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<th>Brjuno Numbers and Non Linearizability of Polynomials of Degree more than two (Problems on complex dynamical systems)</th>
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<tr>
<td>Author(s)</td>
<td>Okuyama, Yuusuke</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1998), 1042: 59-67</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1998-04</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/62099">http://hdl.handle.net/2433/62099</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Brjuno Numbers and Non Linearizability of Polynomials of Degree more than two

Yuusuke Okuyama
Department of Mathematics, Graduate School of Science
Kyoto University, Japan

October 9, 1997

Abstract
We consider an irrational number $\alpha$ which is not a Brjuno number. If there exists a cubic polynomial which has a Siegel point of multiplier $\exp(2\pi i \alpha)$, for any $d \geq 4$ there exists a $d - 2$ dimensional holomorphic family of $\mathcal{P}_{\lambda,d}$ of which all elements have a Siegel point of multiplier $\exp(2\pi i \alpha)$.

1 Introduction

Consider a germ of a holomorphic map $(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$$

with multiplier $\lambda$ at $z = 0$. And we consider the case that $|\lambda| = 1$ but $\lambda$ is not a root of unity. Thus the multiplier $\lambda$ can be written as

$$\lambda = e^{2\pi i \alpha} \text{ for an } \alpha \in \mathbb{R} - \mathbb{Q}.$$ 

The origin is said to be an irrationally indifferent fixed point.

The linearization problem for $f$ is whether or not there exists a holomorphic local change of coordinate $z = h(w)$ with $h(0) = 0$ and $h'(0) \neq 0$ which conjugates $f$ to the irrational rotation $w \mapsto \lambda w$ so that

$$h(\lambda w) = f(h(w))$$

near the origin. We say that an irrationally indifferent fixed point is a Siegel point or a Cremer point according as the local linearization is possible or not (cf. [2]).
For a suitable $t > 0$, the map $z \mapsto \frac{1}{t}f(tz)$ is holomorphic and univalent on the unit disk $\mathbb{D}$. So we consider the special case that $f$ is holomorphic and univalent on the unit disk $\mathbb{D}$. Let

$$S := \{f; \text{ holomorphic and univalent map on } \mathbb{D}, f(0) = 0, \text{ and } |f'(0)| = 1\},$$

$$S_{\lambda} := \{f \in S; \ f'(0) = \lambda\}.$$

Then we can consider the linearization problem for $f \in S_{\lambda}$ at the origin.

**Definition 1.1.** A map $f \in S_{\lambda}$ will be called linearizable at the origin if there exist a neighborhood $U_{f}$ of the origin and a map $H_{f}$ which is holomorphic and univalent on $U_{f}$, and satisfies

$$H_{f}(0) = 0, \quad H'_{f}(0) = 1, \quad f(H_{f}(z)) = H_{f}(\lambda z).$$

In this case $H_{f}$ will be called a linearizing map of $f$ at the origin, and the connected component of the Fatou set of $f$ which contains the origin is called a Siegel disk of $f$ at the origin.

In order to state the results on this problem, we should introduce the following.

**Definition 1.2.** For $\alpha \in \mathbb{R} - \mathbb{Q}$, we consider the continued fraction expansion

$$\alpha = a_{0} + \frac{1}{a_{1} + \frac{1}{a_{2} + \cdots}}$$

where $a_{0}$ is an integral part of $\alpha$, and $\alpha_{1} := \alpha - a_{0}$. $a_{1}$ is an integral part of $1/\alpha_{1}$, and $\alpha_{2} := 1/\alpha_{1} - a_{1}$. $a_{2}$ is an integral part of $1/\alpha_{2}$. Inductively, we define $a_{n}$. And we define the $n$-th approximate fraction

$$\frac{p_{n}}{q_{n}} := a_{0} + \frac{1}{a_{1} + \frac{1}{a_{2} + \cdots + \frac{1}{a_{n-1} + \frac{1}{a_{n}}}}}$$

where $p_{n}/q_{n}$ is an irreducible fraction. An $\alpha \in \mathbb{R} - \mathbb{Q}$ is called a Brjuno number if

$$\sum_{i \geq 0} \frac{\log q_{i+1}}{q_{i}} < +\infty.$$ 

We define $B := \{\alpha \in \mathbb{R} - \mathbb{Q}; \alpha \text{ is a Brjuno number}\}$. 
Next we state the known results about this problem.

**Theorem 1.1 (Brjuno).** If $\alpha \in B$, all maps $f \in S_{\exp(2\pi i \alpha)}$ are linearizable at the origin.

And Yoccoz proved that this result is best possible.

**Theorem 1.2 (Yoccoz [5]).** If $\alpha \not\in B$, there exists a map $f \in S_{\exp(2\pi i \alpha)}$ which is non linearizable at the origin.

Furthermore for quadratic polynomials, Yoccoz proved the following.

**Theorem 1.3 (Yoccoz [5]).** If $\alpha \not\in B$,

$$P(z) = e^{2\pi i \alpha}z + z^2$$

is non linearizable at the origin.

For $d \geq 2$, we define

$$\mathcal{P}_{\lambda,d} := \{P(z) = \lambda z + a_2z^2 + \cdots + a_dz^d, (a_2, \ldots, a_d) \in \mathbb{C}^{d-1}\} \cong \mathbb{C}^{d-1}$$

For $\mathcal{P}_{\lambda,d}$, $d \geq 3$, Pérez-Marco proved the following.

**Theorem 1.4 (Pérez-Marco [4]).** Fix $\lambda = \exp(2\pi i \alpha)$ (\(\alpha \not\in B\)) and $d \geq 3$. There exists an open dense subset of $\mathcal{P}_{\lambda,d}$ of which all elements are non linearizable at the origin.

**Theorem 1.5 (Pérez-Marco [4]).** In the same condition as above, for any $(a_3, \ldots, a_d) \in \mathbb{C}$ ($a_d \neq 0$), There exists an open dense subset $U$ of $\mathbb{C}$ such that if $a_2 \in U$, then $z \mapsto \lambda z + a_2z^2 + a_3z^3 \cdots + a_dz^d$ is non linearizable at the origin.

We would like to consider the linearizability of polynomials of degree more than two in Section 2.

## 2 Linearizability of polynomials of degree more than two

For $\lambda = e^{2\pi i \alpha}$ ($\alpha \in \mathbb{R} - \mathbb{Q}$), and $A \in \mathbb{C}$, let $P_{\lambda,A}$ be a cubic monic polynomial

$$P_{\lambda,A}(z) := \lambda z + Az^2 + z^3.$$

Then $P_{\lambda}$ has a fixed point of multiplier $\lambda$ at $z = 0$. Conversely, for any cubic polynomial with a fixed point of multiplier $\lambda$, there exists some $A \in \mathbb{C}$ such that it is affine conjugate to $P_{\lambda,A}$. 
Theorem 2.1. Fix $\lambda = e^{2\pi i \alpha}$ ($\alpha \notin B$) and $d \geq 4$. Suppose there exists $A \in \mathbb{C}$ such that $P_{\lambda,A}$ is linearizable at the origin, then the family $P_{\lambda,d}$ contains a $d-2$ dimensional holomorphic family of which all elements are linearizable at the origin.

Now we shall prove this main theorem.

2.1 Cubic perturbation of univalent maps

Let $\lambda = e^{2\pi i \alpha}$ for $\alpha \in \mathbb{R} - \mathbb{Q}$, and let $f$ be an element of $S_{\lambda}$. We define, for $a \in \overline{D} - \{0\}$, $A \in \mathbb{C}$ and $b \in \mathbb{C}$,

$$f_{a,A,b}(z) := a^{-1}f(az) + Abz^2 + b^2z^3.$$  

By definition, the triplet $(U', U, f)$ is called a polynomial-like map of degree $d$ if $U'$ and $U$ are simply connected proper subdomains of $\mathbb{C}$, and $U'$ is relatively compact in $U$, and $f : U' \to U$ is a holomorphic and proper map of degree $d$.

Lemma 2.1. For $A \in \mathbb{C}$ and $b \in \mathbb{C}$, we define

$$R_{A,b} := \frac{10}{9}|A||b| + \frac{15}{2},$$

$$B_{A,b} := 27R_{A,b} + 3|A||b| + \frac{81}{4} = 33|A||b| + \frac{891}{4},$$

$$W := \{z; |z| < R_{A,b}\} \text{ and}$$

$$W_{f,a,A,b} := \{z; |z| < \frac{1}{3}\} \cap f_{a,A,b}^{-1}(W).$$

For $f \in S$, $a \in \overline{D} - \{0\}$, $A \in \mathbb{C}$ and $|b|^2 > B_{A,b}$, the triplet $(W_{f,a,A,b}, W, f_{a,A,b})$ is a polynomial-like map of degree 3.

Proof. It is sufficient to prove this in $a = 1$. Since $f$ is univalent in $D$, it follows that $\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$ for $z \in D$. In particular, if $|z| = 1/3$, we have $3/16 \leq |f(z)| \leq 3/4$. For $|z| = 1/3$, it follows that

$$|b^2z^3| - |Abz^2 + f(z)| \geq \frac{|b|^2}{27} - \frac{|A||b|}{9} - \frac{3}{4} > R_{A,b}.$$  

Thus $f_{1,A,b}(\{z; |z| < 1/3\})$ properly contains the disk $W$, so $f_{1,A,b} : W_{f,1,A,b} \to W$ is proper and $W_{f,1,A,b}$ is simply connected by the maximum modulus principle. And for $|z| = 1/3$ and $z_1 \in W$, it follows that

$$|b^2z_1^3 - z_1| \geq |b^2z_1^3| - R_{A,b} > |Abz^2 + f(z)|$$

and

$$\sqrt[3]{\frac{z_1}{b^2}} < \sqrt[3]{\frac{R_{A,b}}{|b|^2}} \leq \sqrt[3]{\frac{R_{A,b}}{27R_{A,b}}} = \frac{1}{3}.$$
since $|b|^2 > B_{A,b} > 27R_{A,b}$ by definition. Thus by the theorem of Rouché, $f_{1,A,b} : W_{f,1,A,b} \to W$ is a proper map of degree three.

If $W_{f,1,A,b}$ were not connected, then the number of connected components of $W_{f,1,A,b}$ would be three or two. First, if it were three, the connected component of $W_{f,1,A,b}$ containing the origin would be conformally mapped to $W$ by $f_{1,A,b}$. However this would contradict Schwarz lemma because $|\lambda| = 1$. Second, if it were two, two cases would occur. If $f_{1,A,b}$ would conformally map the connected component of $W_{f,1,A,b}$ containing the origin to $W$, we could derive a contradiction by the same argument as above. If not, there would exist the connected component $W'$ of $W_{f,1,A,b}$ which would not contain the origin. Then $f_{1,A,b}$ would conformally map $W'$ onto $W$. So there would exist the only point $z_0 \in W'$ such that $f_{1,A,b}(z_0) = 0$. We define $\phi := (f_{1,A,b}|W'|)^{-1}$, and $\psi(z) := \phi(R_{A,b}z)$. Then $\psi$ would conformally map $\mathbb{D}$ to $W'$, and $\psi(0) = z_0$ (See figure 1).

![Figure 1:](attachment:image.png)

By the Koebe one-quarter theorem, it follows that $W'$ contains the open disk of which the radius are $\frac{1}{4}|\psi'(0)| = \frac{1}{4}R_{A,b}|\phi'(0)|$. Since $\mathbb{D}_{1/3} \supset W_{f,1,A,b} \supset W'$ and $W' \neq 0$, we have $\frac{1}{4}R_{A,b}|\phi'(0)| < \frac{1}{6}$. Hence

$$\frac{1}{|\phi'(0)|} > \frac{3R_{A,b}}{2}.$$ 

On the other hand, we have

$$\frac{1}{|\phi'(0)|} = |f'_{1,A,b}(z_0)| \leq |f'(z_0)| + 2|A||b||z_0| + 3|b|^2|z_0|^2$$
and by the Koebe theorem, $|f'(z_0)| < 9/2$ for $|z_0| < 1/3$. Since $f_{1,A,b}(z_0) = f(z_0) + Abz_0^2 + b^2z_0^3 = 0$ and $|f(z_0)| \leq \frac{|z_0|}{(1-|z_0|)^2}$, we have $3|b|^2|z_0|^2 + 2|A||b||z_0| < \frac{27}{4} + \frac{5}{3}|A||b|$ for $|z_0| < 1/3$. Hence

$$\frac{1}{|\phi'(0)|} < \frac{9}{2} + \frac{27}{4} + \frac{5}{3}|A||b| = \frac{3}{2}R_{A,b}.$$ That is a contradiction. So $W_{f,a,A,b}$ is connected, and the proof is completed.

\[\square\]

**Remark 2.1.** $|b|^2 > B_{A,b}$ if and only if $|b| > \frac{33|A| + \sqrt{1089|A| + 891}}{2} =: N(|A|)$.

**2.2 Straightening of the polynomial-like mapping**

Let $M$ be an arbitrary positive number and take a smooth function $\eta : \mathbb{R} \to [0,1]$ identically 1 on $(-\infty, 1/3]$ and identically 0 on $[R_{A,b}, +\infty)$. And we define the round annulus $A(M) := \{b; N(M) < |b| < N(M) + 1\}$.

For $f \in S_\lambda$, $a \in \overline{D} - \{0\}$, $A \in \mathrm{D}_M = \{z; |z| < M\}$ and $b \in A(M)$, we define

$$\tilde{f}_{a,A,b}(z) := \eta(|z|)f_{a,A,b}(z) + (1 - \eta(|z|))(\lambda z + Abz^2 + b^2z^3).$$

Then $\tilde{f}_{a,A,b} : \mathbb{C} \to \mathbb{C}$ is $C^\infty$ on $\mathbb{C}$.

**Lemma 2.2.** If $a \to 0$, then $\tilde{f}_{a,A,b}(z)$ converges to $\lambda z + Abz^2 + b^2z^3$ in $C^\infty$-topology on $\mathbb{C}$, and this convergence is uniform in $f \in S_\lambda$, $A \in \mathrm{D}_M$ and $b \in A(M)$.

**Proof.** The function $f_{a,A,b}$ is uniformly convergent to $\lambda z + Abz^2 + b^2z^3$ on $\{|z| \leq R_{A,b}\}$ as $a \to 0$. Since $f(z)$ is univalent on $\mathbb{D}$, the coefficients of the power series expansion of it can be estimated uniformly in $f$. It is clear that this convergence is uniform in $A \in \mathrm{D}_M$ and $b \in A(M)$. \[\square\]

We can also prove that two critical points of $z \mapsto \lambda z + Abz^2 + b^2z^3$ is included in $\{|z| < 1/3\}$ by the theorem of Rouché. We can conclude the following.

**Lemma 2.3.** There exists an $a_0 \in (0,1]$ and a continuous function $k : [0,a_0] \to [0,1)$ such that $k(0) = 0$ and for any $f \in S_\lambda$, $A \in \mathrm{D}_M$ and $b \in A(M)$ and $a \in \overline{D}_a_0 - \{0\}$, the map $\tilde{f}_{a,A,b}$ is a branched covering map of $\mathbb{C}$ of degree 3 and it satisfies

$$\left| \frac{\partial \tilde{f}_{a,A,b}(z)}{\partial \tilde{f}_{a,A,b}(z)} \right| \leq k(|a|) \quad (1/3 \leq |z| \leq R_{A,b}).$$
Moreover, $\frac{\partial \tilde{f}_{a,A,b}(z)}{\partial \tilde{f}_{a,A,b}(z)}$ holomorphically depends on $A \in \mathbb{D}_M$ and $b \in \mathcal{A}(M)$ and $a \in \mathbb{D}_{a_0} - \{0\}$. If $f$ is a polynomial, this complex dilatation is also holomorphically depends on the coefficients of $f$.

For $f \in S_{\lambda}$, $A \in \mathbb{D}_M$, $b \in \mathcal{A}(M)$ and $a \in \mathbb{D}_{a_0} - \{0\}$, We can define a Beltrami coefficient $\mu = \mu_{f,a,A,b}$ on $\mathbb{C}$ such that it is invariant for a pullback of $\tilde{f}_{a,A,b}$ and it assumes 0 on $\mathbb{C} - W$ and on $\bigcap_{n \geq 0} f_{a,A,b}(W_{f,a,A,b})$. Since supp $\mu \subset W$ and $||\mu||_{\infty} \leq k(a) < 1$, by the Ahlfors-Bers theorem, there exists a unique quasiconformal homeomorphism $\phi = \phi_{f,a,A,b}$ of $\mathbb{C}$ onto itself which satisfies the following

(i) for a.e. $z \in \mathbb{C}$, $\bar{\partial} \phi(z) = \mu(z) \partial \phi(z)$,

(ii) $\phi(0) = 0$ and

(iii) $\phi(z) - z$ is bounded on $\mathbb{C}$.

**Lemma 2.4 (cf. [1]).** There exists an $A' \in \mathbb{C}$ such that $\phi \circ \tilde{f}_{a,A,b} \circ \phi^{-1}(z) = \lambda z + A'z^2 + b^2 z^3$, where $A' \in \mathbb{C}$ holomorphically depends on $A \in \mathbb{D}_M$, $b \in \mathcal{A}(M)$ and $a \in \mathbb{D}_{a_0} - \{0\}$. If $f$ is a polynomial, it also holomorphically depends on the coefficients of $f$.

**Proof.** $\phi \circ \tilde{f}_{a,A,b} \circ \phi^{-1} : \mathbb{C} \to \mathbb{C}$ is holomorphic, fixes 0 and $\infty$. So it is a branched covering map of $\mathbb{C}$ of degree 3 fixing the origin. Thus we can write

$$\phi \circ \tilde{f}_{a,A,b} \circ \phi^{-1}(z) = \lambda' z + A'z^2 + b'z^3 \quad (\lambda', A', b' \in \mathbb{C}).$$

By the theorem of Naisul ([3]), the multiplier of the fixed point of a holomorphic map is topologically invariant when its module is 1. So we have $\lambda' = \lambda$. Next, we would like to show $b' = b^2$. According to (iii), we have

$$\phi_{f,a,A,b}(z) = z + c + \text{(lower terms)}$$

at a neighborhood of the point at infinity. When $|z|$ is sufficiently large, $\tilde{f}_{a,A,b}(z) = \lambda z + Abz^2 + b^2 z^3$ by definition, and we note that $\phi(\tilde{f}_{a,A,b}(z)) = \lambda \phi(z) + A'(\phi(z))^2 + b'(\phi(z))^3$. Therfore it follows that

$$\phi(\lambda z + Abz^2 + b^2 z^3) - (\lambda z + Abz^2 + b^2 z^3)
= (b'-b^2)z^3 + \{(A'-Ab) + 3b'c\} z^2 + \text{(lower terms)}.$$

Since this quantity is bounded as $|z| \to +\infty$, it is necessary that $b' - b^2 = 0$ and $A' - Ab + 3b'c = 0$. Thus it follows that $b' = b^2$ and $A' = Ab - 3b^2c$. \(\square\)

**Remark 2.2.** It is easy to see the following: $c = c(f, a, A, b)$ holomorphically depends on $A \in \mathbb{D}_M$, $b \in \mathcal{A}(M)$ and $a \in \mathbb{D}_{a_0} - \{0\}$. If $f$ is a polynomial, it also holomorphically depends on the coefficients of $f$. And $c \to 0$ uniformly in $f \in S_{\lambda}$, $A \in \mathbb{D}_M$ and $b \in \mathcal{A}(M)$ as $a \to 0$. 

3 Completion of proof

Let $\alpha \not\in B$ and $\lambda = e^{2\pi i \alpha}$. Suppose that $P_{A_0, \lambda}$ is linearizable at the origin. Then for $b \in \mathbb{C}^*$, $\frac{1}{b} P_{A_0, \lambda}(bz) = \lambda z + A_0 b z^2 + b^2 z^3$ is also linearizable at the origin.

We take $M = M_0 := 2|A_0| + 1$. By the Remark 2.2, for any $\epsilon > 0$ there exists an $a_1 \in (0, a_0]$ which is independent of $f \in S_\lambda$, $A \in \mathbb{D}_M$ and $b \in A(M_0)$ such that

$$3|b||c(f, a, A, b)| < \epsilon \quad (0 < |a| < a_1).$$

We can take $\epsilon > 0$ so that $|A_0| < M_0 - 2\epsilon$. We define a holomorphic map $F_{f,a,b}$ on $\mathbb{D}_{M_0}$:

$$A \mapsto A - 3bc(f, a, A, b).$$

By the theorem of Rouché, there exists $A_1 = A_1(f, a, b)$ such that $F_{f,a,b}(A_1) = A_0$. We can see that $A_1 = A_1(f, a, b)$ holomorphically depends on $b \in A(M_0)$ and $a \in \mathbb{D}_{a_1} - \{0\}$, and if $f$ is a polynomial, it also holomorphically depends on the coefficients of $f$. We can conclude the following.

**Proposition 3.1.** For any $f \in S_\lambda$, $b \in A(M_0)$ and $a \in \mathbb{D}_{a_1} - \{0\}$, there exists $A_1 = A_1(f, a, b)$ which is holomorphic in $a \in \mathbb{D}_{a_1} - \{0\}$, $b \in A(M_0)$ and $f \in S_\lambda$ and also exists $\phi = \phi_{f,a,A_1,b}$ which is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ onto itself which is defined in the previous section such that

$$\phi \circ f_{a,A_1,b} \circ \phi^{-1}(z) = \frac{1}{b} P_{A_0, \lambda}(bz).$$

So if there exists $A_0 \in \mathbb{C}$ such that $P_{A_0, \lambda}$ is linearizable at the origin, $f_{a,A_1,b}(z) = a^{-1}f(az) + A_1 b z^2 + b^2 z^3$ is linearizable at the origin.

In particular, we consider for $d > 1$,

$$U_d := \{ P(z) = \lambda z + a_2 z^2 + \cdots + a_d z^d; \sum_{n=2}^d n|a_n| \leq 1 \} \subset S_\lambda.$$ 

Consequently we can at least conclude the Theorem 2.1.

**References**


