数理解析研究所講究録
1043巻 1998年 250-257

A Non-Zero-Sum Repeated Game —— Criminal vs Police

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Abstract In this paper a conflict between a potential criminal offender and a law-enforcement authorities is studied. The model is a non-zero-sum n-period game with perfect information, where each player has to "act" at most a permitted times during the n periods. We formulate the game by dynamic programming and derive equiliblia of the four two-period games each in an explicit form depending on the parameter values of the game. It is shown that in equilibrium the offender is more pushed to making a crime, and the defender invests more effort in law-enforcement, both in the second period than in the first. The fact that the expected payoff to the offender is non-decreasing as his illeagal income coming from an unpunished crime increases, but the expected payoff to the dedender is not necessarily non-increasing, is also established.

1. Formulation of the Multistage Game —— Criminal vs Police.

The game is played as a repeated game over n periods between a potenti al criminal offender (hereafter called a criminal, or player I) and a law-enforcement authorities (hereafter called police, or player II) Being a repeated game implies that the fundamentals of the game are the same in each period. There are two pure strategies available in each period to player I: to commit a crime (C) and to act honestly (H). Similarly, player II has two pure strategies: to enforce the law (E) or to do nothing (N). If player I chooses H he earns his leagal income $r > 0$ (dollars). If he chooses C, illegal income in amount of $\pi > 0$, in addition to his legal income r, may be earned. However if I's crime is detected and arrested by II, I is punished by having to pay a fine in amount of $f > 0$, and inprisoned until the end of the game. When caught in prison, I earns no income at all, of course.

If player II chooses E, with a cost of $c > 0$ (dollars), he can (cannot), catch I's crime with probability $p (\bar{p} = 1 - p)$. In case that I commits crime that goes unpunished, a loss of $l > 0$ is inflicted upon society.

So a single stage of this game has the game tree as shown by Figure 1, and is represented by a bimatrix game with payoff bimatrix (1).

We assume that $c < pl$ i.e. the strategy E for player II has a positive merit of choosing. This condition is very important as is seen in the proofs of the subsequent theorems.

We shall disacuss the n-stage game, where player I wants to commit crime at most k of n periods, and player II attempts to prevent I's illegal act by taking enforcement action at most m times during n periods. After each period is over, the outcome in that period becomes known to both players. The total payoff during n periods is the sum of the payoffs on each period. We assume that all of the above information is known to both players.

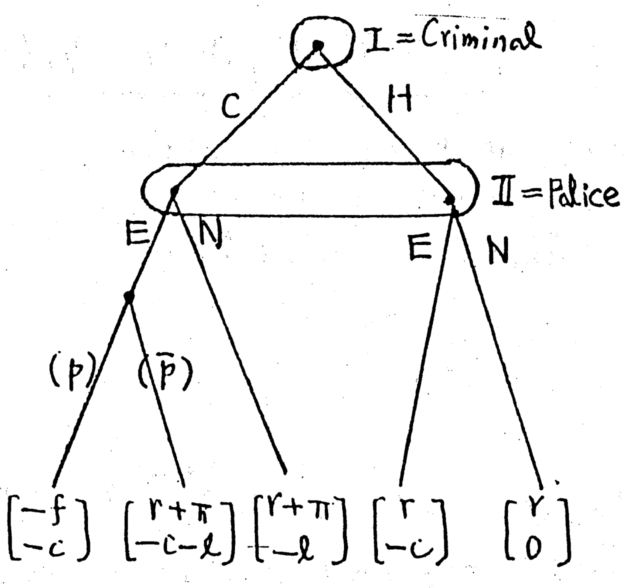

Figure 1. Game tree of a single stage

C: commit a crime
H: be honest
E: law-enforcement
N: do nothing
p: prob. of being punished
f: amount of fine
$r(\pi)$: leagal (illeagal) income to I (if unpunished)
c: II's cost of law-enforcement
ℓ: social loss to II for an unpunished crime

(1)

		(II) E	(II) N
(I)	C	$-pf+\bar{p}(r+\pi)$, $-(c+\bar{p}\ell)$	$r+\pi$, $-\ell$
	H	r, $-c$	r, 0

Let $\Gamma_{k,m}(n)$ denote the game described above. (n, k, m) denotes the state of the system in which players I and II possess k and m times to take actions, respectively, and they have n periods to go as their "mission time." Let $(u_{k,m}(n), v_{k,m}(n))$ represent the equilibrium values of this non-zero-sum n-stage game $\Gamma_{k,m}(n)$. Then the Optimality Equation of dynamic programming gives a system of equations

$$(u_{k,m}(n), v_{k,m}(n)) = \text{Eq.Val.}$$

$$(2)\quad \begin{array}{c|c|c} & E & N \\ \hline C & -pf+\bar{p}\{r+\pi+u_{k-1,m-1}(n-1)\},\ -(c+\bar{p}\ell)+\bar{p}v_{k-1,m-1}(n-1) & r+\pi+u_{k-1,m}(n-1),\ -\ell+v_{k-1,m}(n-1) \\ \hline H & r+u_{k,m-1}(n-1),\ -c+v_{k,m-1}(n-1) & r+u_{k,m}(n-1),\ v_{k,m}(n-1) \end{array}$$

(if the equilibrium values exist uniquely), with the boundary conditions :

(2a) $\quad (u_{0,m}(n), v_{0,m}(n)) = (nr, 0), \quad \text{for } 1 \leqq m \leqq n,$

(2b) $\quad (u_{k,0}(n), v_{k,0}(n)) = (nr + k\pi, -k\ell), \quad \text{for } 1 \leqq k \leqq n,$

(2c) $\quad (u_{0,0}(n), v_{0,0}(n)) = (nr, 0), \quad \text{for } n \geqq 1,$

(2d) $\quad u_{k,m}(0) = v_{k,m}(0) = 0, \quad \forall k, m \geqq 0,$

and

(2e) $\quad (u_{k,m}(n), v_{k,m}(n)) = (u_{k',m'}(n), v_{k',m'}(n)), \text{ with } k' = k \wedge n,\ m' = m \wedge n.$

The four conditions (2a) ~ (2d) imply that: (a) If II has m times of law-enforcement and his opponent has none of the opportunity of violation, then the decision-pair H-N is repeated throughout the whole period, (b) If I has k times of violating law and his opponent cannot do anything becaause of lack of budget, then I chooses C and H k and $n-k$ times, respectively, during the n periods, (c) If both players do not ~~have~~ any law-violation and law-enforcement intentions, the decision -pair H-N is repeated throughghout the whole period, and (d) The problem with $n=1$ reduces to the bimatrix game with payoff matrix (1).

If release from prison and a second offense are not taken into account, we need not consider large n, and the optimality equation (2), with (2a) ~ (2e), can be, in principle solved by backward induction. The two-period games $\Gamma_{n,n}(n)$, $\Gamma_{1,1}(n)$, $\Gamma_{1,n}(n)$ and $\Gamma_{n,1}(n)$, all for n=2, are explicitly solved in the subsequent sections 2,3,4 and 5, respectively.

Concerning the n-period games of the kind discussed in this paper, Sakaguchi [3] studies a zero-sum game of smuggler vs customs, and Dawid, Feichtinger and Jorgensen [1] and Sakaguchi [4] investigate non-zero-sum games, with $k=m=n$,the latter being related to full-informattion optimal stopping games. Also Kilgour [2] studies a zero-sum game with $k=m=n$, where the offender is not restricted to the two pure strategies and is asked to choose the "level of violation" $q \in [0,1]$ in each stage.

2. The Game $\Gamma_{n,n}(n)$.

First we consider the case $k=m=n$, which is discussed by Dawid, Feichtinger and Jorgensen [1]. Let us simply write $U_n = u_{n,n}(n)$ and $V_n = v_{n,n}(n)$. Then (2) becomes

$$(U_n, V_n) = \text{Eq.Val.}\begin{array}{|c|c|}\hline -pf+\bar{p}(r+\pi+U_{n-1}),\ -(c+\bar{p}l)+\bar{p}V_{n-1} & r+\pi+U_{n-1},\ -l+V_{n-1} \\ \hline r+U_{n-1},\quad -c+V_{n-1} & r+U_{n-1},\quad V_{n-1} \\ \hline \end{array}$$

$$(3) \qquad = (U_{n-1}, V_{n-1}) + \text{Eq.Val.}\, M_n$$

where M_n is a bimatrix

$$(4) \qquad M_n = \begin{array}{|c|c|}\hline \bar{p}(r+\pi)-p(f+U_{n-1}),\ -(c+\bar{p}l+pV_{n-1}) & r+\pi,\quad -l \\ \hline r,\qquad -c & r,\quad 0 \\ \hline \end{array}$$

$$(\, n \geqq 1 ;\ U_0 = V_0 = 0\,)$$

which is identical to(1)for $n=1$.

The two theorems that follow are not new, and essentially reproduction of the main results in [1]but with more simpler description of the proofs.

Theorem 1. Let $\pi_1 = \left(\frac{p}{\bar{p}}\right)(f+r)$. Then the solution to the game $\Gamma_{1,1}(1)$ is :

Case	$0<\pi<\pi_1$	$\pi=\pi_1$	$\pi>\pi_1$
Eq. play	$x_1^*-y_1^*$, with $x=\frac{c}{\bar{p}l}$, $y=\frac{\pi}{\bar{p}(f+r+\pi)}$	$(zC+\bar{z}H)$-E $\forall z\in[0,1]$	C-E
Eq.values	$U_1=r$ $V_1=-c/p$	$U_1=r$ $V_1=-(c+\bar{p}l\bar{z})$	$U_1=-pf+\bar{p}(r+\pi)$ $V_1=-(c+\bar{p}l)$

(The mixed-strategy equilibrium is denoted by $x_1^*-y_1^*$ with $x_1^*=\langle x,\bar{x}\rangle$ and y_1^* $\langle y,\bar{y}\rangle$. The strategy $zC+\bar{z}H$ means the mixture of the pure strategies C and H, with probabilities z and $\bar{z}$, respectively.)

Proof We have $-(c+\bar{p}l)>-l$, since we assumed $c<pl$. So by the "circular rule" of finding the eq.of 2×2 bimatrix game, which of $-pf+\bar{p}(r+\pi)$ and r is larger becomes important. Clearly $-pf+\bar{p}(r+\pi)\{\gtreqless\}r$, if $\pi\{\gtreqless\}\pi_1$. If $0<\pi<\pi_1$ there is a mixed-strategy eq. $x_1^*-y_1^*$ which together with the eq.values is found by solving

$$\begin{cases} -(c+\bar{p}l)-c\bar{x}=-lx=V_1 \ , \\ \{-pf+\bar{p}(r+\pi)\}y+(r+\pi)\bar{y}=r=U_1 \ , \end{cases}$$

If $\pi>\pi_1$,there is a pure-strategy eq.C-E. □

Theorem 2 Let $\pi_2 = \left(\frac{p}{\bar{p}}\right)(f+r)+\left(\frac{p}{\bar{p}^2}\right)r$. Then the solution tto the game $\Gamma_{2,2}(2)$ is :

Case	$0<\pi<\pi_1$	$\pi_1<\pi<\pi_2$	$\pi>\pi_2$
Eq play	x_2^*-y_2^* with $x=c/(pl+c)$ $y=\pi/p(f+2r+\pi)$	x_2^*-y_2^* with $x=c/p\{c+(1+\bar{p})l\}$ $y=(\pi/p)/\{\bar{p}f+(1+\bar{p})(r+\pi)\}$	C-E
Eq values	$U_2=2r$ $V_2=-\dfrac{c(2pl+c)}{p(pl+c)}$	$U_2=-pf+\bar{p}(r+\pi)+r$ $V_2=-\left[\dfrac{cl}{p\{c+(1+\bar{p})l\}}+c+\bar{p}l\right]$	$U_2=(1+\bar{p})\{-pf+\bar{p}(r+\pi)\}$ $V_2=-(1+\bar{p})(c+\bar{p}l)$

(The solutions for the bordering cases $\pi=\pi_1$ and $\pi=\pi_2$ are omitted. The mixed-strategy equilibrium is denoted by x_2^*-y_2^* with $x_2^*=\langle x,\bar{x}\rangle$ and $y_2^*=\langle y,\bar{y}\rangle$.)

Proof Substituing the values of U_1 and V_1, which was found in Theorem 1, into M_2 in (4) we obtain the bimatrices

$$(5_1)\qquad \left[\begin{array}{cc|cc} \bar{p}(r+\pi)-p(f+r), & -\bar{p}l & r+\pi, & -l \\ \hline r, & -c & r, & 0 \end{array}\right], \qquad \text{if } 0<\pi<\pi_1 \ ;$$

$$(5_2)\qquad \left[\begin{array}{cc|cc} \bar{p}\{-pf+\bar{p}(r+\pi)\}, & -\bar{p}(c+\bar{p}l) & r+\pi, & -l \\ \hline r, & -c & r, & 0 \end{array}\right], \qquad \text{if } \pi>\pi_1 .$$

Since $0<\pi<\pi_1\equiv(p/\bar{p})(f+r) \Rightarrow \pi<(p/\bar{p})(f+2r) \Leftrightarrow \bar{p}(r+\pi)-p(f+r)<r$, the bimatrix (5_1) has the mixed-strategy solution and it is found,together with eq. values by solving

$$\begin{cases} -\bar{p}lx-c\bar{x}=-lx=V_2-V_1 \\ \{\bar{p}(r+\pi)-p(f+r)\}y+(r+\pi)\bar{y}=r=U_2-U_1 \end{cases}$$

and giving the result mentioned in the theorem (1st column of the table).

Since $c<pl<pl(1+1/\bar{p}) \Leftrightarrow \bar{p}(c+\bar{p}l)<l$ and

$$\bar{p}\{-pf+\bar{p}(r+\pi)\}\begin{Bmatrix}\le\\=\\>\end{Bmatrix}r, \quad \text{if} \quad \pi\begin{Bmatrix}<\\=\\>\end{Bmatrix}\pi_2\equiv(p/\bar{p})(f+r)+(p/\bar{p}^2)r,$$

the bimatrix (5_2) has the mixed-strategy eq. x_2^*-y_2^* if $\pi_1<\pi<\pi_2$, and pure-strategy eq. C-E, if $\pi>\pi_2$. This mixed-strategy eq. together with eq. values is found by solving

$$\begin{cases} -\bar{p}(c+\bar{p}l)x-c\bar{x}=-lx=V_2-V_1 \\ \bar{p}\{-pf+\bar{p}(r+\pi)\}y+(r+\pi)\bar{y}=r=U_2-U_1 \end{cases}$$

and giving the result mentioned in the theorem (2nd column of the table). The pure-strategy eq. C-E for $\pi>\pi_2$ gives the eq. values

$$(U_2, V_2)=(U_1+\bar{p}\{-pf+\bar{p}(r+\pi)\},\ V_1-\bar{p}(c+\bar{p}l)),$$

which are mentioned in the 3rd column of the table in the theorem. □

Hereafter we shall omit considerations about the bordering cases, where a continuum of equilibria in the first period and correspondingly in the whole game exist. Direct calculations show that the game value for the offender is continuous, non-decreasing in π, and the game value for the defender involves the mixing parameter $z\in[0,1]$ chosen arbitrarily by the offender.

In the game $\Gamma_{1,1}(1)$, $\pi > \pi_1 = (p/\bar{p})(f+r)$ means that commiting crime for I has a positive expected payoff to I. Theorem 1 implies that I is not motivated to commit crime as long as $\pi < \pi_1$ and C-E is in equilibrium if $\pi > \pi_1$. II should take p and f larger, in order to make law-violater get low profit.

In the game $\Gamma_{2,2}(2)$, $\pi > \pi_2$ means that commiting a crime for I in the first stage has a positive profit in that stage since $\bar{p}\pi > p\{f+2r+\bar{p}\pi-p(f+r)\}$. Theorem 2 implies that I is not motivatedto commit crime in both stages as long as $\pi < \pi_1$, the choice-pair C-E is in eq.in the first stage if $\pi > \pi_2$, and the intermediate situation arises if $\pi_1 < \pi < \pi_2$.

3. The Game $\Gamma_{1,1}(n)$.

4 The Game $\Gamma_{1,n}(n)$.

5, The Game $\Gamma_{n,1}(n)$.

6, Remarks and a Numerical Example

1°) From Theorems 1 and 2 , the two-period equilibrium play in the game $\Gamma_{2,2}(2)$ is as follows :

Case	Two-period eq play in $\Gamma_{2,2}(2)$
$0 < \pi < \pi_1$	$\{x_2^*\text{-}y_2^*;\ x_1^*\text{-}y_1^*\}$, with $(x_2^*)_1 = c/(pl+c)$, $(y_2^*)_1 = \pi/\{p(f+2r+\pi)\}$ and $(x_1^*)_1 = c/(pl)$, $(y_1^*)_1 = \pi/\{p(f+r+\pi)\}$.
$\pi_1 < \pi < \pi_2$	$\{x_2^*\text{-}y_2^*;\ \text{C-E}\}$, with $(x_2^*)_1 = c/p\{c+(1+\bar{p})l\}$, and $(y_2^*)_1 = (\pi/p)/\{\bar{p}f+(1+\bar{p})(r+\pi)\}$.
$\pi > \pi_2$	$\{\text{C-E};\ \text{C-E}\}$

(For example $\{x_2^*\text{-}y_2^*;\ x_1^*\text{-}y_1^*\}$ means that players employ mixed-strategy pair $x_2^*\text{-}y_2^*$ in the 1sat period, and if the game is not over(cf. the game is over if and only if it results in C-E and I is punished by II) $x_1^*\text{-}y_1^*$ is employed in the 2nd period.)

2°) For the game $\Gamma_{2,2}(2)$ we easily find, from Theorems 1 and 2, that

$$(x_2^*)_1 \leqq (x_1^*)_1 \quad \text{and} \quad (y_2^*)_1 \leqq (y_1^*)_1 \quad \text{for all} \quad \pi > 0.$$

This means that player I is more pushed to making a crime, and II invests more effort in law-enforcement, both in the 2nd period than in the 1st. From Theorems 4 and 5, the above is true in the games $\Gamma_{1,2}(2)$ and $\Gamma_{2,1}(2)$ also.

3°) Throughout Theorems 2~5, the following fact is observed. As functions of $\pi > 0$ the two-period eq.payoff U_2 for I is continuous and non-decreasing, but V_2 for II is piece-wise constant and not necessarily decreasing as π increases.

4°) We give, by Table 2, a numerical example of the solutions to the four two-period games for the parameters $r=1/2, p=2/3, c=1, f=\ell=2$ and therefore $c < p\ell$ being satisfied and $\pi_1 = 5 < \pi_2 = 8$.

Table2. Example of Solutions to the Five Games

Game	Case	Eq.play x^*-y^* in the 1st period	Eq.payoffs	Based on
$\Gamma_{1,1}(1)$	1	$\langle 3/4, 1/4\rangle - \langle \frac{3\pi}{2\pi+5}, \frac{5-\pi}{2\pi+5}\rangle$	$1/2 - (-3/2)$	Th. 1
	2	C-E	$(1/6)(2\pi-7) - (-5/3)$	
$\Gamma_{2,2}(2)$	1	$\langle 3/7, 4/7\rangle - \langle \frac{3\pi}{2(\pi+3)}, \frac{6-\pi}{2(\pi+3)}\rangle$	$1 - (-33/14)$	Th. 2
	2′	$\langle 9/22, 13/22\rangle - \langle \frac{9\pi}{8(\pi+1)}, \frac{8-\pi}{8(\pi+1)}\rangle$	$(1/3)(\pi-2) - (-82/33)$	
	2″	C-E	$(2/9)(2\pi-7) - (-20/9)$	
$\Gamma_{1,1}(2)$	1	$\langle 9/11, 2/11\rangle - \langle \frac{3\pi}{5\pi+6}, \frac{2\pi+6}{5\pi+6}\rangle$	$\frac{3\pi^2+5\pi+6}{5\pi+6} - (-\frac{21}{11})$	Th. 3
	2	$\langle 4/5, 1/5\rangle - \langle \frac{2\pi+5}{4\pi+11}, \frac{2\pi+6}{4\pi+11}\rangle$	$\frac{2\pi^2+9\pi+11}{4\pi+11} - (-\frac{29}{15})$	
$\Gamma_{1,2}(2)$	1	$\langle 3/4, 1/4\rangle - \langle \frac{3\pi}{2(\pi+3)}, \frac{6-\pi}{2(\pi+3)}\rangle$	$1 - (-15/8)$	Th. 4
	2	$\langle 3/4, 1/4\rangle - \langle \frac{2\pi+5}{2(\pi+3)}, \frac{1}{2(\pi+3)}\rangle$	$(1/3)(\pi-2) - (-23/12)$	
$\Gamma_{2,1}(2)$	1	$\langle 9/16, 7/16\rangle - \langle \frac{3\pi}{2(2\pi+3)}, \frac{\pi+6}{2(2\pi+3)}\rangle$	$\frac{3\pi^2+4\pi+6}{2(2\pi+3)} - (-21/8)$	Th. 5
	2	$\langle 1/2, 1/2\rangle - \langle \frac{3\pi}{2(2\pi+3)}, \frac{\pi+6}{2(2\pi+3)}\rangle$	$\frac{10\pi^2+13\pi-12}{6(2\pi+3)} - (-8/3)$	

(Case 1,2,2′,2″ means $0<\pi<5$, $\pi>5$, $5<\pi<8$, $\pi>8$, respectively.)

We can make sure that, for all $\pi > 0$,

$$u_{1,2}(2) \leq \left\{ \begin{matrix} u_{1,1}(2) \\ u_{2,2}(2) \end{matrix} \right\} \leq u_{2,1}(2), \qquad \text{for I},$$

and

$$v_{2,1}(2) \leq \left\{ \begin{matrix} v_{1,1}(2) \\ v_{2,2}(2) \end{matrix} \right\} \leq v_{1,2}(2), \qquad \text{for II}.$$

Furthermore, we can check that the facts mentioned in 3°) are true.

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