The Optimal Stopping Problem for Fuzzy Random Sequences

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1. Introduction and notations

Fuzzy random variables was first studied by Puri and Ralescu [7] and have been studied by many authors. Stojaković [9] discussed fuzzy conditional expectation and Puri and Ralescu [8] studied fuzzy martingales. This paper discusses optimal stopping problems of a sequence of fuzzy random variables.

Let \((\Omega, \mathcal{M}, P)\) be a probability space, \(\mathcal{M}\) is a \(\sigma\)-field and \(P\) is a probability measure. Let \(\mathbb{R}\) be the set of all real numbers and let \(\mathbb{N}\) be the set of all nonnegative integers. \(\mathcal{B}\) denotes the Borel \(\sigma\)-field of \(\mathbb{R}\) and \(\mathcal{I}\) denotes the set of all bounded closed sub-intervals of \(\mathbb{R}\). A fuzzy set \(\tilde{a}\) is called a fuzzy number if the membership function \(\tilde{a} : \mathbb{R} \mapsto [0, 1]\) is normal, upper-semicontinuous, convex and has a compact support. \(\mathcal{R}\) denotes the set of all fuzzy numbers. We write the \(\alpha\)-cut \((\alpha \in [0, 1])\) of a fuzzy number \(\tilde{a} \in \mathcal{R}\) by

\[
\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+], \quad \alpha \in [0, 1].
\]

A map \(\tilde{X} : \Omega \mapsto \mathcal{R}\) is called a fuzzy random variable if

\[
\{(\omega, x) \mid \tilde{X}(\omega)(x) \geq \alpha\} = \{(\omega, x) \mid x \in \tilde{X}_\alpha(\omega)\} \in \mathcal{M} \times \mathcal{B} \quad \text{for all } \alpha \in [0, 1],
\]

where \(\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] := \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\} (\in \mathcal{I})\) is \(\alpha\)-cut of fuzzy numbers \(\tilde{X}(\omega)\) for \(\omega \in \Omega\).

Lemma 1.1 ([10, Theorems 2.1 and 2.2]). For a map \(\tilde{X} : \Omega \mapsto \mathcal{R}\), the following (i) and (ii) are equivalent:

(i) \(\tilde{X}\) is a fuzzy random variable.

(ii) The maps \(\omega \mapsto \tilde{X}_\alpha^-(\omega)\) and \(\omega \mapsto \tilde{X}_\alpha^+(\omega)\) are measurable for all \(\alpha \in [0, 1]\).

A fuzzy random variable \(\tilde{X}\) is called integrably bounded if \(\omega \mapsto \tilde{X}_\alpha^-(\omega)\) and \(\omega \mapsto \tilde{X}_\alpha^+(\omega)\) are integrable for all \(\alpha \in [0, 1]\). For an integrably bounded fuzzy random variable \(\tilde{X}\), we define closed intervals

\[
E(\tilde{X})_{\alpha} = \left[\int_{\mathbb{R}} \tilde{X}_\alpha^-(\omega) \, dP(\omega), \int_{\mathbb{R}} \tilde{X}_\alpha^+(\omega) \, dP(\omega)\right], \quad \alpha \in [0, 1].
\]

Then the map \(\alpha \mapsto E(\tilde{X})_{\alpha}\) is left-continuous by the dominated convergence theorem. Therefore, the expectation \(E(\tilde{X})\) is a fuzzy number defined by

\[
E(\tilde{X})(x) := \sup_{\alpha \in [0, 1]} \min \left\{\alpha, 1_{E(\tilde{X})_{\alpha}}(x)\right\} \quad \text{for } x \in \mathbb{R}.
\]
For an integrably bounded fuzzy random variable $\tilde{X}$ and a sub-$\sigma$-field $\mathcal{N}(\subset \mathcal{M})$, the conditional expectation $E(\tilde{X}|\mathcal{N})$ is defined as follows: For $\alpha \in [0,1]$, there exist unique classical conditional expectations $E(\tilde{X}_\alpha^{-}|\mathcal{N})$ and $E(\tilde{X}_\alpha^{+}|\mathcal{N})$ such that

$$\int_\Lambda E(\tilde{X}_\alpha^{-}|\mathcal{N})(\omega) \, dP(\omega) = \int_\Lambda \tilde{X}_\alpha^{-}(\omega) \, dP(\omega) \quad \text{for all } \Lambda \in \mathcal{N},$$

and

$$\int_\Lambda E(\tilde{X}_\alpha^{+}|\mathcal{N})(\omega) \, dP(\omega) = \int_\Lambda \tilde{X}_\alpha^{+}(\omega) \, dP(\omega) \quad \text{for all } \Lambda \in \mathcal{N}. \quad (1.5)$$

Then we can easily check the maps $\alpha \mapsto E(\tilde{X}_\alpha^{-}|\mathcal{N})(\omega)$ and $\alpha \mapsto E(\tilde{X}_\alpha^{+}|\mathcal{N})(\omega)$ are left-continuous by the monotone convergence theorem. Therefore, we define

$$E(\tilde{X}_\alpha|\Lambda')(\omega) := [E(\tilde{X}_\alpha^{-}|\mathcal{N})(\omega), E(\tilde{X}_\alpha^{+}|\mathcal{N})(\omega)] \quad \text{for } \omega \in \Omega. \quad (1.7)$$

and we give a conditional expectation by a fuzzy random variable

$$E(\tilde{X}|\mathcal{N})(\omega)(\mathcal{N}) := \sup \min \{ \alpha, 1_{E(\tilde{X}_\alpha|\mathcal{N})(\omega) \in [0,1]} \} \quad \text{for } x \in \mathbb{R}. \quad (1.8)$$

2. An optimal stopping problem

Let $\{\tilde{X}_n\}_{n \in \mathbb{N}}$ be a sequence of fuzzy random variables. $\mathcal{M}_n (n \in \mathbb{N})$ denotes the smallest $\sigma$-field on $\Omega$ generated by $\{\tilde{X}_{k,\alpha}^{-}, \tilde{X}_{k,\alpha}^{+} | k = 0, 1, 2, \cdots, n; \alpha \in [0,1] \}$, and $\mathcal{M}_\infty$ denotes the smallest $\sigma$-field generated by $\bigcup_{n \in \mathbb{N}} \mathcal{M}_n$. A map $\tau : \Omega \mapsto \mathbb{N} \cup \{\infty\}$ is called a stopping time if

$$\{\tau = n\} \in \mathcal{M}_n \quad \text{for all } n \in \mathbb{N}. \quad (2.1)$$

Lemma 2.1. For a finite stopping time $\tau$, we define

$$\tilde{X}_\tau(\omega) := \tilde{X}_n(\omega), \quad \omega \in \{\tau = n\} \quad \text{for } n \in \mathbb{N}. \quad (2.2)$$

Then, $\tilde{X}_\tau$ is a fuzzy random variable.

Let $g : \mathcal{I} \mapsto \mathbb{R}$ be a weighting function, which is continuous and monotone (see Fortemps and Roubens [3]). Using this $g$, the scalarization of the fuzzy reward will be done by

$$G_\tau(\omega) := \begin{cases} \int_0^1 g(\tilde{X}_{\tau,\alpha}(\omega)) \, d\alpha, & \text{if } \tau(\omega) < \infty \\ \limsup_{n \to \infty} \int_0^1 g(\tilde{X}_{n,\alpha}(\omega)) \, d\alpha & \text{if } \tau(\omega) = \infty. \end{cases} \quad (2.3)$$

Note that $g(\tilde{X}_{\tau,\alpha}(\omega)) \in \mathbb{R}$ and the map $\alpha \mapsto g(\tilde{X}_{\tau,\alpha}(\omega))$ is left-continuous on $[0,1]$, so that the right-hand integral of (2.3) is well-defined. From the linearity of the weighting function $g$, we define

$$E(G_\tau) := E \left( \int_0^1 g(\tilde{X}_{\tau,\alpha}(\cdot)) \, d\alpha \right) = \int_0^1 g(\tilde{E}(\tilde{X}_\alpha)) \, d\alpha \quad \text{for stopping times } \tau. \quad (2.4)$$
Definition 2.1. A stopping time $\tau^*$ is called optimal if $E(G_{\tau^*}) \geq E(G_{\tau})$ for all stopping times $\tau$.

Define

$$Z_n(\omega) := \text{ess sup}_{\tau \geq n} E(G_{\tau} | \mathcal{M}_n) = \text{ess sup}_{\tau \geq n} E \left( \int_0^1 g(\tilde{X}_{\tau,n}(\cdot)) \, d\alpha | \mathcal{M}_n \right),$$

for $\omega \in \Omega$, $n \in \mathbb{N}$.

Lemma 2.2. Define

$$\sigma^*(\omega) := \inf \{ n | G_n(\omega) = Z_n(\omega) \}, \quad \omega \in \Omega,$$

where the infimum of the empty set is understood to be $+\infty$. If $\sigma^* < \infty$, then $\sigma^*$ is an optimal stopping time for Definition 2.1.

3. A fuzzy stopping problem

Definition 3.1. A fuzzy stopping time is a map $\tilde{\tau} : \mathbb{N} \times \Omega \mapsto [0,1]$ satisfying the following (i) – (iii):

(i) For each $n \in \mathbb{N}$, $\tilde{\tau}(n, \cdot)$ is $\mathcal{M}_n$-measurable.

(ii) For each $\omega \in \Omega$, $n \mapsto \tilde{\tau}(n,\omega)$ is non-increasing.

(iii) For each $\omega \in \Omega$, there exists an integer $n_0$ such that $\tilde{\tau}(n,\omega) = 0$ for all $n \geq n_0$.

In the grade of membership of stopping times, '0' and '1' represent 'stop' and 'continue' respectively. The following lemmas imply the properties of fuzzy stopping times.

Lemma 3.1.

(i) Let $\tilde{\tau}$ be a fuzzy stopping time. Define a map $\tilde{\tau}_{\alpha} : \Omega \mapsto \mathbb{N}$ by

$$\tilde{\tau}_{\alpha}(\omega) = \inf \{ n \in \mathbb{N} | \tilde{\tau}(n,\omega) < \alpha \} \quad (\omega \in \Omega) \quad \text{for } \alpha \in (0,1],$$

where the infimum of the empty set is understood to be $+\infty$. Then, we have:

(a) $\{ \tilde{\tau}_{\alpha} \leq n \} \in \mathcal{M}_n \quad (n \in \mathbb{N})$;

(b) $\tilde{\tau}_{\alpha}(\omega) \leq \tilde{\tau}_{\alpha'}(\omega)$ \quad ($\omega \in \Omega$) \quad if $\alpha \geq \alpha'$;

(c) $\lim_{\alpha' \uparrow \alpha} \tilde{\tau}_{\alpha'}(\omega) = \tilde{\tau}_{\alpha}(\omega)$ \quad ($\omega \in \Omega$) \quad if $\alpha > 0$;

(d) $\tilde{\tau}_0(\omega) := \lim_{\alpha \downarrow 0} \tilde{\tau}_{\alpha}(\omega) < \infty$ \quad ($\omega \in \Omega$).
(ii) Let \( \{ \tilde{\tau}_\alpha \}_{\alpha \in [0,1]} \) be maps \( \tilde{\tau}_\alpha : \Omega \mapsto \mathbb{N} \) satisfying the above (a) – (d). Define a map \( \tilde{\tau} : \mathbb{N} \times \Omega \mapsto [0,1] \) by

\[
\tilde{\tau}(n, \omega) := \sup_{\alpha \in [0,1]} \{ \alpha \wedge 1_{\{\tilde{\tau}_\alpha > n\}}(\omega) \}, \quad n \in \mathbb{N}, \ \omega \in \Omega. \tag{3.2}
\]

Then \( \tilde{\tau} \) is a fuzzy stopping time.

Let \( g : \mathcal{I} \mapsto \mathbb{R} \) be a weighting function (see [3]). For a fuzzy stopping time \( \tilde{\tau}(n, \omega) \), the scalarization of the fuzzy reward will be done by

\[
G_{\tilde{\tau}}(\omega) := \int_0^1 g(\tilde{X}_{\tilde{\tau}_\alpha}(\omega)) \, d\alpha, \quad \omega \in \Omega, \tag{3.3}
\]

where \( \tilde{\tau}_\alpha \) is defined by (3.1). Note that \( g(\tilde{X}_{\tilde{\tau}_\alpha}(\omega)) \in \mathbb{R} \) and the map \( \alpha \mapsto g(\tilde{X}_{\tilde{\tau}_\alpha}(\omega)) \) is left-continuous on \((0,1]\), so that the integral of (3.3) is well-defined. From the linearity of the weighting function \( g \), we define

\[
E(G_{\tilde{\tau}}) := E \left( \int_0^1 g(\tilde{X}_{\tilde{\tau}_\alpha}(\cdot)) \, d\alpha \right) = \int_0^1 g(E(\tilde{X}_{\tilde{\tau}_\alpha})) \, d\alpha \tag{3.4}
\]

for fuzzy stopping times \( \tilde{\tau} \).

**Definition 3.2.**

(i) Let \( \alpha \in [0,1] \). A stopping time \( \tau^* \) is called \( \alpha \)-optimal if \( g(E(\tilde{X}_{\tau^*})_{\alpha}) \geq g(E(\tilde{X}_{\tau})_{\alpha}) \) for all stopping times \( \tau \).

(ii) A fuzzy stopping time \( \tilde{\tau}^* \) is called optimal if \( E(G_{\tilde{\tau}^*}) \geq E(G_{\tilde{\tau}}) \) for all fuzzy stopping times \( \tilde{\tau} \).

Define a sequence of subsets \( \{ \Lambda_n \}_{n=0}^{\infty} \) of \( \Omega \) by

\[
\Lambda_n := \{ \omega \in \Omega \mid g(\tilde{X}_{n,\alpha})(\omega) \geq E(g(\tilde{X}_{n+1,\alpha})|\mathcal{M}_n)(\omega) \}, \quad n \in \mathbb{N}. \tag{3.5}
\]

**Assumption A** (Monotone case).

\[ \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots \quad \text{and} \quad \bigcup_{n=0}^{\infty} \Lambda_n = \Omega. \]

In order to characterize \( \alpha \)-optimal stopping times, let

\[
\gamma_n^\alpha := \text{ess sup}_{\tilde{\tau} : \tilde{\tau}_\alpha \geq n} E(g(\tilde{X}_{\tilde{\tau}_\alpha})|\mathcal{M}_n) \quad \text{for } n \in \mathbb{N}. \tag{3.5}
\]

And we define a map \( \hat{\sigma}_\alpha^* : \Omega \mapsto \mathbb{N} \) by

\[
\hat{\sigma}_\alpha^*(\omega) := \inf \left\{ n \mid g(\tilde{X}_{n,\alpha})(\omega) = \gamma_n^\alpha(\omega) \right\} \tag{3.6}
\]
for $\omega \in \Omega$ and $\alpha \in [0,1]$, where the infimum of the empty set is understood to be $+\infty$. Then, the next lemma is given by Chow et al. [2].

**Lemma 3.2** ([2, Theorems 4.1 and 4.5]). Suppose Assumption A holds. Then, the following (i) and (ii) hold:

(i) $\gamma_n^\alpha(\omega) = \max\{g(\tilde{X}_{n,\omega}),\gamma_{n+1}^\alpha(\omega)\}$ a.a. $\omega \in \Omega$ for $n \in \mathbb{N}$.

(ii) Let $\alpha \in [0,1]$. If $\tilde{\sigma}_n^\alpha < \infty$ a.s., then $\tilde{\sigma}_n^\alpha$ is $\alpha$-optimal and $E(\gamma_0^\alpha) = E(g(\tilde{X}_{\tilde{\sigma}_n^\alpha,\omega}))$.

In order to construct an optimal fuzzy stopping time from $\alpha$-optimal stopping times $\{\tilde{\sigma}_n^\alpha\}_{\alpha \in [0,1]}$, we need a regularity condition.

**Assumption B** (Regularity of fuzzy stopping times). A fuzzy stopping time $\tilde{\sigma}$ is called regular if the map $\alpha \mapsto \tilde{\sigma}_\alpha^*(\omega)$ is non-increasing for each $\omega \in \Omega$.

Under Assumption B, we can assume the left-continuity of the map $\alpha \mapsto \tilde{\sigma}_\alpha^*(\omega)$ and we can define a map $\tilde{\sigma}^* : \mathbb{N} \times \Omega \mapsto [0,1]$ by

$$\tilde{\sigma}^*(n,\omega) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\tilde{\sigma}_\alpha^*(n,\omega) > n\}}\}, \quad n \in \mathbb{N}, \, \omega \in \Omega. \quad (3.7)$$

**Theorem 3.1.** Suppose Assumptions A and B hold. Then $\tilde{\sigma}^*$ is an optimal fuzzy stopping time.

**References**


