\|f\|_{\infty, \alpha, \lambda} = \|f\|_{\infty, \alpha} + \sup_{x \in F, t, s \in [0, T], t \neq s} \frac{|f(x, t) - f(x, s)|}{|t - s|^\lambda}.
\]

We will prove the following lemma in §3.
Lemma 1.1. Let $d - 1 \leq \beta < d$ and $F$ be a compact $\beta$-set in $\mathbb{R}^d$ satisfying (1.4) and $F \subset B(0,R/2)$. Then there exists a bounded operator $\mathcal{E}$ from $\Lambda_\alpha(F \times [0,T])$ to $\Lambda_\alpha(\mathbb{R}^d)$ with the following properties:

(i) $\mathcal{E}(f)(\cdot,t)$ is a $C^1$-function on $\mathbb{R}^d \setminus F$ for each $t \in [0,T]$, and both of $\mathcal{E}(f)(x,\cdot)$ and $\left(\frac{\partial \mathcal{E}(f)}{\partial x_j}\right)(x,\cdot)$ ($j = 1, \ldots, d$) are measurable for each $x \in \mathbb{R}^d$ and for each $x \in \mathbb{R}^d \setminus F$, respectively,

(ii) $\mathcal{E}(f) = f$ on $F$ and supp $\mathcal{E}(f)(\cdot,t) \subset B(0,2R)$ for each $t \in [0,T]$.

(iii) $\left| \frac{\partial \mathcal{E}(f)}{\partial y_i}(y,s) \right| \leq c\|f\|_{\infty,\alpha}\text{dist}(y,\partial D)^{\alpha-1}$, $\left| \frac{\partial^2 \mathcal{E}(f)}{\partial y_i \partial y_k}(y,s) \right| \leq c\|f\|_{\infty,\alpha}\text{dist}(y,\partial D)^{\alpha-2}$ for every $(y,s) \in (\mathbb{R}^d \setminus F) \times [0,T]$.

(iv) If $f \in \Lambda_\alpha,\lambda(F \times [0,T])$, then $\mathcal{E}(f) \in \Lambda_\alpha,\lambda(\mathbb{R}^d \times [0,T])$.

Using Lemma 1.1 we define, for $f \in \Lambda_\alpha(S_D)$,

\[
\Phi f(X) = \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \langle \nabla \mathcal{E}(f)(y,s), \nabla_y W(X-Y) \rangle dy + \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \mathcal{E}(f)(y,s) \Delta_y W(X-Y) dy
\]

for $X = (x,t) \in D \times \mathbb{R}$ and

\[
\Phi f(X) = -\int_0^T ds \int_D \langle \nabla \mathcal{E}(f)(y,s), \nabla_y W(X-Y) \rangle dy - \int_0^T ds \int_D \mathcal{E}(f)(y,s) \Delta_y W(X-Y) dy
\]

for $X = (x,t) \in (\mathbb{R}^d \setminus \overline{D}) \times \mathbb{R}$.

Furthermore we also define the operator $K$ by

\[
Kf(Z) = \frac{1}{2} (I_1(Z) + I_2(Z)),
\]

where

\[
I_1(Z) = \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} \langle \nabla \mathcal{E}(f)(y,s), \nabla_y W(Z-Y) \rangle dy
\]

\[
+ \int_0^T ds \int_{\mathbb{R}^d \setminus \overline{D}} (\mathcal{E}(f)(Y) - f(Z)) \Delta_y W(Z-Y) dy
\]

\[
+ f(Z) \int_{(\mathbb{R}^d \setminus \overline{D}) \times \{0\}} W(Z-Y) dy
\]
and
\[
I_2(Z) = - \int_0^T ds \int_D \langle \nabla \mathcal{E}(f)(y, s), \nabla y W(Z - Y) \rangle dy \\
- \int_0^T ds \int_D (\mathcal{E}(f)(Y) - f(Z)) \Delta_y W(Z - Y) dy - f(Z) \int_{D \times \{0\}} W(Y - Z) dy
\]

Under these notations we will prove the following theorem in §3.

**Theorem.** Assume that \( D \) is a bounded domain in \( \mathbb{R}^d \) such that \( \partial D \) is a \( \beta \)-set. If \( 0 \leq \beta - (d - 1) < \alpha < 1 \) and \( f \in \Lambda_{\alpha, \alpha/2}(S_D) \), then, for each \( Z \in \partial D \times [0, T] \),

\[
\lim_{X \to Z, X \in D \times (0, T)} \Phi f(X) = Kf(Z) + \frac{1}{2}f(Z)
\]

and

\[
\lim_{X \to Z, X \in (\mathbb{R}^d \setminus \overline{D}) \times (0, T)} \Phi f(X) = Kf(Z) - \frac{1}{2}f(Z).
\]

Thus we see that our double layer heat potentials have the same boundary behavior as the usual ones for a bounded cylinder with smooth lateral boundary.

**Remark.** In this paper we shall treat the double layer heat potentials of Hölder continuous functions on \( S_D \). But under a similar consideration we can also the double layer heat potentials of functions in a Besov space on \( S_D \) and prove that they have the parabolically non-tangential limit at a.e. \( Z \in S_D \).

2. Properties of \( W \)

In this section we recall and study properties of the function \( W \). To do so, we use the parabolic metric \( \delta \) defined by

\[
\delta(X, Y) = (|x - y|^2 + |t - s|)^{1/2} \quad \text{for } X = (x, t) \text{ and } Y = (y, s).
\]

**Lemma 2.1.** (i) \( W(X) \leq c\delta(X, 0)^{-d} \),

(ii) \( |\nabla_x W(X)| \leq c\delta(X, 0)^{-d-1} \) if \( X \neq 0 \),

(iii) \( \left| \frac{\partial^2}{\partial x_i \partial x_j} W(X) \right| \leq c\delta(X, 0)^{-d-2} \), \( \left| \frac{\partial}{\partial t} W(X) \right| \leq c\delta(X, 0)^{-d-2} \) if \( X \neq 0 \),

(iv) \( \left| \frac{\partial^2}{\partial x_i \partial x_j \partial x_k} W(X) \right| \leq c\delta(X, 0)^{-d-3} \), \( \left| \frac{\partial^2}{\partial x_i \partial t} W(X) \right| \leq c\delta(X, 0)^{-d-3} \) if \( X \neq 0 \),

(v) \( |W(X - Y) - W(Z - Y)| \leq c\delta(X, Z) \epsilon \{ \delta(X, Y)^{-d-\epsilon} + \delta(Z, Y)^{-d-\epsilon} \} \)

if \( 0 \leq \epsilon \leq 1 \) and \( X \neq Y, Z \neq Y \),

(vi) \( |\nabla_y W(X - Y) - \nabla_y W(Z - Y)| \leq c\delta(X, Z) \epsilon \{ \delta(X, Y)^{-d-1-\epsilon} + \delta(Z, Y)^{-d-1-\epsilon} \} \)

if \( 0 \leq \epsilon \leq 1 \) and \( X \neq Y, Z \neq Y \).
Proof. The assertions (i), (ii), (iii) and (iv) are well known (cf. [B2, p.5]). The assertions (v) and (vi) will be shown by the same method as in the proof of Lemma 2.3 in [W2].

Let $\Omega_0$ be a bounded piecewise smooth domain in $\mathbb{R}^d$ and $u, v$ be smooth functions on $\overline{\Omega}_0 \times [0, \rho]$. Using the divergence theorem, we obtain

\begin{equation}
\int_0^\rho \int_{\Omega_0} (uL^*v - vLu) \, dx \, dt
= \int_0^\rho dt \int_{\partial \Omega_0} \langle u\nabla_x v - v\nabla_x u, n_x \rangle \, d\sigma(x) - \int_{\Omega_0 \times \{t=0\}} uv \, dx + \int_{\Omega_0 \times \{t=\rho\}} uv \, dx,
\end{equation}

where

$L = \triangle - \frac{\partial}{\partial t}$ and $L^* = \triangle + \frac{\partial}{\partial t}$.

If $Lu = L^*v = 0$ in $\Omega_0 \times (0, \rho)$, then (2.1) implies

\begin{equation}
\int_0^\rho dt \int_{\partial \Omega_0} \langle u\nabla_x v - v\nabla_x u, n_x \rangle \, d\sigma(x) - \int_{\Omega_0 \times \{t=0\}} uv \, dx + \int_{\Omega_0 \times \{t=\rho\}} uv \, dx = 0.
\end{equation}

Let $X = (x, t)$ ($0 \leq t \leq T$) be an exterior point of $\Omega_0 \times (0, T)$. Then, setting $u = 1$ and $v(Y) = W(X - Y)$ and noting that $W(X - Y) = 0$ for $Y = (y, T)$, we deduce from (2.2)

\begin{equation}
\int_0^T ds \int_{\partial \Omega_0} \langle \nabla_y W(X - Y), n_y \rangle \, d\sigma(y) - \int_{\Omega_0 \times \{s=0\}} W(X - Y) \, dy = 0.
\end{equation}

Hereafter we assume that $D$ is a bounded domain in $\mathbb{R}^d$ such that $\partial D$ is a $\beta$-set satisfying $\overline{D} \subset B(O, R/2)$.

Let us use the Whitney decomposition to approximate $D$ and $\mathbb{R}^d \setminus \overline{D}$ (cf. [S, p.167]). Let $\mathcal{W}(D)$ be the Whitney decomposition of $D$ and define

\[ A_n = \bigcup_{k=k_0}^n \bigcup_{Q \in \mathcal{W}_k(D)} Q, \]

where $\mathcal{W}_k(D) = \{Q \in \mathcal{W}(D); Q \text{ is a } k\text{-cube}\}$ and $k_0$ is the smallest integer $k$ such that $\mathcal{W}_k(D) \neq \emptyset$.

Similarly we also define

\[ B_n = \left( \bigcup_{k=-\infty}^n \bigcup_{Q \in \mathcal{W}_k(\mathbb{R}^d \setminus \overline{D})} Q \right). \]

Then we have the following lemma.
Lemma 2.2. Set 

\[ g_n(X) = \int_0^T \int_{A_n} \Delta_y W(X - Y) dy \quad \text{and} \quad h_n(X) = \int_0^T \int_{B_n} \Delta_y W(X - Y) dy. \]

Then \( \lim_{n \to \infty} g_n(X) \) and \( \lim_{n \to \infty} h_n(X) \) exist on \( \mathbb{R}^d \times [0, T] \) and for \( X \in \mathbb{R}^d \times (0, T] \)

\[ \lim_{n \to \infty} g_n(X) = \int_{D \times \{0\}} W(X - Y) dy - \chi_D(X) \]

and

\[ \lim_{n \to \infty} h_n(X) = \int_{(\mathbb{R}^d \setminus \overline{D}) \times \{0\}} W(X - Y) dy - \chi_{\mathbb{R}^d \setminus \overline{D}}(X). \]

Proof. Let \( X = (x, t) \in \mathbb{R}^d \times (0, T] \) and \( t > \rho > 0 \). Applying (2.2) to \( A_n \times (0, \rho) \), we have

\[ \int_0^\rho ds \int_{\partial A_n} \langle \nabla_y W(X - Y), n_y \rangle d\sigma(y) - \int_{A_n \times \{0\}} W(X - Y) dy + \int_{A_n \times \{\rho\}} W(X - Y) dy = 0. \]

Using the divergence theorem for \( A_n \) in \( \mathbb{R}^d \), we have

\[ \int_0^\rho ds \int_{A_n} \Delta_y W(X - Y) dy - \int_{A_n \times \{0\}} W(X - Y) dy + \int_{A_n \times \{\rho\}} W(X - Y) dy = 0. \]

As \( \rho \to t \) and \( n \to \infty \), we obtain,

\[ \lim_{n \to \infty} g_n(X) = \int_{D \times \{0\}} W(X - Y) dy - \chi_D(X). \]

On the other hand \( g_n(X) = 0 \) for \( t = 0 \). Hence \( \lim_{n \to \infty} g_n(X) \) exists for each \( X \in \mathbb{R}^d \times [0, T] \).

Similarly we can also prove the conclusion for \( h_n \).

\[ \square \]

3. Double layer heat potentials

In this section we first prove Lemma 1.1 in §1.
Proof of Lemma 1.1 We use the extension operator $\mathcal{E}_{0}$ in [S, p.172] and choose a $C^{\infty}$-function $\phi_{0}$ such that

$$\phi_{0} = 1 \text{ on } B(0, R), \quad \text{supp } \phi_{0} \subset B(0, 2R) \quad \text{and} \quad 0 \leq \phi_{0} \leq 1.$$ 

We define

$$\mathcal{E}(f)(x,t) = \mathcal{E}_{0}(f(\cdot, t))(x)\phi_{0}(x)$$

for $f \in \Lambda_{\alpha}(F)$ and $(x,t) \in (\mathbb{R}^{d} \setminus F) \times [0,T]$ and

$$\mathcal{E}(f)(x,t) = f(x,t) \quad \text{on } (x,t) \in F \times [0,T].$$

Then properties (i), (ii), (iii) follow from the definition and (13) on p.174 in [S]. Since the operator $\mathcal{E}_{0}$ is linear, positive and maps the constant function 1 to 1, (iv) is also valid.

In [W1] we gave the following lemma.

**Lemma A.** Let $\delta, k$ be non-negative numbers satisfying $d-\beta > \delta$ and $d-\delta-k > 0$. Then

$$\int_{B(z,r)} \text{dist}(y, \partial D)^{-\delta} |y-z|^{-k} dy \leq cr^{d-\delta-k}$$

for every $z \in \partial D$ and $r > 0$.

We next show that the double layer heat potential defined by (1.5) and (1.6) converges.

**Lemma 3.1.** Let $0 \leq \beta - (d-1) < \alpha < 1$ and $f \in \Lambda_{\alpha}(S_{D})$. Then $\Phi f$ is caloric in $(\mathbb{R}^{d} \setminus \partial D) \times \mathbb{R}$.

**Proof.** Set, for $X = (x,t) \in D \times \mathbb{R}$,

$$(3.1) \quad J_{1}(X) = \int_{0}^{T} ds \int_{\mathbb{R}^{d} \setminus \overline{D}} \langle \nabla_{y} \mathcal{E}(f)(y,s), \nabla_{y} W(X-Y) \rangle dy$$

and let $X_{0} = (x_{0},t_{0}) \in D$. Choose $\rho > 0$ satisfying $\overline{B(x_{0},2\rho)} \subset D$. If $X = (x,t) \in B(x_{0},\rho) \times \mathbb{R}$, then we deduce from Lemmas 2.1 and 1.1 and Lemma A

$$|J_{1}(X)| \leq \int_{0}^{T} ds \int_{\mathbb{R}^{d} \setminus \overline{D}} \text{dist}(y, \partial D)^{\alpha-1} \delta(X,Y)^{-1-d} dy \leq c_{1} \rho^{-1-d} \|f\|_{\infty,\alpha}$$

whence $J_{1}$ converges locally uniformly in $D$. We denote by $g_{1}$ the integrand of the right-hand side on (3.1). Since

$$|\nabla_{y} \frac{\partial^{2}}{\partial x_{i} \partial x_{j}} W(X-Y)| \leq c_{2} \delta(X,Y)^{-d-3} \text{ and } |\nabla_{y} \frac{\partial}{\partial t} W(X-Y)| \leq c_{3} \delta(X,Y)^{-d-3},$$

we deduce from the above integrals that $J_{1}$ converges locally uniformly in $D$.
we see that the integral of $Lg_1$ over $(\mathbb{R}^d \backslash \overline{D}) \times [0,T]$ also converges locally uniformly on $D$. Therefore $J_1$ satisfies the heat equation in $D \times \mathbb{R}$.

Next, set

$$J_2(X) = \int_0^T ds \int_{\mathbb{R}^d \backslash \overline{D}} \mathcal{E}(f)(y,s) \Delta W(X-Y)dy.$$ 

Using Lemma 1.1, (iii), we can show by the above method that $J_2$ also converges locally uniformly in $D$ and satisfies the heat equation. Thus we conclude that $\Phi f = J_1 + J_2$ has the same properties in $D \times \mathbb{R}$. We can show that $\Phi f$ also has the same properties in $(\mathbb{R}^d \backslash \overline{D}) \times \mathbb{R}$. \qed

Using Lemma 2.1, (iv), (v) and Lemma A, we can prove the following lemma by a similar method to that in the proof of [W1, Lemma 3.3].

**Lemma 3.2.** Let $0 \leq \beta - (d - 1) < \alpha < 1$ and $f \in \Lambda_{\alpha, \alpha/2}(S_D)$. Then both of the function $J_1$ defined by (3.1) and the function $J_3$ defined by

$$J_3(X) = \int_0^T ds \int_{\mathbb{R}^d \backslash \overline{D}} (\mathcal{E}(f)(Y) - \mathcal{E}(f)(X)) \Delta W(X-Y)dy$$

are are continuous on $\mathbb{R}^d \times [0,T]$. Furthermore the function $J_1'$ (resp. $J_3'$) obtained by replacing $\mathbb{R}^d \backslash \overline{D}$ with $D$ in the definition of $J_1$ (resp. $J_3$) is also continuous on $\mathbb{R}^d \times [0,T]$.

**Lemma 3.3.** Let $0 \leq \beta - (d - 1) < \alpha < 1$ and $g \in \Lambda_{\alpha, \alpha/2}(\mathbb{R}^d \times [0,T])$ such that $g(\cdot, t) \in C^1(\mathbb{R}^d)$, $\text{supp} \ g(\cdot, s) \subset B(0, r_0)$ for every $t \in [0,T]$ and $\frac{\partial g}{\partial x_j}(x, \cdot)$ is bounded for every $x \in \mathbb{R}^d$. Let $X=(x,t) \in \mathbb{R}^d \times (0,T]$ and set, for $0 < \rho \leq T$,

$$A_{\rho}g(X) = \int_0^\rho ds \int_{\mathbb{R}^d \backslash \overline{D}} \langle \nabla g(Y), \nabla_y W(X-Y) \rangle dy$$

$$+ \int_0^\rho ds \int_{\mathbb{R}^d \backslash \overline{D}} (g(Y) - g(X)) \Delta_y W(X-Y)dy$$

$$+ g(X) \int_{(\mathbb{R}^d \backslash \overline{D}) \times \{0\}} W(X-Y)dy$$

and

$$B_{\rho}g(X) = - \int_0^\rho ds \int_D \langle \nabla g(Y), \nabla_y W(X-Y) \rangle dy$$

$$- \int_0^\rho ds \int_D (g(Y) - g(X)) \Delta_y W(X-Y)dy$$

$$- g(X) \int_{D \times \{0\}} W(X-Y)dy.$$
Then

\[ A_T g(X) = B_T g(X) + g(X) \text{ for } X \in \mathbb{R}^d \times (0, T] \]

**Proof.** To simplify the notation, we use \( A_\rho(x) \) and \( B_\rho(X) \) instead of \( A_\rho g(X) \) and \( B_\rho g(X) \), respectively. We first show (3.2) in case \( D = D_0 \) is a bounded piecewise smooth domain. Let \( X = (x, t) \) and set, for \( 0 < \rho < t \),

\[ I_\rho(X) = - \int_0^\rho ds \int_{\partial D_0} g(Y) \langle \nabla_y W(X - Y), n_y \rangle d\sigma(y). \]

The Green formula for \( D_0 \) yields

\[ I_\rho(X) = - \int_0^\rho ds \int_{\partial D_0} \langle \nabla g(Y), \nabla_y W(X - Y) \rangle dy \]

\[ - \int_0^\rho ds \int_{D_0} (g(Y) - g(X)) \triangle_y W(X - Y) dy \]

\[ - g(X) \int_0^\rho ds \int_{D_0} \triangle_y W(X - Y) dy \]

From (2.2) we deduce

\[ \int_0^\rho ds \int_{D_0} \triangle_y W(X - Y) dy \]

\[ = \int_{(\mathbb{R}^d \setminus \partial D_0) \times (0, \rho]} W(X - Y) dy - \int_{D_0 \times \{0\}} W(X - Y) dy, \]

whence

\[ \int_0^t ds \int_{D_0} \triangle_y W(X - Y) dy = \int_{D_0 \times \{0\}} W(X - Y) dy - \chi_{D_0}(x). \]

This and (3.3) imply

\[ I_t(X) = B_t(X) + g(X) \chi_{D_0}(x) \text{ for } X \in (\mathbb{R}^d \setminus \partial D_0) \times (0, T]. \]

Similarly, using the Green formula for \( B(0, r) \setminus \overline{D}_0 \) and \( r \to \infty \), we obtain

\[ I_t(X) = A_t(X) - g(X) \chi_{\mathbb{R}^d \setminus \overline{D}_0}(x) \]

for \( X \in (\mathbb{R}^d \setminus \partial D_0) \times (0, T] \). This and (3.4) lead to

\[ A_t(X) = B_t(X) + g(X) \text{ for } X \in (\mathbb{R}^d \setminus \partial D_0) \times (0, T]. \]
Noting that $A_t(X) = A_T(X)$ and $B_t(X) = B_T(X)$, we obtain (3.2) for $X \in (\mathbb{R}^d \setminus \partial D) \times (0, T]$. Since $A_T$ and $B_T$ are continuous on $\mathbb{R}^d \times (0, T]$ by Lemma 3.2, (3.2) holds for a bounded piecewise smooth domain $D = D_0$.

We next show (3.2) for a bounded domain such that $\partial D$ is a $\beta$-set. We use (3.2) for $D_0 = A_n$. Since $A_T$ and $B_T$ are continuous on $\mathbb{R}^d \times (0, T]$ by Lemma 3.2, (3.2) holds for a bounded piecewise smooth domain $D = D_0$.

Since $\int_0^T \int_{\mathbb{R}^d} |\nabla g(Y)||\nabla_y W(X-Y)|dyds < \infty$, $\int_0^T ds \int_{\mathbb{R}^d} \|g(Y) - g(X)||\triangle W(X-Y)|dy < \infty$ and $\int_{\mathbb{R}^d \times \{0\}} W(X-Y)dy < \infty$, we see that (3.2) holds for the domain $D$ as $n \to \infty$. \hfill \Box

Lemma 3.4. Let $0 \leq \beta - (d-1) < \alpha$ and $f \in \Lambda_{\alpha,\alpha/2}(S_D)$. Then (3.2) holds for $g = \mathcal{E}(f)$.

Sketch of Proof. Let $f \in \Lambda_{\alpha,\alpha/2}(S_D)$ and $\{v_m\}$ be a mollifier on $\mathbb{R}^d$ such that $\supp v_m \subset B(0, 1/m)$. We define, for $Y = (y, s) \in \mathbb{R}^d \times [0, T]$,

$$g_m(Y) = (\mathcal{E}(f)(\cdot, s) * v_m)(y).$$

Lemma 3.3 yields

$$A_Tg_m(X) = B_Tg_m(X) + g_m(X) \text{ for } X \in \mathbb{R}^d \times (0, T].$$

Using $g_m(X) \to \mathcal{E}(f)(X)$ uniformly as $m \to \infty$ and Lemmas A, 1.1 and 2.1, we can show that

$$A_Tg_m(X) \to A_T\mathcal{E}(f)(X)$$

and

$$B_Tg_m(X) \to B_T\mathcal{E}(f)(X)$$

for $X \in \mathbb{R}^d \times [0, T]$ as $m \to \infty$. \hfill \Box

We can also show the following lemma.

Lemma 3.5. Let $0 \leq \beta - (d-1) < \alpha < 1$. Then the operator $K$ defined by (1.7) is a bounded operator from $\Lambda_{\alpha,\alpha/2}(S_D)$ to $\Lambda_{\alpha,\alpha/2}(S_D)$.

Let us prove our theorems.
Proof of Theorem. Let $X \in D \times (0, T]$. Using Lemma 2.2, we have $\Phi f(X) = A_T f(X)$. Since $A_T f$ is continuous on $\mathbb{R}^d \times (0, T]$ by Lemma 3.2, we have

$$\lim_{X \to Z, X \in D \times (0, T)} \Phi f(X) = A_T f(Z).$$

On the other hand Lemma 3.4 yields

$$K f(Z) = \frac{1}{2} (A_T f(Z) + B_T f(Z)) = A_T f(Z) - \frac{1}{2} f(Z).$$

Therefore we have (1.8). Similarly we can show (1.9). \hfill \square

References


