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Dynamics of Sub-hyperbolic and Semi-hyperbolic Rational Semigroups and Conformal Measures of Rational Semigroups

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Abstract

We consider dynamics of semigroups of rational functions on Riemann sphere. First, we will define hyperbolic rational semigroups and show the metrical property. We will also define sub-hyperbolic and semi-hyperbolic rational semigroups and show no wandering domain theorems. By using these theorems, we can show the continuity of the Julia set with respect to the perturbation of the generators. By using a method similar to that in [Y], we can show that if a finitely generated rational semigroup is semi-hyperbolic and satisfies the open set condition with the open set O satisfying $\sharp(\partial O\cap J(G))<\infty$, then 2-dimensional Lebesgue measure of the Julia set is equal to 0.

Next, we will consider constructing δ-subconformal measures. If a rational semigroup has at most countably many elements, then we can construct δ-subconformal measures. We will see that if a finitely generated rational semigroup is semi-hyperbolic, then the Hausdorff dimension of the Julia set is less than the exponent δ.

Considering conformal measures in a skew product, with a method of the thermodynamical formalism, we can get another upper estimate of the Hausdorff dimension of the Julia sets of finitely generated expanding semigroups.

In more general cases than the cases in which semigroups are hyperbolic or satisfy the strong open set condition, we can construct generalized Brolin-Lyubich's invariant measures or self-similar measures in the Julia sets and can show the uniqueness. We will get a lower estimate of the metric entropy of the invariant measures. With these facts and a generalization of Mañé's result, we get a lower estimate of the Hausdorff dimension of any finitely generated rational semigroups such that the backward images of the Julia sets by the generators are mutually disjoint.

For a Riemann surface S, let $\mathrm{End}(S)$ denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of functions. A *rational semigroup* is a subsemigroup of $\mathrm{End}(\overline{\mathbb{C}})$ without any constant elements.

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Definition 0.1. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\},\ J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the *Fatou set* for G and $J(G)$ is called the *Julia set* for G.

$J(G)$ is backward invariant under G but not forward invariant in general. If $G = \langle f_1, f_2, \dots f_n \rangle$ is a finitely generated rational semigroup , then $J(G)$ has the *backward self-similarity.* That is, we have $J(G) = \cup_{i=1}^{n} f_i^{-1}(J(G))$. The Julia set of any rational semigroup is a perfect set, backward orbit of any point of the Julia set is dense in the Julia set and the set of repelling fixed points of the semigroup is dense in the Julia set. For more detail about these properties, see [ZR], [GR], [HM1], [HM2], [S1] and [S2].

1 Sub-hyperbolic and Semi-hyperbolic Rational Semigroups

Definition 1.1. Let G be a rational semigroup. We set

$$P(G) = \overline{\bigcup_{g \in G} \{ \text{ critical values of } g\}}.$$

We call $P(G)$ the post critical set of G. We say that G is *hyperbolic* if $P(G) \subset F(G)$. Also we say that G is *sub-hyperbolic* if $\sharp\{P(G) \cap J(G)\} < \infty$ and $P(G) \cap F(G)$ is a compact set.

Theorem 1.2 ([S4]). *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated hyperbolic rational semigroup . Assume that G contains an element with the degree at least two and each Möbius transformation in G is neither the identity nor an elliptic element. Let K be a compact subset of $\overline{\mathbb{C}} \setminus P(G)$. Then there are a positive number c, a number $\lambda > 1$ and a conformal metric ρ on an open subset V of $\overline{\mathbb{C}} \setminus P(G)$ which contains $K \cup J(G)$ and is backward invariant under G such that for each k*

$$\inf\{\|(f_{i_k} \circ \dots \circ f_{i_1})'(z)\|_\rho \mid z \in (f_{i_k} \circ \dots \circ f_{i_1})^{-1}(K), (i_k, \dots, i_1) \in \{1, \dots, n\}^k\}$$

$\geq c\lambda^k$, here we denote by $\| \cdot \|_\rho$ the norm of the derivative measured from the metric ρ to it.

Now we will show the converse of Theorem 1.2.

Theorem 1.3 ([S4]). *Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup. If there are a positive number c, a number $\lambda > 1$ and a conformal metric ρ on an open subset U containing $J(G)$ such that for each k*

$$\inf\{\|(f_{i_k} \circ \dots \circ f_{i_1})'(z)\|_\rho \mid z \in (f_{i_k} \circ \dots \circ f_{i_1})^{-1}(J(G)), (i_k, \dots, i_1) \in \{1, \dots, n\}^k\}$$

$\geq c\lambda^k$, where we denote by $\| \cdot \|_\rho$ the norm of the derivative measured from the metric ρ on V to it, then G is hyperbolic and for each $h \in G$ such that $\deg(h)$ *is one the map h is not elliptic.*

Remark 1. Because of the compactness of $J(G)$, we can show, with an easy argument, which is familiar to us in the iteration theory of rational functions, that even if we exchange the metric ρ to another conformal metric ρ_1, the enequality of the assumption holds with the same number λ and a different constant c_1.

Definition 1.4. Let $G = \langle f_1, f_2, \dots f_n \rangle$ be a finitely generated rational semigroup. We say that G is expanding if the assumption in Theorem 1.3 holds.

We denote by $B(x, \epsilon)$ a ball of center x and radius ϵ in the spherical metric. Also for any rational map g, we denote by $B_g(x, \epsilon)$ a connected component of $g^{-1}(B(x, \epsilon))$.

Definition 1.5. Let G be a rational semigroup. We say that G is *semi-hyperbolic* (resp. *weakly semi-hyperbolic*) if there are positive number δ and positive integer N such that for any $x \in J(G)$ (resp. $\partial J(G)$), any element $g \in G$ and any connected component $B_g(x, \delta)$ of $g^{-1}(B(x, \delta))$,

$$\deg(g : B_g(x, \delta) \to B(x, \delta)) \leq N.$$

Remark 2.

1. If G is semi-hyperbolic and $N = 1$, then G is hyperbolic.
2. If G is sub-hyperbolic and for each $g \in G$, there is no super attracting fixed point of g in $J(G)$, then G is semi-hyperbolic.
3. For a rational map f with the degree at least two, $\langle f \rangle$ is semi-hyperbolic if and only if f has no parabolic orbits and each critical point in the Julia set is non-recurrent([CJY], [Y]). If $\langle f \rangle$ is semi-hyperbolic, then there are no indifferent cycles and Hermann rings.

Definition 1.6. Let G be a rational semigroup and U be a component of $F(G)$. For every element g of G, we denote by U_g the connected component of $F(G)$ containing $g(U)$. We say that U is a wandering domain if $\{U_g\}$ is infinite.

Theorem 1.7. *Let G be a rational semigroup. Assume that G is weakly semi-hyperbolic and there is a point $z \in F(G)$ such that the closure of the G-orbit $\overline{G(z)}$ is included in $F(G)$. Then for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$ and there is no wandering domain.*

With this result, we get

Theorem 1.8. *Let G be a rational semigroup. Assume that*

- *G is weakly semi-hyperbolic and for each $g \in G, \deg(g) \geq 2$, or*
- *G is semi-hyperbolic and there is an element $h \in G$ such that $\deg(h) \geq 2$.*

Then for each $x \in F(G)$, $\overline{G(x)} \subset F(G)$ and there is no wandering domain.

By Theorem 1.8, we can show the following result.

Theorem 1.9. *Let G be a finitely generated rational semigroup which is sub-hyperbolic or semi-hyperbolic. Assume that $F(G) \neq \emptyset$, there is an element $g \in G$ such that $\deg(g) \geq 2$ and for each Möbius transformation in G is loxodromic or hyperbolic. Then there is a non-empty compact subset K of $P(G) \cap F(G)$ such that K is an attractor i.e. for any open neighborhood U of K and each $z \in F(G)$, $g(z) \in U$ for all but finitely many $g \in G$.*

Theorem 1.10. *Let G be a finitely generated rational semigroup which contains an element with the degree at least two. Assume that $\sharp P(G) < \infty$ and $P(G) \subset J(G)$. Then $J(G) = \overline{\mathbb{C}}$.*

By Theorem 1.9 and Theorem 2.3.4 in [S3], we get the following result.

Theorem 1.11. *Let M be a complex manifold. Let $\{G_a\}_{a \in M}$ be a holomorphic family of rational semigroups (See the definition in [S3]) where $G_a = \langle f_{1,a}, \cdots, f_{n,a} \rangle$. We assume that for a point $b \in M$, G_b is sub-hyperbolic or semi-hyperbolic, contains an element of the degree at least two and each Möbius transformation in G_b is hyperbolic or loxodromic. Then the map*

$$a \mapsto J(G_a)$$

is continuous at the point $a = b$ with respect to the Hausdorff metric.

Definition 1.12. Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. We say that G satisfies the *open set condition* with respect to the generators $f_1, f_2, \dots f_m$ if there is an open set O such that for each $j = 1, \dots, m$, $f_j^{-1}(O) \subset O$ and $\{f_j^{-1}(O)\}_{j=1,\dots,m}$ are mutually disjoint.

Proposition 1.13. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that G satisfies the open set condition with respect to the generators $f_1, f_2, \dots f_m$ and $O \setminus J(G) \neq \emptyset$ where O is the open set in the definition of open set condition. Then $J(G)$ has empty interior points.*

We get the next lemma by a modification of the arguments in [Y] or [CJY].

Lemma 1.14. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic and satisfies the open set condition with respect to the generators $f_1, f_2, \dots f_m$. Let O be the open set in Definition 1.12 and δ be a number in the definition of semi-hyperbolicity. Then for any ϵ there is a positive integer n_0 such that for each $g \in G$ with the word length greater than n_0, each point $y \in J(G) \setminus B(\partial O, \delta)$ and each connected component $B_g(y, \frac{1}{2}\delta)$ of $g^{-1}(B(y, \frac{1}{2}\delta))$, the diameter of $B_g(y, \frac{1}{2}\delta)$ is less than ϵ.*

By Proposition 1.13, Lemma 1.14 and a modification of the arguments in [Y], we get the next result.

Theorem 1.15. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup which is semi-hyperbolic and satisfies the open set condition with respect to the generators $f_1, f_2, \dots f_m$. Let O be the open set in Definition 1.12. Assume that $\sharp(\partial O \cap J(G)) < \infty$. Then the 2-dimensional Lebesgue measure of $J(G)$ is equal to 0.*

2 δ-subconformal measure

Definition 2.1. Let G be a rational semigroup and δ be a non-negative number. We say that a probability measure μ on $\overline{\mathbb{C}}$ is δ-subconformal if for each $g \in G$ and for each measurable set A

$$\mu(g(A)) \leq \int_A \|g'(z)\|^\delta d\mu.$$

For each $x \in \overline{\mathbb{C}}$ and each real number s we set

$$S(s,\ x) = \sum_{g \in G} \sum_{g(y)=x} \|g'(y)\|^{-s}$$

counting multiplicities and

$$S(x) = \inf\{s \mid S(s,\ x) < \infty\}.$$

If there is not s such that $S(s,\ x) < \infty$, then we set $S(x) = \infty$.Also we set

$$s_0(G) = \inf\{S(x)\},\ s(G) = \inf\{\delta \mid \exists\mu : \delta\text{-subconformal measure}\}$$

Theorem 2.2 ([S4]). *Let G be a rational semigroup which has at most countably many elements. If there exists a point $x \in \overline{\mathbb{C}}$ such that $S(x) < \infty$ then there is a $S(x)$-subconformal measure.*

Proposition 2.3 ([S4]). *Let G be a rational semigroup and τ a δ-subconformal measure for G where δ is a real number. Assume that $\sharp J(G) \geq 3$ and for each $x \in E(G)$ there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. Then the support of τ contains $J(G)$.*

By Theorem 1.9 and Proposition 2.3, we can show the next result.

Theorem 2.4. *Let $G = \langle f_1, f_2, \ldots f_n\rangle$ be a finitely generated rational semigroup. Assume that G is sub-hyperbolic, for each $g \in G$ there is no super attracting fixed point of g in $J(G)$, there is an element of G with the degree at least two and each Möbius transformation in G is hyperbolic or loxodromic. Then*

$$\dim_H(J(G)) \leq s(G) \leq s_0(G).$$

3 Conformal Measures in a Skew Product

Let m be a positive integer. We denote by Σ_m the one-sided word space, that is

$$\Sigma_m = \{1, \ldots, m\}^{\mathbb{N}}$$

and denote by $\sigma : \Sigma_m \to \Sigma_m$ the shift map, that is

$$(w_1, \ldots) \mapsto (w_2, \ldots).$$

Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. We define a map $\tilde{f} : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ by

$$\tilde{f}((w,x)) = (\sigma w, f_{w_1} x).$$

$\tilde{f}$ is a finite-to-one and open map. We have that a point $(w,x) \in \Sigma_m \times \overline{\mathbb{C}}$ satisfies $f'_{w_1}(x) \neq 0$ if and only if $\tilde{f}$ is a homeomorphism in a small neighborhood of (w,x). Hence the map $\tilde{f}$ has infinitely many critical points. We set $\tilde{J} = \cap_{n=0}^{\infty}(\Sigma_m \times J(G))$. Then by definition, $\tilde{f}^{-1}(\tilde{J}) = \tilde{J}$. Also from the backward self-similarity of $J(G)$, we can show that $\pi(\tilde{J}) = J(G)$ where $\pi : \Sigma_m \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the second projection.

For each $j = 1, \dots, m$, let φ_j be a Hölder continuous function on $f_j^{-1}(J(G))$. We set for each $(w,x) \in \tilde{J}$, $\varphi((w,x)) = \varphi_{w_1}(x)$. Then φ is a Hölder continuous function on $\tilde{J}$. We define an operater L on $C(\tilde{J}) = \{\psi : \tilde{J} \to \mathbb{C} \mid \text{continuous}\}$ by

$$L\psi((w,x)) = \sum_{\tilde{f}((w',y))=(w,x)} \frac{\exp(\varphi((w',y)))}{\exp(P)} \psi((w',y)),$$

counting multiplicities, where we denote by $P = P(\tilde{f}|_{\tilde{J}}, \varphi)$ the pressure of $(\tilde{f}|_{\tilde{J}}, \varphi)$.

Lemma 3.1. *With the same notations as the above, let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then for each set of Hölder continuous functions $\{\varphi_j\}_{j=1,\dots,m}$, there exists a unique probability measure τ on $\tilde{J}$ such that*

- $L^*\tau = \tau$,
- *for each $\psi \in C(\tilde{J})$, $\|L^n\psi - \tau(\psi)\alpha\|_{\tilde{J}} \to 0, n \to \infty$, where we set $\alpha = \lim_{l\to\infty} L^l(1) \in C(\tilde{J})$ and we denote by $\|\cdot\|_{\tilde{J}}$ the supremum norm on $\tilde{J}$,*
- *$\alpha\tau$ is an equilibrium state for $(\tilde{f}|_{\tilde{J}}, \varphi)$.*

Lemma 3.2. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then there exists a unique number $\delta > 0$ such that if we set $\varphi_j(x) = -\delta \log(\|f_j'(x)\|), j = 1, \dots, m,$ then $P = 0$.*

From Lemma 3.1, for this δ there exists a unique probability measure τ on $\tilde{J}$ such that $L_\delta^*\tau = \tau$ where L_δ is an operator on $C(\tilde{J})$ defined by

$$L_\delta\psi((w,x)) = \sum_{\tilde{f}((w',y))=(w,x)} \frac{\psi((w',y))}{\|(f_{w_1'})'(y)\|^\delta}.$$

Also δ satisfies that

$$\delta = \frac{h_{\alpha\tau}(\tilde{f})}{\int_{\tilde{J}} \tilde{\varphi}\alpha d\tau} \leq \frac{\log(\sum_{j=1}^m \deg(f_j))}{\int_{\tilde{J}} \tilde{\varphi}\alpha d\tau},$$

where $\alpha = \lim_{l\to\infty} L_\delta^l(1)$, we denote by $h_{\alpha\tau}(\tilde{f})$ the metric entropy of $(\tilde{f}, \alpha\tau)$ and $\tilde{\varphi}$ is a function on $\tilde{J}$ defined by $\tilde{\varphi}((w,x)) = \log(\|f'_{w_1}(x)\|)$.

By these argument, we get the following result.

Theorem 3.3. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup and δ the number in the above argument. Then*

$$\dim_H(J(G)) \leq s(G) \leq \delta.$$

Moreover, if the sets $\{f_j^{-1}(J(G))\}$ are mutually disjoint, then $\dim_H(J(G)) = \delta < 2$ and $0 < H_\delta(J(G)) < \infty$, where we denote by H_δ the δ-Hausdorff measure.

Corollary 3.4. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated expanding rational semigroup. Then*

$$\dim_H(J(G)) \leq \frac{\log(\sum_{j=1}^m \deg(f_j))}{\log \lambda},$$

where λ denotes the number in Definition 1.4.

4 Generalized Brolin-Lyubich's Invariant Measure, Self-Similar Measure

With the same notation as the previous section, we define an operator $\tilde{A}$ on $C(\tilde{J})$ by

$$\tilde{A}\tilde{\psi}((w,x)) = \frac{1}{\sum_{j=1}^m \deg(f_j)} \sum_{\tilde{f}((w',y))=(w,x)} \tilde{\psi}((w',y)), \text{ for each } \tilde{\psi} \in C(\tilde{J}),$$

and an operator A on $C(J(G)) = \{\psi : J(G) \to \mathbb{C} \mid \text{ continuous}\}$ by

$$A\psi(x) = \frac{1}{\sum_{j=1}^m \deg(f_j)} \sum_{j=1}^m \sum_{f_j(y)=x} \psi(y), \text{ for each } \psi \in C(J(G)).$$

Then $\tilde{A} \circ \pi^* = \pi^* \circ A$, where π^* is the map from $C(J(G))$ to $C(\tilde{J})$ defined by $(\pi^*\psi)((w,x)) = \psi(x)$. Note that since $\pi(\tilde{J}) = J(G)$, we have that for each $\psi \in C(J(G))$,

$$\|\pi^*\psi\|_{\tilde{J}} = \|\psi\|_{J(G)}. \tag{1}$$

Now we consider a condition such that the invariant measures are unique.

Definition 4.1. Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. With the same notation as the previous section, we say that G satisfies condition $*$ if for any $z \in \tilde{J} \setminus \mathrm{per}(\tilde{f})$, for any $\epsilon > 0$, there exists a positive integer $n_0 = n_0(z, \epsilon)$ such that

$$\frac{\sharp\{\tilde{f}^{-n_0}(z) \cap Z_\infty\}}{(\sum_{j=1}^m \deg(f_j))^{n_0}} < \epsilon, \tag{2}$$

counting multiplicities, where we set

$$Z_\infty = \cup_{n=1}^\infty \tilde{f}^n(\{ \text{ critical points of } \tilde{f}\} \cap \tilde{J}). \tag{3}$$

Remark 3. Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. In each case of the following, the condition $*$ holds.

- There exists an element f such that for each $j = 1, \dots, m$, $f_j = f$.
- The sets $\{f_i^{-1}(J(G))\}_{i=1,\dots,m}$ are mutually disjoint.
- $J(G) \setminus \overline{\cup_{g \in G}\{\text{critical values of } g\} \cap J(G)} \neq \emptyset$.

Therefore we have many finitely generated rational semigroups satisfying condition $*$. It seems to be true that the condition $*$ holds if a finitely generated rational semigroup G satisfies that $J(G) \cap E(G) = \emptyset$, where $E(G)$ denotes the exceptional set of G, that is $E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp\{\cup_{g \in G} g^{-1}(z)\} < \infty\}$.

Theorem 4.2. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that $F(H) \supset J(G)$, where we set $H = \{g^{-1} \in Aut\ (\overline{\mathbb{C}}) \mid g \in Aut\ (\overline{\mathbb{C}}) \cap G\}$, and condition $*$ holds. Then we have the following:*

1. *There exists a unique probability measure $\tilde{\mu}$ on $\tilde{J}$ such that*

$$\|\tilde{A}^n \tilde{\varphi} - \tilde{\mu}(\tilde{\varphi}) \mathbf{1}_{\tilde{J}}\|_{\tilde{J}} \to 0, \ n \to \infty, \ \text{for any } \tilde{\varphi} \in C(\tilde{J}),$$

   where we denote by $\mathbf{1}_{\tilde{J}}$ the constant function on $\tilde{J}$ taking its value 1, *and exists a unique probability measure μ on $J(G)$ such that*

$$\|A^n \varphi - \mu(\varphi) \mathbf{1}_{J(G)}\|_{J(G)} \to 0, \ n \to \infty, \ \text{for any } \varphi \in C(J(G)),$$

   where we denote by $\mathbf{1}_{J(G)}$ the constant function on $J(G)$ taking its value 1.

2. *$\pi_* \tilde{\mu} = \mu$ and $\tilde{\mu}$ is $\tilde{f}$-invariant.*
3. *$(\tilde{f}, \tilde{\mu})$ is exact. In particular, $\tilde{\mu}$ is ergodic.*
4. *μ is non-atomic. supp (μ) is equal to $J(G)$.*
5. *$h(\tilde{f}|_{\tilde{J}}) \geq h_{\tilde{\mu}}(\tilde{f}) \geq \log(\sum_{j=1}^m \deg(f_j))$, where $h(\tilde{f}|_{\tilde{J}})$ denotes the topological entropy of $\tilde{f}$ on $\tilde{J}$.*

Proof. We will show the statement in the similar way to [L]. By [HM3], the family of all holomorphic inverse branches of any elements of G in any open set U which has non-empty intersection with $J(G)$ is normal in U. With this fact, we can show that the operator $\tilde{A}$ is almost periodic, i.e. for each $\tilde{\psi} \in C(\tilde{J})$, $\{\tilde{A}^n \tilde{\psi}\}_n$ is relative compact in $C(\tilde{J})$. Hence, by [L], $C(\tilde{J})$ is the direct sum of the attractive basin of 0 for $\tilde{A}$ and the closure of the space generated by unit eigenvectors. It is easy to see that 1 is the unique eigenvalue and the eigenvectors are constant. Therefore 1. holds.

Because of the condition $*$, $E(G)$ is included in $F(G)$. With the fact, we can show that μ is non-atomic, which implies 5. □

Remark 4. If $\tilde{f}|_{\tilde{J}}$ is expansive,(in particular, if G is expanding,) then

$$h(\tilde{f}|_{\tilde{J}}) = h_{\tilde{\mu}}(\tilde{f}) = \log(\sum_{j=1}^{m} \deg(f_j)).$$

Remark 5. We can also construct *self-similar measures* on $J(G)$ and show the uniqueness under a similar assumption to condition $*$. For example, in each case of the Remark after Definition 4.1, we can show that.

Now we consider a generalization of Mañé's result([Ma]).

Theorem 4.3. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that the sets $\{f_i^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint. We define a map $f : J(G) \to J(G)$ by $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$. If μ is an ergodic invariant probability measure for $f : J(G) \to J(G)$ with $h_\mu(f) > 0$, then*

$$\int_{J(G)} \log(\|f'\|)\, d\mu > 0$$

and

$$HD(\mu) = \frac{h_\mu(f)}{\int_{J(G)} \log(\|f'\|)\, d\mu},$$

where we set

$$HD(\mu) = \inf\{\dim_H(Y) \mid Y \subset J(G),\ \mu(Y) = 1\}.$$

Proof. We can show the statement in the same way as [Ma]. Note that the Ruelle's inequality([Ru]) also holds for the map $f : J(G) \to J(G)$. □

From the remark after Definition 4.1, Theorem 4.2 and Theorem 4.3, we get the following result. This solves the Problem 12 in [Re] of F.Ren's.

Theorem 4.4. *Let $G = \langle f_1, f_2, \dots f_m \rangle$ be a finitely generated rational semigroup. Assume that the sets $\{f_i^{-1}(J(G))\}_{j=1,\dots,m}$ are mutually disjoint. Then*

$$\dim_H(J(G)) \geq \frac{\log(\sum_{j=1}^{m} \deg(f_j))}{\int_{J(G)} \log(\|f'\|)\, d\mu},$$

where μ denotes the probability measure in Theorem 4.2 and $f(x) = f_i(x)$ if $x \in f_i^{-1}(J(G))$.

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