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Dynamics of skew tent maps

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1 Introduction

We consider skew tent maps defined by

$$\begin{aligned} f_{a,b}(x) &= \begin{cases} ax+1, & (x \leq 0) \\ -bx+1, & (x \geq 0) \end{cases} \\ (a,b) &\in D := \{(a,b) : a>0, b>1, a+b \geq ab\}. \end{aligned}$$

By the dynamical behavior of $f_{a,b}$, the parameter domain D is divided into subdomains D_k defined by some algebraic curves : $D=\sum_{k=2}^{\infty} D_k$. In each D_k there are subdomains D_k^A where $f_{a,b}$ has unique attracting periodic orbit of period k and D_k^B where $f_{a,b}$ has $2k$ or k chaotic intervals, which is analyzed by the method of renormalization ([Ich][Ito]). In this paper we give the relation between $\star$-product for kneading sequence and renormalization as Theorem 1. We also give an explicit proof of Theorem A in [MV91], which states monotonicity of kneading sequence of this family. Moreover we correct Corollary of Theorem 2 in [MV92] as Proposition 4.

We denote kneading sequence of $f_{a,b}$ by $K(a,b)$ and topological entropy of $f_{a,b}$ by $h(a,b)$. We refer basic definitions and notations from [Ich][CE80].

2 Renormalization and $\star$-product

We denote $f_{a,b}$ by f in this section. For getting maximal level of renormalization, we assume sequence $\underline{B}$ is prime. Let $|\underline{A}|$ be the length of sequence $\underline{A}$ and $int(J)$ interior of an interval J. For the definitions of renormalization and $\star$-product, see [Ito] and [CE80] respectively.

Definition $\underline{S}$ is called *prime* if $\underline{S}$ does not have any finite sequence $\underline{A}\,(\neq 0)$ of L's and R's and any finite or infinite sequence $\underline{B}\,(\neq C)$ such that $\underline{S}=\underline{A}\star\underline{B}$.

Theorem 1 $K(a,b) = \underline{A} \star \underline{B}$ where $\underline{A}(\neq \emptyset)$ is finite sequence of L's and R's and $\underline{B}(\neq C)$ is prime if and only if there exist invariant closed intervals $\{J_i\}_{i=0,\cdots,|\underline{A}|}$ such that $J_{|\underline{A}|} \ni 0$, $fJ_i = J_{i+1}\,(i = 0,\cdots,|\underline{A}|-1)$, $fJ_{|\underline{A}|} = J_0$ and $int(J_i) \cap int(J_{i'}) = \emptyset\,(i \neq i')$. f can not have any refinement of $\{J_i\}$.

Proof Assume $K(a,b) = \underline{A} \star \underline{B}$ where $\underline{A}$ $(\neq \emptyset)$ is finite sequence of L's or R's and $\underline{B}(\neq C)$ is prime. Set $x_n = f^n(1)\,(n \geq 0)$, $p = |\underline{A}|$ and $\underline{A} = A_0A_1\cdots A_{p-1}$. Let J_i be convex hull of $\{x_{i+k(p+1)} : k = 0, 1, \cdots\}$ for $i = 0,\cdots,p$. Then, we have $fJ_i = J_{i+1}\,(i = 0,\cdots,p-1)$ and f is monotone on each J_i except of $i = p$. Remark that f^{p+1} on each J_i has same slopes. It follows that $f^{p+1}|_{J_i} \sim f^{p+1}|_{J_{i'}}\,(i \neq i')$. We consider the following two cases.

The first case : $\underline{B}$ does not contain both L and R.

$\underline{B}$ is finite in this case. It follows that J_p contains a turning point 0 as an end point of it. Hence, $fJ_p = J_0$. As f is monotone on J_i for all $i\,(0 \leq i \leq p)$, f^{p+1} restricted on J_i is monotone and surjective on J_i. Hence, its slope is -1. Then $\{J_i\}$'s are disjoint or there would exist some i, i' such that $J_i = J_{i'}$ from continuity of f. The latter can not occur
because of the assumption of $\underline{A}$. Therefore $\{J_i\}$ are disjoint. Notice that the first case corresponds to boundary curve of D_k^A and D_k^B.

The second case : $\underline{B}$ contains both L and R.

In this case $fJ_p = J_0$ and $f^{p+1}|_{J_i}$ has unique turning point c_i inside J_i. We set two slopes of $f^{p+1}|_{J_i}$ $\alpha(>0)$, $\beta(<0)$. We divide J_i into two subinterval I_{α_i} and I_{β_i} corresponding to slope α and β. As $f^{p+1}|_{J_i}$ is surjective on J_i, we have that $\sup\{|\alpha|, |\beta|\} > 1$.

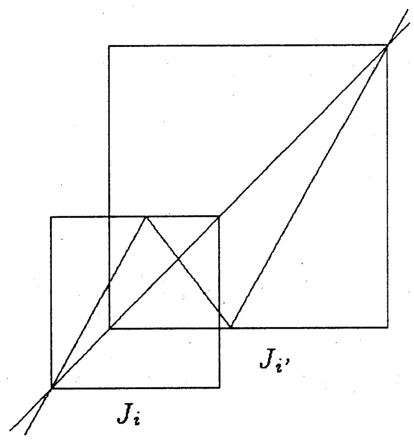

Figure 1: The graph of f^{p+1} on $J_{i'}$ having two turning points.

If $|\beta| < 1$, then turning point is attracted to a fixed point on I_β. It follows that $\underline{B} = L^\infty$ or R^∞. This contradicts assumption of this case.

If $|\beta| = 1$, we reduce to the first case.

If $|\beta| > 1$ and $int(J_i) \cap int(J_{i'}) \neq \emptyset$, there exists $J_{i'}$ such that $f^{p+1}|_{J_{i'}}$ has two turning points (see Figure 1) or there exist i, i' such that $J_i = J_{i'}$. In the latter case we have J_m equals J_p for some $m\,(m \neq p)$. This contradicts that f is monotone on J_m because J_p includes turning point in it. Hence we obtain $int(J_i) \cap int(J_{i'}) = \emptyset$. Notice that the second case corresponds to D_k^B.

Conversely, if there exist disjoint invariant closed intervals $\{J_i\}_{i=0,\cdots,|\underline{A}|}$ in theorem, we have $K(a,b) = \underline{A} \star \underline{B}$ with $\underline{A} = A_{J_0} A_{J_1} \cdots A_{J_{p-1}}$. If $\underline{B}$ is not prime, f has refinement of $\{J_i\}$. Hence, $\underline{B}$ is prime. □

Now we have the relation of our renormalization (i.e., $(|\underline{A}|+1)$ - renormalization is a skew tent map of D) and $\star$-product.

Corollary 1 If $|\underline{B}| \neq 2$ in above theorem, then f is renormalizable of level $|\underline{A}| + 1$.

Proof Let p be $|\underline{A}|$. In the first case, we have $|\underline{B}| = 2$ because a turning point of f on J_p is 2-periodic point of f^{p+1}. In the second case, we have $|\beta| > 1$ and $f^{p+1} J_i = J_i$. It follows $(\alpha, \beta) \in D$. Therefore f is $(p+1)$-renormalizable on $[c_i, f^{p+1}(c_i)]$ (resp. $[f^{p+1}(c_i), c_i]$) if $c_i < f^{p+1}(c_i)$ (resp. $f^{p+1}(c_i) < c_i$). □

It is well known that for a smooth unimodal map g, n-periodic g-admissible sequence implies the existence of n or $2n$-periodic point ([Dev89]). This fact is proved by Schwarzian derivative. But we have the following analogous fact for skew tent maps.

Corollary 2 If $K(a,b) = \underline{A} \star \underline{B}$ where $\underline{A}\,(\neq \emptyset)$ is finite sequence of L's and R's and $\underline{B}\,(\neq C)$ is prime, then f has periodic points of period $|A| + 1$. Moreover if $|\underline{B}| = 2$, then f also has periodic points of period $2(|A| + 1)$.

Remark For showing Corollary 1 and 2, we need only the assumption $\underline{B} \neq C, L^\infty, R^\infty$ instead of primarity of $\underline{B}$.

3 Monotonicity of kneading sequences

In this section we will mention the monotonicity property of kneading sequence in the domain

$$\tilde{D} = \{(a,b) \in D;\, a \geq 1\}.$$

Let us define the order for parameter pairs as follows, according to M. Misiurewicz and E. Visinescu [MV91] :

$(a,b) \succ (a',b') \Leftrightarrow a' \geq a,\ b' \geq b,$ and at least one of these inequalities is strict.

Kneading sequences are monotone increasing with respect to this order.

Monotonicity Theorem (Theorem A in [MV91]) For (a',b'), (a,b) in $\tilde{D}$ with $(a',b') \succ (a,b)$, it holds that $K(a',b') > K(a,b)$.

This theorem is already proved in [MV91]. M. Misiurewicz and E. Visinescu showed the claim by using the estimation of topological entropy. But we shall reprove it by using only thier results for $D^\star$ in [MV91], and renormalization method, not via the topological entropy. For that purpose, we prepare Proposition 1, Proposition 2 and Proposition 3 (for the detailed proofs, see [Ich]).

As to $\star$-*product*, we have the following.

Proposition 1 Let $\underline{A}$ and $\underline{B}$ be symbolic sequences of L, R, and C with $\underline{A} \succ \underline{B}$. Then for all $n \geq 1$, $R^{\star n} \star \underline{A} \succ R^{\star n} \star \underline{B}$.

3.1 Monotonicity in $D^\star$

Let $D^\star$ be the domain $\{(a,b) \in D\,;\, a+b < ab^2, a > 1\}$.

M. Misiurewicz and E. Visinescu proved in [MV91] that $K(a',b') > K(a,b)$ for (a',b'), $(a,b) \in D^\star$ such that $(a',b') \succ (a,b)$. This domain $D^\star$ is characterlized by the following.

Fact 1 (Lemma 2.1 in [MV91]) $(a,b) \in D^\star \Leftrightarrow K(a,b) \succ RLR^\infty$.

First, monotone increasing property of kneading sequence is proved in $D^\star$.

Fact 2 (Proposition 4.3 in [MV91]) If (a,b) and (a',b') are in $D^\star$ with $(a,b) \prec (a',b')$, then $K(a,b) \prec K(a',b')$.

3.2 Renormalization and $\star$-product (for $\tilde{D}$)

Proposition 2 Let (a,b) be in D. The following three conditions are equivalent mutually.

(i) $(a,b) \in D_0$.

(ii) There exists a unique number $m \geq 1$ and a prime sequence $\underline{B}$ whose length is longer than 2 such that $K(a,b) = R^{\star m} \star \underline{B}$.

(iii) There exists some number $m \geq 1$ such that $\varphi^m(a,b) \in D^\star$, where $\varphi(a,b) = (b^2, ab)$.

Furthermore, there exist closed subintervals of $I_{a,b}$, $\{I_i\}_{i=0,\cdots,2^m-1}$ such that their interiors are disjoint mutually, $f_{a,b}I_i = I_{i+1}$ for $0 \leq i \leq 2^m - 2$ and $f_{a,b}I_{2^m-1} = I_0$, $I_{2^m-1} \ni 0$, and $f_{a,b}^{2m}|_{I_i} \sim f_{\varphi^m(a,b)}$.

Proposition 3 Let $(a,b),(a',b') \in \tilde{D} \setminus D^\star$ such that $(a,b) \prec (a',b')$. If $\varphi^m(a,b) \in D^\star$ and $\varphi^n(a',b') \in D^\star$, then $m \geq n$.

3.3 Proof of Monotonicity Theorem

Assume that $(a,b) \prec (a',b')$.

(i) If both (a,b) and (a',b') belong to $D^\star$, then the proof is already given by Fact 2.

(ii) Assume that either (a,b) or (a',b') belongs to $D^\star$. Then (a',b') is in $D^\star$ because $(a,b) \prec (a',b')$. By virtue of Fact 1, $K(\varphi^n(a,b)) \preceq RLR^\infty \prec K(\varphi^n(a',b'))$. We have that $K(a,b) \prec K(a',b')$ as an order relation "$\prec$" is total.

(iii) Assume that (a,b) and (a',b') both belong to $\tilde{D} \setminus D^\star$. Then, by Proposition 3, their kneading sequences are written as, for some $n \leq m$,

$$K(a,b) = R^{\star m} \star K(\varphi^m(a,b)) \quad \text{and} \quad K(a',b') = R^{\star n} \star K(\varphi^n(a,b)).$$

If $m = n$, then we have that $\varphi^n(a,b) \prec \varphi^n(a',b')$ since φ is an increasing function. Because $K(\varphi^n(a,b)) \prec K(\varphi^n(a',b'))$ and from Proposition 1, we have that $K(a,b) \prec K(a',b')$.

If $n < m$, then we have that $\varphi^n(a,b) \neq D^\star$ and $\varphi^n(a',b') \in D^\star$. By virtue of Fact 1, it follows that

$$K(\varphi^n(a,b)) \preceq RLR^\infty \prec K(\varphi^n(a',b')).$$

By Proposition 1, we have that $K(a,b) \prec K(a',b')$. □

4 Renormalization and topological entropy

Now we correct two statements of [MV92].

First : **kneading sequence for boundary curve of** $A_m(= D^A_{m+1})$ **and** $B_m(= D^B_{m+1})$.

In Theorem 1 of the paper [MV92], they say ;

$$(\lambda, \beta)(= (a, b)) \in A_m \Leftrightarrow K(\lambda, \beta) = (RL^m)^\infty,$$

$$(\lambda, \beta) \in B_m \Leftrightarrow K(\lambda, \beta) = RL^{m-1} \star \underline{B} \text{ with } \underline{B} \in M$$

where M is set of kneading sequence for tent map $f_{\lambda,\lambda}$ $(1 < \lambda \leq 2)$.

A_m and B_m have common boundary curve : $\lambda^m \mu = 1$. In our opinion this curve should be discussed separately from A_m and from B_m. We find our reason in the fact that the kneading sequence on this curve is $RL^m RL^{m-1}C$, not admitted by one on A_m and on B_m.

Second : **topological entropy of** $B_1(= D_0)$ **is not constant.**

In Corollary in [MV92], they say ;

$$\text{let } (\lambda, \beta), (\lambda', \beta') \in \{(\lambda, \beta) \in D\,;\, \lambda \leq 1\} \text{ such that } (\lambda, \beta) < (\lambda', \beta'),$$

$$(\lambda, \beta), (\lambda', \beta') \in A_m \cup B_m \;\Rightarrow\; h(\lambda, \beta) = h(\lambda', \beta').$$

Namely, topological entropy on B_m is constant for all $m\,(\geq 1)$. But we can show the followings :

Proposition 4 Let (λ, β), $(\lambda', \beta') \in \{(\lambda, \beta) \in D\,;\, \lambda \leq 1\}$. If $(\lambda, \beta) < (\lambda', \beta')$,

$$h(\lambda, \beta) < h(\lambda', \beta').$$

Proof From [MT88] we obtain that topological entropy of $f_{a,b}$ for B_1 naturally follows from one of its renormalized map of subdomain $a \geq 1$ where the strictly monotonicity holds. □

A counter example to this statement is given in [Ich].

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