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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1998-04: 99-101</td>
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<tr>
<td>Issue Date</td>
<td>1998-04</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62094">http://hdl.handle.net/2433/62094</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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A transversality condition for quadratic family at Collet-Eckmann parameter

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January 6, 1998

We consider real quadratic maps $Q_t : \mathbb{R} \rightarrow \mathbb{R}, \ x \mapsto t - x^2$, where $t \in \mathbb{R}$ is a parameter. We say that $Q_t$ satisfies Collet-Eckmann condition if

$$
\liminf_{n \rightarrow \infty} \sqrt[n]{|DQ_t^n(Q_t(0))|} > 1.
$$

This condition implies that the dynamics of $Q_t$ is 'chaotic' (existence of absolutely continuous invariant measure, decay of correlation, etc.). We give

**Theorem 1** If $Q_t$ satisfies Collet-Eckmann condition, then

$$
\lim_{n \rightarrow \infty} \frac{\partial}{\partial s} [Q_s^n(0)]_{s=t} \quad > 0.
$$

(1)

In a sense, the condition (1) implies that the quadratic family is transversal to the "manifold" of the maps which is topologically conjugate to $Q_t$.

Combining theorem 1 with Jacobson's theorem [2], we get

**Proposition 2** Let $A$ be the set of parameters $t$ for which $Q_t$ satisfies Collet-Eckmann condition and

$$
\liminf_{n \rightarrow \infty} n^{-1} \log |DQ_t^n(Q_t^n(0))| = 0.
$$

(2)

Then every point in $A$ is a density point of $A$ itself in the interval $[0, 2]$. 
Remark that $A$ contains $t = 2$. The condition (2) holds if the critical point 0 is not recurrent.

We prove theorem 1 as follows. Take $r > 1$ such that

$$\lim_{n \to \infty} \inf \sqrt[n]{|DQ_{t}^{n}(Q_{t}(0))|} > r > 1.$$ 

We consider $Q_{t}$ as a map from the complex plain to itself. Let $A$ be a Ruelle operator $A$ on the quadratic differentials:

$$A(\varphi)(x) = \sum_{Q_{t}(y) = x} \frac{\varphi(y)}{[DQ_{t}(y)]^{2}},$$

acting on the space

$$S = \left\{ x = \sum_{i=1}^{\infty} x_{i}\psi_{i} \mid \sum_{i} |x_{i}DQ_{t}^{i}(Q(0))|r^{-i} < \infty \right\}$$

where $\psi_{i}(z) = (z - Q_{i}(0))^{-1}$. We endowe $S$ with a norm

$$|x| = \sum_{i} |x_{i}DQ_{t}^{i}(Q(0))|r^{-i}.$$ 

Then we have, formally,

$$\lim_{n \to \infty} \frac{\partial}{\partial s} Q_{s}^{n}(0)\big|_{s=t} = \det(\text{Id} - A).$$

Comparing $A$ with the Perron-Frobenius operator, we see that the spectral radius of $A$ is smaller than 1. Hence, if $A$ were a finite-dimensional operator, these would imply (1). Actually, we can’t give any appropriate definition for the determinant in (3) since $A$ is an infinite dimensional operator. Instead, we approximate $A$ by a sequence of finite-dimensional operators. For detail, see [3].

References
