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Brjuno Numbers and Non Linearizability of Polynomials of Degree more than two

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Abstract

We consider an irrational number α which is not a Brjuno number. If there exists a cubic polynomial which has a Siegel point of multiplier $\exp(2\pi i\alpha)$, for any $d \geq 4$ there exists a $d-2$ dimensional holomorphic family of $\mathcal{P}_{\lambda,d}$ of which all elements have a Siegel point of multiplier $\exp(2\pi i\alpha)$.

1 Introduction

Consider a germ of a holomorphic map $(\mathbb{C},0) \to (\mathbb{C},0)$

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$$

with multiplier λ at $z=0$. And we consider the case that $|\lambda| = 1$ but λ is not a root of unity. Thus the multiplier λ can be written as

$$\lambda = e^{2\pi i\alpha} \text{ for an } \alpha \in \mathbb{R} - \mathbb{Q}.$$

The origin is said to be an irrationally indifferent fixed point.

The linearization problem for f is whether or not there exists a holomorphic local change of coordinate $z = h(w)$ with $h(0) = 0$ and $h'(0) \neq 0$ which conjugates f to the irrational rotation $w \mapsto \lambda w$ so that

$$h(\lambda w) = f(h(w))$$

near the origin. We say that an irrationally indifferent fixed point is a Siegel point or a Cremer point according as the local linearization is possible or not (cf. [2]).

For a suitable $t > 0$, the map $z \mapsto \frac{1}{t} f(tz)$ is holomorphic and univalent on the unit disk $\mathbb{D}$. So we consider the special case that f is holomorphic and univalent on the unit disk $\mathbb{D}$. Let

$$S := \{f;\ \text{holomorphic and univalent map on } \mathbb{D}\ ,\ f(0) = 0\ ,\ \text{and } |f'(0)| = 1\},$$
$$S_\lambda := \{f \in S;\quad f'(0) = \lambda\}.$$

Then we can consider the linearization problem for $f \in S_\lambda$ at the origin.

Definition 1.1. *A map $f \in S_\lambda$ will be called linearizable at the origin if there exist a neighborhood U_f of the origin and a map H_f which is holomorphic and univalent on U_f, and satisfies*

$$H_f(0) = 0, \quad H_f'(0) = 1, \quad f(H_f(z)) = H_f(\lambda z).$$

In this case H_f will be called a linearizing map of f at the origin, and the connected component of the Fatou set of f which contains the origin is called a Siegel disk of f at the origin.

In order to state the results on this problem, we should introduce the following.

Definition 1.2. *For $\alpha \in \mathbb{R} - \mathbb{Q}$, we consider the continued fraction expantion*

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

where a_0 is an integral part of α, and $\alpha_1 := \alpha - a_0$. a_1 is an integral part of $1/\alpha_1$, and $\alpha_2 := 1/\alpha_1 - a_1$. a_2 is an integral part of $1/\alpha_2$. Inductively, we define a_n. And we define the n-th approximate fraction

$$\frac{p_n}{q_n} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

where p_n/q_n is an irreducible fraction. An $\alpha \in \mathbb{R} - \mathbb{Q}$ is called a Brjuno number if

$$\sum_{i \geq 0} \frac{\log q_{i+1}}{q_i} < +\infty.$$

We define $\mathcal{B} := \{\alpha \in \mathbb{R} - \mathbb{Q}; \alpha \text{ is a Brjuno number.}\}$.

Next we state the known results about this problem.

Theorem 1.1 (Brjuno). *If $\alpha \in \mathcal{B}$, all maps $f \in S_{\exp(2\pi i\alpha)}$ are linearizable at the origin.*

And Yoccoz proved that this result is best possible.

Theorem 1.2 (Yoccoz [5]). *If $\alpha \notin \mathcal{B}$, there exists a map $f \in S_{\exp(2\pi i\alpha)}$ which is non linearizable at the origin.*

Furthermore for quadratic polynomials, Yoccoz proved the following.

Theorem 1.3 (Yoccoz [5]). *If $\alpha \notin \mathcal{B}$,*

$$P(z) = e^{2\pi i\alpha}z + z^2$$

is non linearizable at the origin.

For $d \geq 2$, we define

$$\mathcal{P}_{\lambda,d} := \{P(z) = \lambda z + a_2 z^2 + \cdots + a_d z^d; (a_2, \ldots, a_d) \in \mathbb{C}^{d-1}\} \cong \mathbb{C}^{d-1}$$

For $\mathcal{P}_{\lambda,d}$, $d \geq 3$, Pérez-Marco proved the following.

Theorem 1.4 (Pérez-Marco [4]). *Fix $\lambda = \exp(2\pi i\alpha)$ ($\alpha \notin \mathcal{B}$) and $d \geq 3$. There exists an open dense subset of $\mathcal{P}_{\lambda,d}$ of which all elements are non linearizable at the origin.*

Theorem 1.5 (Pérez-Marco [4]). *In the same condition as above, for any $(a_3, \ldots, a_d) \in \mathbb{C}$ ($a_d \neq 0$), There exists an open dense subset U of $\mathbb{C}$ such that if $a_2 \in U$, then $z \mapsto \lambda z + a_2 z^2 + a_3 z^3 \cdots + a_d z^d$ is non linearizable at the origin.*

We would like to consider the linearizability of polynomials of degree more than two in Section 2.

2 Linearizability of polynomials of degree more than two

For $\lambda = e^{2\pi i\alpha}$ ($\alpha \in \mathbb{R} - \mathbb{Q}$), and $A \in \mathbb{C}$, let $P_{\lambda,A}$ be a cubic monic polynomial

$$P_{\lambda,A}(z) := \lambda z + Az^2 + z^3.$$

Then P_λ has a fixed point of multiplier λ at $z = 0$. Conversely, for any cubic polynomial with a fixed point of multiplier λ, there exists some $A \in \mathbb{C}$ such that it is affine conjugate to $P_{\lambda,A}$.

Theorem 2.1. *Fix $\lambda = e^{2\pi i\alpha}$ $(\alpha \notin \mathcal{B})$ and $d \geq 4$. Suppose there exists $A \in \mathbb{C}$ such that $P_{\lambda,A}$ is linerizable at the origin, then the family $\mathcal{P}_{\lambda,d}$ contains a $d-2$ dimensional holomorphic family of which all elements are linearizable at the origin.*

Now we shall prove this main theorem.

2.1 Cubic perturbation of univalent maps

Let $\lambda = e^{2\pi i\alpha}$ for $\alpha \in \mathbb{R} - \mathbb{Q}$, and let f be an element of S_λ. We define, for $a \in \overline{\mathbb{D}} - \{0\}$, $A \in \mathbb{C}$ and $b \in \mathbb{C}$,

$$f_{a,A,b}(z) := a^{-1}f(az) + Abz^2 + b^2z^3.$$

By definition, the triplet (U', U, f) is called a polynomial-like map of degree d if U' and U are simply connected proper subdomains of $\mathbb{C}$, and U' is relatively compact in U, and $f : U' \to U$ is a holomorphic and proper map of degree d.

Lemma 2.1. *For $A \in \mathbb{C}$ and $b \in \mathbb{C}$, we define*

$$\begin{aligned} R_{A,b} :=& \frac{10}{9}|A||b| + \frac{15}{2}, \\ B_{A,b} :=& 27R_{A,b} + 3|A||b| + \frac{81}{4} = 33|A||b| + \frac{891}{4}, \\ W :=& \{z; |z| < R_{A,b}\} \text{ and} \\ W_{f,a,A,b} :=& \{z; |z| < \frac{1}{3}\} \cap f^{-1}_{a,A,b}(W). \end{aligned}$$

For $f \in S$, $a \in \overline{\mathbb{D}} - \{0\}$, $A \in \mathbb{C}$ and $|b|^2 > B_{A,b}$, the triplet $(W_{f,a,A,b}, W, f_{a,A,b})$ is a polynomial-like map of degree 3.

Proof. It is sufficient to prove this in $a = 1$. Since f is univalent in $\mathbb{D}$, it follows that $\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$ for $z \in \mathbb{D}$. In particular, if $|z| = 1/3$, we have $3/16 \leq |f(z)| \leq 3/4$. For $|z| = 1/3$, it follows that

$$|b^2z^3| - |Abz^2 + f(z)| \geq \frac{|b|^2}{27} - \frac{|A||b|}{9} - \frac{3}{4} > R_{A,b}.$$

Thus $f_{1,A,b}(\{z; |z| < 1/3\})$ properly contains the disk W, so $f_{1,A,b} : W_{f,1,A,b} \to W$ is proper and $W_{f,1,A,b}$ is simply connected by the maximum modulus principle. And for $|z| = 1/3$ and $z_1 \in W$, it follows that

$$|b^2z^3 - z_1| \geq |b^2z^3| - R_{A,b} > |Abz^2 + f(z)| \quad \text{and}$$

$$\sqrt[3]{\left|\frac{z_1}{b^2}\right|} < \sqrt[3]{\frac{R_{A,b}}{|b|^2}} \leq \sqrt[3]{\frac{R_{A,b}}{27R_{A,b}}} = \frac{1}{3}$$

since $|b|^2 > B_{A,b} > 27R_{A,b}$ by definition. Thus by the theorem of Rouché, $f_{1,A,b} : W_{f,1,A,b} \to W$ is a proper map of degree three.

If $W_{f,1,A,b}$ were not connected, then the number of connected components of $W_{f,1,A,b}$ would be three or two. First, if it were three, the connected component of $W_{f,1,A,b}$ containing the origin would be conformally mapped to W by $f_{1,A,b}$. However this would contradict Schwarz lemma because $|\lambda| = 1$. Second, if it were two, two cases would occur. If $f_{1,A,b}$ would conformally map the connected component of $W_{f,1,A,b}$ containing the origin to W, we could derive a contradiction by the same argument as above. If not, there would exist the connected component W' of $W_{f,1,A,b}$ which would not contain the origin. Then $f_{1,A,b}$ would conformally map W' onto W. So there would exist the only one point $z_0 \in W'$ such that $f_{1,A,b}(z_0) = 0$. We define $\phi := (f_{1,A,b}|W')^{-1}$, and $\psi(z) := \phi(R_{A,b}z)$. Then ψ would conformally map $\mathbb{D}$ to W', and $\psi(0) = z_0$ (See figure 1).

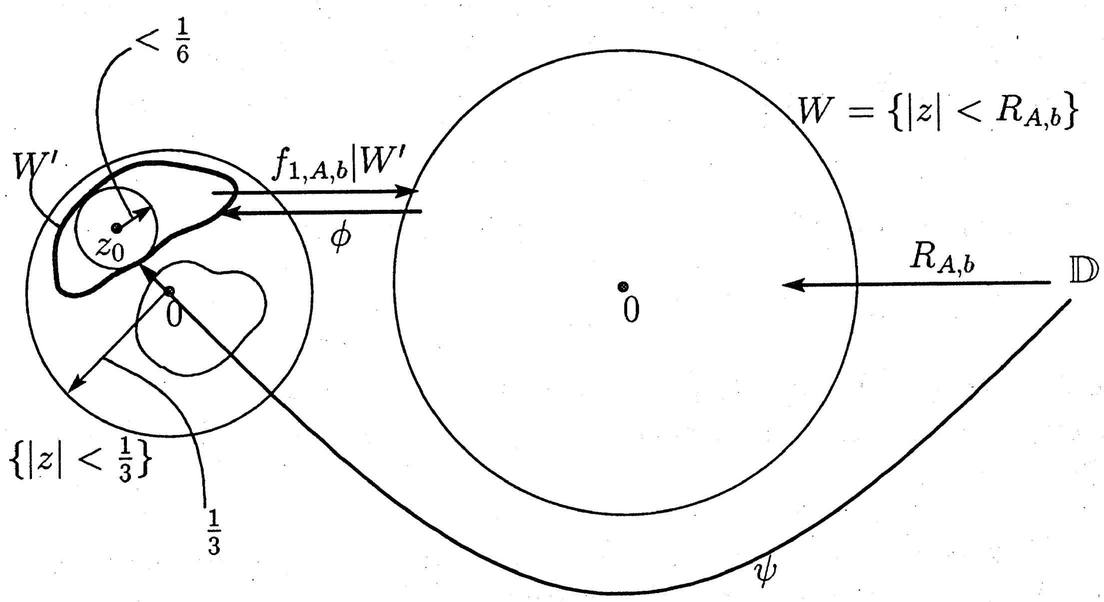

Figure 1:

By the Koebe one-quarter theorem, it follows that W' contains the open disk of which the radius are $\frac{1}{4}|\psi'(0)| = \frac{1}{4}R_{A,b}|\phi'(0)|$. Since $\mathbb{D}_{1/3} \supset W_{f,1,A,b} \supset W'$ and $W' \not\ni 0$, we have $\frac{1}{4}R_{A,b}|\phi'(0)| < \frac{1}{6}$. Hence

$$\frac{1}{|\phi'(0)|} > \frac{3R_{A,b}}{2}.$$

On the other hand, we have

$$\frac{1}{|\phi'(0)|} = |f'_{1,A,b}(z_0)| \leq |f'(z_0)| + 2|A||b||z_0| + 3|b|^2|z_0|^2$$

and by the Koebe theorem, $|f'(z_0)| < 9/2$ for $|z_0| < 1/3$. Since $f_{1,A,b}(z_0) = f(z_0)+Abz_0^2+b^2z_0^3 = 0$ and $|f(z_0)| \le \frac{|z_0|}{(1-|z_0|)^2}$, we have $3|b|^2|z_0|^2+2|A||b||z_0| < \frac{27}{4}+\frac{5}{3}|A||b|$ for $|z_0| < 1/3$. Hence

$$\frac{1}{|\phi'(0)|} < \frac{9}{2}+\frac{27}{4}+\frac{5}{3}|A||b| = \frac{45}{4}+\frac{5}{3}|A||b| = \frac{3}{2}R_{A,b}.$$

That is a contradicton. So $W_{f,a,A,b}$ is connected, and the proof is completed. □

Remark 2.1. $|b|^2 > B_{A,b}$ *if and only if* $|b| > \frac{33|A|+\sqrt{1089|A|+891}}{2} =: N(|A|)$.

2.2 Straightenning of the polynomial-like mapping

Let M be an arbitrary positive number and take a smooth function $\eta : \mathbb{R} \to [0,1]$ identically 1 on $(-\infty, 1/3]$ and identically 0 on $[R_{A,b}, +\infty)$. And we define the round annulus $\mathcal{A}(M) := \{b; N(M) < |b| < N(M)+1\}$.

For $f \in S_\lambda$, $a \in \overline{\mathbb{D}} - \{0\}$, $A \in \mathbb{D}_M = \{z; |z| < M\}$ and $b \in \mathcal{A}(M)$, we define

$$\tilde{f}_{a,A,b}(z) := \eta(|z|)f_{a,A,b}(z) + (1-\eta(|z|))(\lambda z + Abz^2 + b^2z^3).$$

Then $\tilde{f}_{a,A,b} : \mathbb{C} \to \mathbb{C}$ is C^∞ on $\mathbb{C}$.

Lemma 2.2. *If $a \to 0$, then $\tilde{f}_{a,A,b}(z)$ converges to $\lambda z + Abz^2 + b^2z^3$ in C^∞-topology on $\mathbb{C}$, and this convergence is uniform in $f \in S_\lambda$, $A \in \mathbb{D}_M$ and $b \in \mathcal{A}(M)$.*

Proof. The function $f_{a,A,b}$ is uniformly convergent to $\lambda z + Abz^2 + b^2z^3$ on $\{|z| \le R_{A,b}\}$ as $a \to 0$. Since $f(z)$ is univalent on $\mathbb{D}$, the coefficients of the power series expantion of it can be estimated uniformly in f. It is clear that this convergence is unifoum in $A \in \mathbb{D}_M$ and $b \in \mathcal{A}(M)$. □

We can also prove that two critical points of $z \mapsto \lambda z+Abz^2+b^2z^3$ is included in $\{|z| < 1/3\}$ by the theorem of Rouché. We can conclude the following.

Lemma 2.3. *There exists an $a_0 \in (0,1]$ and a continuous function $k : [0,a_0] \to [0,1)$ such that $k(0) = 0$ and for any $f \in S_\lambda$, $A \in \mathbb{D}_M$ and $b \in \mathcal{A}(M)$ and $a \in \overline{\mathbb{D}}_{a_0} - \{0\}$, the map $\tilde{f}_{a,A,b}$ is a branched covering map of $\mathbb{C}$ of degree 3 and it satisfies*

$$\left|\frac{\bar{\partial}\tilde{f}_{a,A,b}(z)}{\partial\tilde{f}_{a,A,b}(z)}\right| \le k(|a|) \quad (1/3 \le |z| \le R_{A,b}).$$

Moreover, $\frac{\bar{\partial}\tilde{f}_{a,A,b}(z)}{\partial\tilde{f}_{a,A,b}(z)}$ *holomorphically depends on* $A \in \mathbb{D}_M$ *and* $b \in \mathcal{A}(M)$ *and* $a \in \mathbb{D}_{a_0} - \{0\}$*. If* f *is a polynomial, this complex dilatation is also holomorphically depends on the coeffecients of* f*.*

For $f \in S_\lambda$, $A \in \mathbb{D}_M$, $b \in \mathcal{A}(M)$ and $a \in \mathbb{D}_{a_0} - \{0\}$, We can define a Beltrami coefficient $\mu = \mu_{f,a,A,b}$ on $\mathbb{C}$ such that it is invariant for a pullback of $\tilde{f}_{a,A,b}$ and it assumes 0 on $\mathbb{C} - W$ and on $\bigcap_{n\geq 0} \tilde{f}_{a,A,b}^{-n}(W_{f,a,A,b})$. Since $\operatorname{supp}\mu \subset W$ and $\|\mu\|_\infty \leq k(a) < 1$, by the Ahlfors-Bers theorem, there exists a unique quasiconformal homeomorphism $\phi = \phi_{f,a,A,b}$ of $\mathbb{C}$ onto itself which satisfies the following

(i) for a.e.$z \in \mathbb{C}$, $\bar{\partial}\phi(z) = \mu(z)\partial\phi(z)$,

(ii) $\phi(0) = 0$ and

(iii) $\phi(z) - z$ is bounded on $\mathbb{C}$.

Lemma 2.4 (cf. [1]). *There exists an* $A' \in \mathbb{C}$ *such that* $\phi \circ \tilde{f}_{a,A,b} \circ \phi^{-1}(z) = \lambda z + A'z^2 + b^2z^3$*, where* $A' \in \mathbb{C}$ *holomorphically depends on* $A \in \mathbb{D}_M$*,* $b \in \mathcal{A}(M)$ *and* $a \in \mathbb{D}_{a_0} - \{0\}$*. If* f *is a polynomial, it also holomorphically depends on the coeffecients of* f*.*

Proof. $\phi \circ \tilde{f}_{a,b} \circ \phi^{-1} : \mathbb{C} \to \mathbb{C}$ is holomorphic, fixes 0 and ∞. So it is a branched covering map of $\mathbb{C}$ of degree 3 fixing the origin. Thus we can write

$$\phi \circ \tilde{f}_{a,A,b} \circ \phi^{-1}(z) = \lambda' z + A'z^2 + b'z^3 \quad (\lambda',\ A',\ b' \in \mathbb{C}).$$

By the theorem of Naisul ([3]), the multiplier of the fixed point of a holomorphic map is topologically invariant when its module is 1. So we have $\lambda' = \lambda$. Next, we would like to show $b' = b^2$. According to (iii), we have

$$\phi_{f,a,A,b}(z) = z + c + \text{(lower terms)}$$

at a neighborhood of the point at infinity. When $|z|$ is sufficiently large, $\tilde{f}_{a,A,b}(z) = \lambda z + Abz^2 + b^2z^3$ by definition, and we note that $\phi(\tilde{f}_{a,A,b}(z)) = \lambda\phi(z) + A'(\phi(z))^2 + b'(\phi(z))^3$. Therfore it follows that

$$\begin{aligned}\phi(\lambda z + Abz^2 + b^2z^3) &- (\lambda z + Abz^2 + b^2z^3) \\ &= (b' - b^2)z^3 + \{(A' - Ab) + 3b'c\}z^2 + \text{(lower terms)}.\end{aligned}$$

Since this quantity is bounded as $|z| \to +\infty$, it is necessary that $b' - b^2 = 0$ and $A' - Ab + 3b'c = 0$. Thus it follows that $b' = b^2$ and $A' = Ab - 3b^2c$. □

Remark 2.2. *It is easy to see the following:* $c = c(f, a, A, b)$ *holomorphically depends on* $A \in \mathbb{D}_M$*,* $b \in \mathcal{A}(M)$ *and* $a \in \mathbb{D}_{a_0} - \{0\}$*. If* f *is a polynomial, it also holomorphically depends on the coeffecients of* f*. And* $c \to 0$ *uniformly in* $f \in S_\lambda$*,* $A \in \mathbb{D}_M$ *and* $b \in \mathcal{A}(M)$ *as* $a \to 0$*.*

3 Completion of proof

Let $\alpha \notin \mathcal{B}$ and $\lambda = e^{2\pi i\alpha}$. Suppose that $P_{A_0,\lambda}$ is linearizable at the origin. Then for $b \in \mathbb{C}^*$, $\frac{1}{b}P_{A_0,\lambda}(bz) = \lambda z + A_0bz^2 + b^2z^3$ is also linearizable at the origin.

We take $M = M_0 := 2|A_0| + 1$. By the Remark 2.2, for any $\epsilon > 0$ there exists an $a_1 \in (0, a_0]$ which is independent of $f \in S_\lambda$, $A \in \mathbb{D}_M$ and $b \in \mathcal{A}(M_0)$ such that

$$3|b||c(f, a, A, b)| < \epsilon \quad (0 < |a| < a_1).$$

We can take $\epsilon > 0$ so that $|A_0| < M_0 - 2\epsilon$. We define a holomorphic map $F_{f,a,b}$ on $\mathbb{D}_{M_0}$:

$$A \mapsto A - 3bc(f, a, A, b).$$

By the theorem of Rouché, there exists $A_1 = A_1(f, a, b)$ such that $F_{f,a,b}(A_1) = A_0$. We can see that $A_1 = A_1(f, a, b)$ holomorphically depends on $b \in \mathcal{A}(M_0)$ and $a \in \mathbb{D}_{a_1} - \{0\}$, and if f is a polynomial, it also holomorphically depends on the coeffecients of f. We can conclude the following.

Proposition 3.1. *For any $f \in S_\lambda$, $b \in \mathcal{A}(M_0)$ and $a \in \mathbb{D}_{a_1} - \{0\}$, there exists $A_1 = A_1(f, a, b)$ which is holomorphic in $a \in \mathbb{D}_{a_1} - \{0\}$, $b \in \mathcal{A}(M_0)$ and $f \in S_\lambda$ and also exists $\phi = \phi_{f,a,A_1,b}$ which is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ onto itself which is defined in the previous section such that*

$$\phi \circ \tilde{f}_{a,A_1,b} \circ \phi^{-1}(z) = \frac{1}{b}P_{A_0,\lambda}(bz).$$

So if there exists $A_0 \in \mathbb{C}$ such that $P_{A_0,\lambda}$ is linearizable at the origin, $f_{a,A_1,b}(z) = a^{-1}f(az) + A_1bz^2 + b^2z^3$ is linearizable at the origin.

In particular, we consider for $d > 1$,

$$\mathcal{U}_d := \{P(z) = \lambda z + a_2z^2 + \cdots + a_dz^d; \sum_{n=2}^{d} n|a_n| \leq 1\} \subset S_\lambda.$$

Consequently we can at least conclude the Theorem 2.1.

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