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Brjuno Numbers and Non Linearizability of Polynomials of Degree more than two

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Abstract
We consider an irrational number $\alpha$ which is not a Brjuno number. If there exists a cubic polynomial which has a Siegel point of multiplier $\exp(2\pi i \alpha)$, for any $d \geq 4$ there exists a $d - 2$ dimensional holomorphic family of $P_{\lambda,d}$ of which all elements have a Siegel point of multiplier $\exp(2\pi i \alpha)$.

1 Introduction
Consider a germ of a holomorphic map $(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$$

with multiplier $\lambda$ at $z = 0$. And we consider the case that $|\lambda| = 1$ but $\lambda$ is not a root of unity. Thus the multiplier $\lambda$ can be written as

$$\lambda = e^{2\pi i \alpha} \text{ for an } \alpha \in \mathbb{R} - \mathbb{Q}.$$ 

The origin is said to be an irrationally indifferent fixed point.

The linearization problem for $f$ is whether or not there exists a holomorphic local change of coordinate $z = h(w)$ with $h(0) = 0$ and $h'(0) \neq 0$ which conjugates $f$ to the irrational rotation $w \mapsto \lambda w$ so that

$$h(\lambda w) = f(h(w))$$

near the origin. We say that an irrationally indifferent fixed point is a Siegel point or a Cremer point according as the local linearization is possible or not (cf. [2]).
For a suitable $t > 0$, the map $z \mapsto \frac{1}{t} f(tz)$ is holomorphic and univalent on the unit disk $\mathbb{D}$. So we consider the special case that $f$ is holomorphic and univalent on the unit disk $\mathbb{D}$. Let

$$S := \{f; \text{ holomorphic and univalent map on } \mathbb{D}, \ f(0) = 0, \text{ and } |f'(0)| = 1\},$$

$$S_\lambda := \{f \in S; \ f'(0) = \lambda\}.$$

Then we can consider the linearization problem for $f \in S_\lambda$ at the origin.

**Definition 1.1.** A map $f \in S_\lambda$ will be called linearizable at the origin if there exist a neighborhood $U_f$ of the origin and a map $H_f$ which is holomorphic and univalent on $U_f$, and satisfies

$$H_f(0) = 0, \ H'_f(0) = 1, \ f(H_f(z)) = H_f(\lambda z).$$

In this case $H_f$ will be called a linearizing map of $f$ at the origin and the connected component of the Fatou set of $f$ which contains the origin is called a Siegel disk of $f$ at the origin.

In order to state the results on this problem, we should introduce the following.

**Definition 1.2.** For $\alpha \in \mathbb{R} - \mathbb{Q}$, we consider the continued fraction expansion

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

where $a_0$ is an integral part of $\alpha$, and $\alpha_1 := \alpha - a_0$. $a_1$ is an integral part of $1/\alpha_1$, and $\alpha_2 := 1/\alpha_1 - a_1$. $a_2$ is an integral part of $1/\alpha_2$. Inductively, we define $a_n$. And we define the $n$-th approximate fraction

$$\frac{p_n}{q_n} := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}$$

where $p_n/q_n$ is an irreducible fraction. An $\alpha \in \mathbb{R} - \mathbb{Q}$ is called a Brjuno number if

$$\sum_{i \geq 0} \frac{\log q_{i+1}}{q_i} < +\infty.$$ 

We define $\mathcal{B} := \{\alpha \in \mathbb{R} - \mathbb{Q}; \alpha \text{ is a Brjuno number}\}$. 
Next we state the known results about this problem.

**Theorem 1.1 (Brjuno).** If $\alpha \in B$, all maps $f \in S_{\exp(2\pi i\alpha)}$ are linearizable at the origin.

And Yoccoz proved that this result is best possible.

**Theorem 1.2 (Yoccoz [5]).** If $\alpha \notin B$, there exists a map $f \in S_{\exp(2\pi i\alpha)}$ which is non linearizable at the origin.

Furthermore for quadratic polynomials, Yoccoz proved the following.

**Theorem 1.3 (Yoccoz [5]).** If $\alpha \notin B$, 
\[ P(z) = e^{2\pi i\alpha}z + z^2 \]
is non linearizable at the origin.

For $d \geq 2$, we define 
\[ \mathcal{P}_{\lambda,d} := \{ P(z) = \lambda z + a_2 z^2 + \cdots + a_d z^d, (a_2, \ldots, a_d) \in \mathbb{C}^{d-1} \} \cong \mathbb{C}^{d-1} \]
For $\mathcal{P}_{\lambda,d}$, $d \geq 3$, Pérez-Marco proved the following.

**Theorem 1.4 (Pérez-Marco [4]).** Fix $\lambda = \exp(2\pi i\alpha)$ ($\alpha \notin B$) and $d \geq 3$. There exists an open dense subset of $\mathcal{P}_{\lambda,d}$ of which all elements are non linearizable at the origin.

**Theorem 1.5 (Pérez-Marco [4]).** In the same condition as above, for any $(a_3, \ldots, a_d) \in \mathbb{C}$ ($a_d \neq 0$), There exists an open dense subset $U$ of $\mathbb{C}$ such that if $a_2 \in U$, then $z \mapsto \lambda z + a_2 z^2 + a_3 z^3 \cdots + a_d z^d$ is non linearizable at the origin.

We would like to consider the linearizability of polynomials of degree more than two in Section 2.

## 2 Linearizability of polynomials of degree more than two

For $\lambda = e^{2\pi i\alpha}$ ($\alpha \in \mathbb{R} - \mathbb{Q}$), and $A \in \mathbb{C}$, let $P_{\lambda,A}$ be a cubic monic polynomial 
\[ P_{\lambda,A}(z) := \lambda z + A z^2 + z^3. \]
Then $P_\lambda$ has a fixed point of multiplier $\lambda$ at $z = 0$. Conversely, for any cubic polynomial with a fixed point of multiplier $\lambda$, there exists some $A \in \mathbb{C}$ such that it is affine conjugate to $P_{\lambda,A}$. 
Theorem 2.1. Fix $\lambda = e^{2\pi i\alpha}$ ($\alpha \notin B$) and $d \geq 4$. Suppose there exists $A \in \mathbb{C}$ such that $P_{\lambda,A}$ is linearizable at the origin, then the family $P_{\lambda,A}$ contains a $d-2$ dimensional holomorphic family of which all elements are linearizable at the origin.

Now we shall prove this main theorem.

2.1 Cubic perturbation of univalent maps

Let $\lambda = e^{2\pi i\alpha}$ for $\alpha \in \mathbb{R} - \mathbb{Q}$, and let $f$ be an element of $S_{\lambda}$. We define, for $a \in \overline{D} - \{0\}$, $A \in \mathbb{C}$ and $b \in \mathbb{C}$,

$$f_{a,A,b}(z) := a^{-1}f(az) + Abz^2 + b^2z^3.$$

By definition, the triplet $(U', U, f)$ is called a polynomial-like map of degree $d$ if $U'$ and $U$ are simply connected proper subdomains of $\mathbb{C}$, and $U'$ is relatively compact in $U$, and $f : U' \rightarrow U$ is a holomorphic and proper map of degree $d$.

Lemma 2.1. For $A \in \mathbb{C}$ and $b \in \mathbb{C}$, we define

$$R_{A,b} := \frac{10}{9}|A||b| + \frac{15}{2},$$

$$B_{A,b} := 27R_{A,b} + 3|A||b| + \frac{81}{4} = 33|A||b| + \frac{891}{4},$$

$$W := \{z; |z| < R_{A,b}\} \text{ and}$$

$$W_{f,a,A,b} := \{z; |z| < \frac{1}{3}\} \cap f_{a,A,b}^{-1}(W).$$

For $f \in S_{f}, a \in \overline{D} - \{0\}$, $A \in \mathbb{C}$ and $|b|^2 > B_{A,b}$, the triplet $(W_{f,a,A,b}, W, f_{a,A,b})$ is a polynomial-like map of degree 3.

Proof. It is sufficient to prove this in $a = 1$. Since $f$ is univalent in $\mathbb{D}$, it follows that $\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$ for $z \in \mathbb{D}$. In particular, if $|z| = 1/3$, we have $3/16 \leq |f(z)| \leq 3/4$. For $|z| = 1/3$, it follows that

$$|b^2z^3| - |Abz^2 + f(z)| \geq \frac{|b^2|}{27} - \frac{|A||b|}{9} - \frac{3}{4} > R_{A,b}.$$

Thus $f_{1,A,b}(\{z; |z| < 1/3\})$ properly contains the disk $W$, so $f_{1,A,b} : W_{f,1,A,b} \rightarrow W$ is proper and $W_{f,1,A,b}$ is simply connected by the maximum modulus principle. And for $|z| = 1/3$ and $z_1 \in W$, it follows that

$$|b^2z^3 - z_1| \geq |b^2z^3| - R_{A,b} > |Abz^2 + f(z)| \text{ and}$$

$$\sqrt[3]{\frac{|z_1|}{b^2}} < \sqrt[3]{\frac{R_{A,b}}{|b|^2}} \leq \frac{1}{3}.$$
since $|b|^2 > B_{A,b} > 27R_{A,b}$ by definition. Thus by the theorem of Rouché, $f_{1,A,b} : W_{f,1,A,b} \rightarrow W$ is a proper map of degree three.

If $W_{f,1,A,b}$ were not connected, then the number of connected components of $W_{f,1,A,b}$ would be three or two. First, if it were three, the connected component of $W_{f,1,A,b}$ containing the origin would be conformally mapped to $W$ by $f_{1,A,b}$. However this would contradict Schwarz lemma because $|\lambda| = 1$. Second, if it were two, two cases would occur. If $f_{1,A,b}$ would conformally map the connected component of $W_{f,1,A,b}$ containing the origin to $W$, we could derive a contradiction by the same argument as above. If not, there would exist the connected component $W'$ of $W_{f,1,A,b}$ which would not contain the origin. Then $f_{1,A,b}$ would conformally map $W'$ onto $W$. So there would exist the only one point $z_0 \in W'$ such that $f_{1,A,b}(z_0) = 0$. We define $\phi := (f_{1,A,b}|W'|^{-1}$, and $\psi(z) := \phi(R_{A,b}z)$. Then $\psi$ would conformally map $\mathbb{D}$ to $W'$, and $\psi(0) = z_0$ (See figure 1).

![Diagram](image)

Figure 1:

By the Koebe one-quarter theorem, it follows that $W'$ contains the open disk of which the radius are $\frac{1}{4}|\phi'(0)| = \frac{1}{4}R_{A,b}|\phi'(0)|$. Since $\mathbb{D}_{1/3} \supset W_{f,1,A,b} \supset W'$ and $W' \neq 0$, we have $\frac{1}{4}R_{A,b}|\phi'(0)| < \frac{1}{6}$. Hence

$$\frac{1}{|\phi'(0)|} > \frac{3R_{A,b}}{2}.$$ 

On the other hand, we have

$$\frac{1}{|\phi'(0)|} = |f'_{1,A,b}(z_0)| \leq |f'(z_0)| + 2|A||b||z_0| + 3|b|^2|z_0|^2$$
and by the Koebe theorem, \( |f'(z_0)| < 9/2 \) for \( |z_0| < 1/3 \). Since \( f_{1,A,b}(z_0) = f(z_0) + Abz_0^2 + b^2z_0^3 = 0 \) and \( |f(z_0)| \leq \frac{|z_0|}{(1-|z_0|)^2} \), we have \( 3|b|^2|z_0|^2 + 2|A||b||z_0| < \frac{27}{4} + \frac{5}{3}|A||b| \) for \( |z_0| < 1/3 \). Hence
\[
\frac{1}{|\phi'(0)|} < \frac{9}{2} + \frac{27}{4} + \frac{5}{3}|A||b| = \frac{45}{4} + \frac{5}{3}|A||b| = \frac{3}{2}R_{A,b}.
\]
That is a contradiction. So \( W_{f,a,A,b} \) is connected, and the proof is completed.

\[\square\]

Remark 2.1. \( |b|^2 > B_{A,b} \) if and only if \( |b| > \frac{33|A| + \sqrt{1089|A| + 891}}{2} =: N(|A|) \).

2.2 Straightening of the polynomial-like mapping

Let \( M \) be an arbitrary positive number and take a smooth function \( \eta : \mathbb{R} \to [0,1] \) identically \( 1 \) on \( (-\infty, 1/3] \) and identically \( 0 \) on \( [R_{A,b}, +\infty) \). And we define the round annulus \( A(M) := \{ b; N(M) < |b| < N(M) + 1 \} \).

For \( f \in S_\lambda \), \( a \in \overline{D} - \{0\} \), \( A \in \mathbb{D}_M = \{ z; |z| < M \} \) and \( b \in A(M) \), we define
\[
\tilde{f}_{a,A,b}(z) := \eta(|z|)f_{a,A,b}(z) + (1 - \eta(|z|))(\lambda z + Abz^2 + b^2z^3).
\]

Then \( \tilde{f}_{a,A,b} : \mathbb{C} \to \mathbb{C} \) is \( C^\infty \) on \( \mathbb{C} \).

Lemma 2.2. If \( a \to 0 \), then \( \tilde{f}_{a,A,b}(z) \) converges to \( \lambda z + Abz^2 + b^2z^3 \) in \( C^\infty \)-topology on \( \mathbb{C} \), and this convergence is uniform in \( f \in S_\lambda \), \( A \in \mathbb{D}_M \) and \( b \in A(M) \).

Proof. The function \( f_{a,A,b} \) is uniformly convergent to \( \lambda z + Abz^2 + b^2z^3 \) on \( \{ |z| \leq R_{A,b} \} \) as \( a \to 0 \). Since \( f(z) \) is univalent on \( \mathbb{D} \), the coefficients of the power series expansion of it can be estimated uniformly in \( f \). It is clear that this convergence is uniform in \( A \in \mathbb{D}_M \) and \( b \in A(M) \). \( \square \)

We can also prove that two critical points of \( z \mapsto \lambda z + Abz^2 + b^2z^3 \) is included in \( \{ |z| < 1/3 \} \) by the theorem of Rouché. We can conclude the following.

Lemma 2.3. There exists an \( a_0 \in (0,1] \) and a continuous function \( k : [0,a_0] \to [0,1) \) such that \( k(0) = 0 \) and for any \( f \in S_\lambda \), \( A \in \mathbb{D}_M \) and \( b \in A(M) \) and \( a \in \mathbb{D}_{a_0} - \{0\} \), the map \( \tilde{f}_{a,A,b} \) is a branched covering map of \( \mathbb{C} \) of degree 3 and it satisfies
\[
\left| \frac{\partial \tilde{f}_{a,A,b}(z)}{\partial \tilde{f}_{a,A,b}(z)} \right| \leq k(|a|) \quad (1/3 \leq |z| \leq R_{A,b}).
\]
Moreover, \( \frac{\partial f_{a,A,b}(z)}{\partial f_{a,A,b}(z)} \) holomorphically depends on \( A \in \mathbb{D}_{M} \) and \( b \in \mathcal{A}(M) \) and \( a \in \mathbb{D}_{a_{0}} - \{0\} \). If \( f \) is a polynomial, this complex dilatation is also holomorphically depends on the coefficients of \( f \).

For \( f \in S_{\lambda}, A \in \mathbb{D}_{M}, b \in \mathcal{A}(M) \) and \( a \in \mathbb{D}_{a_{0}} - \{0\} \), We can define a Beltrami coefficient \( \mu = \mu_{f,a,A,b} \) on \( \mathbb{C} \) such that it is invariant for a pullback of \( \tilde{f}_{a,A,b} \) and it assumes 0 on \( \mathbb{C} - W \) and on \( \bigcap_{n \geq 0} f_{a,A,b}^{-n}(W_{f,a,A,b}) \). Since \( \text{supp} \mu \subset W \) and \( \|\mu\|_{\infty} \leq k(a) < 1 \), by the Ahlfors-Bers theorem, there exists a unique quasiconformal homeomorphism \( \phi = \phi_{f,a,A,b} \) of \( \mathbb{C} \) onto itself which satisfies the following

(i) for a.e.\( z \in \mathbb{C} \), \( \overline{\partial} \phi(z) = \mu(z) \partial \phi(z) \),

(ii) \( \phi(0) = 0 \) and

(iii) \( \phi(z) - z \) is bounded on \( \mathbb{C} \).

**Lemma 2.4 (cf. [1]).** There exists an \( A' \in \mathbb{C} \) such that \( \phi \circ \tilde{f}_{a,A,b} \circ \phi^{-1}(z) = \lambda z + A'z^{2} + b'z^{3} \), where \( A' \in \mathbb{C} \) holomorphically depends on \( A \in \mathbb{D}_{M} \), \( b \in \mathcal{A}(M) \) and \( a \in \mathbb{D}_{a_{0}} - \{0\} \). If \( f \) is a polynomial, it also holomorphically depends on the coefficients of \( f \).

**Proof.** \( \phi \circ \tilde{f}_{a,A,b} \circ \phi^{-1} : \mathbb{C} \rightarrow \mathbb{C} \) is holomorphic, fixes 0 and \( \infty \). So it is a branched covering map of \( \mathbb{C} \) of degree 3 fixing the origin. Thus we can write

\[ \phi \circ \tilde{f}_{a,A,b} \circ \phi^{-1}(z) = \lambda'z + A'z^{2} + b'z^{3} \quad (\lambda', A', b' \in \mathbb{C}). \]

By the theorem of Naisul ([3]), the multiplier of the fixed point of a holomorphic map is topologically invariant when its module is 1. So we have \( \lambda' = \lambda \).

Next, we would like to show \( b' = b^{2} \). According to (iii), we have

\[ \phi_{f,a,A,b}(z) = z + c + (\text{lower terms}) \]

at a neighborhood of the point at infinity. When \( |z| \) is sufficiently large, \( \tilde{f}_{a,A,b}(z) = \lambda z + Abz^{2} + b^{2}z^{3} \) by definition, and we note that \( \phi(\tilde{f}_{a,A,b}(z)) = \lambda \phi(z) + A'(\phi(z))^{2} + b'(\phi(z))^{3} \). Therefore it follows that

\[ \phi(\lambda z + Abz^{2} + b^{2}z^{3}) - (\lambda z + Abz^{2} + b^{2}z^{3}) \]
\[ = (b' - b^{2})z^{3} + \{(A' - Ab) + 3b'c\}z^{2} + (\text{lower terms}). \]

Since this quantity is bounded as \( |z| \rightarrow +\infty \), it is necessary that \( b' - b^{2} = 0 \) and \( A' - Ab + 3b'c = 0 \). Thus it follows that \( b' = b^{2} \) and \( A' = Ab - 3b'(c) \).

**Remark 2.2.** It is easy to see the following: \( c = c(f, a, A, b) \) holomorphically depends on \( A \in \mathbb{D}_{M}, b \in \mathcal{A}(M) \) and \( a \in \mathbb{D}_{a_{0}} - \{0\} \). If \( f \) is a polynomial, it also holomorphically depends on the coefficients of \( f \). And \( c \rightarrow 0 \) uniformly in \( f \in S_{\lambda}, A \in \mathbb{D}_{M} \) and \( b \in \mathcal{A}(M) \) as \( a \rightarrow 0 \).
3 Completion of proof

Let $\alpha \not\in B$ and $\lambda = e^{2\pi i \alpha}$. Suppose that $P_{A_0, \lambda}$ is linearizable at the origin. Then for $b \in \mathbb{C}^*$, $\frac{1}{b}P_{A_0, \lambda}(bz) = \lambda z + A_0 b z^2 + b^2 z^3$ is also linearizable at the origin.

We take $M = M_0 := 2|A_0| + 1$. By the Remark 2.2, for any $\epsilon > 0$ there exists an $a_1 \in (0, a_0]$ which is independent of $f \in S_\lambda$, $A \in D_M$ and $b \in A(M_0)$ such that

$$3|b||c(f, a, A, b)| < \epsilon \quad (0 < |a| < a_1).$$

We can take $\epsilon > 0$ so that $|A_0| < M_0 - 2\epsilon$. We define a holomorphic map $F_{f, a, b}$ on $D_{M_0}$:

$$A \mapsto A - 3b c(f, a, A, b).$$

By the theorem of Rouché, there exists $A_1 = A_1(f, a, b)$ such that $F_{f, a, b}(A_1) = A_0$. We can see that $A_1 = A_1(f, a, b)$ holomorphically depends on $b \in A(M_0)$ and $a \in D_{a_1} - \{0\}$, and if $f$ is a polynomial, it also holomorphically depends on the coefficients of $f$. We can conclude the following.

**Proposition 3.1.** For any $f \in S_\lambda$, $b \in A(M_0)$ and $a \in D_{a_1} - \{0\}$, there exists $A_1 = A_1(f, a, b)$ which is holomorphic in $a \in D_{a_1} - \{0\}$, $b \in A(M_0)$ and $f \in S_\lambda$ and also exists $\phi = \phi_{f, a, A_1, b}$ which is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ onto itself which is defined in the previous section such that

$$\phi \circ f_{a, A_1, b} \circ \phi^{-1}(z) = \frac{1}{b}P_{A_0, \lambda}(bz).$$

So if there exists $A_0 \in \mathbb{C}$ such that $P_{A_0, \lambda}$ is linearizable at the origin, $f_{a, A_1, b}(z) = a^{-1}f(az) + A_1 b z^2 + b^2 z^3$ is linearizable at the origin.

In particular, we consider for $d > 1$,

$$U_d := \{ P(z) = \lambda z + a_2 z^2 + \cdots + a_d z^d; \sum_{n=2}^{d} n |a_n| \leq 1 \} \subset S_\lambda.$$

Consequently we can at least conclude the Theorem 2.1.

**References**


