A New Parametric Method for Finding Efficient Solutions in Biobjective Shortest Route Problems (Decision Theory and Its Related Fields)

Author(s)
Furukawa, Nagata

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A New Parametric Method for Finding Efficient Solutions in Biobjective Shortest Route Problems

Nagata Furukawa (Soka University)

§ 1. Preliminaries

Denote the variables representing two quantities we want to minimize in the biobjective programs by \( x \) and \( y \). We call the coordinates plane having its orthogonal coordinates \( x \) and \( y \) the original plane, which is denoted by \( \mathcal{P} \). The points of the plane \( \mathcal{P} \) are written as \( a = (x, y) \) and \( b = (x', y') \), etc.

An order relation among the points of the plane \( \mathcal{P} \) is given by the usual manner. Namely, for two points \( a = (x, y) \) and \( b = (x', y') \), we write

\[
a \leq b \quad \text{iff} \quad x \leq x' \text{ and } y \leq y'.
\]

Let \( \Omega \) be a non-empty finite subset of \( \mathcal{P} \). We consider the optimization problem:

\[
(P_0) \quad \text{Minimize } a \text{ subject to } a \in \Omega.
\]

A point \( a \in \Omega \) is said to be an efficient solution to the problem \((P_0)\), if there is no point \( b \in \Omega \) such that \( a \geq b \) and \( a \neq b \).

For \( t \in (0, 1) \), define a \( 2 \times 2 \) matrix \( G(t) \) by

\[
G(t) = \begin{bmatrix} t & 1 \\ t & -1 \end{bmatrix}.
\]

We call the matrix \( G(t) \) a transformation matrix, and \( t \) is called a transformation parameter. Trivially, the matrix \( G(t) \) is nonsingular for every \( t \in (0, 1) \). Let

\[
\begin{bmatrix} u \\ \alpha \end{bmatrix} = G(t) \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{for} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{P}.
\]

The coordinates plane whose orthogonal coordinates are given by \( u \) and \( \alpha \) of (4) is called the transformed plane, which is denoted by \( \mathcal{H} \). The points of the plane \( \mathcal{H} \) are written as \( A = (u, \alpha) \) and \( B = (v, \beta) \), etc..

**Proposition 1.** Let \( a = (x, y) \) and \( b = (x', y') \) be two points of \( \mathcal{P} \). Let \( 0 < t \leq 1 \) be arbitrary but fixed. Let \( A = (u, \alpha) \) and \( B = (v, \beta) \) be the points of \( \mathcal{H} \) transformed from \( a \) and \( b \) by the transformation matrix \( G(t) \),
respectively. Then the order relation $a \leq b$ is equivalently transformed to a relation on $\mathfrak{H}$ as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} x' \\ y' \end{bmatrix} \iff |\alpha - \beta| \leq v-u.$$  \hfill (4)

§ 2. An order relation on the transformed plane

**Definition 1.** Let $0 \leq \lambda \leq 1$ be arbitrary. For two points $A=(u, \alpha)$ and $B=(v, \beta)$ of the transformed plane, we define an order relation $\leq^\lambda$ with the parameter $\lambda$ by the following:

$$A \leq^\lambda B \iff \begin{cases} (i) & |\alpha - \beta| \leq v-u, \\
 & \text{or} \\
(ii) & 0 < \lambda(\beta-\alpha) \leq |v-u| < \beta - \alpha, \\
 & \text{or} \\
(iii) & 0 < v-u < \lambda|\alpha-\beta|, \\
 & \text{or} \\
(iv) & u=v \text{ and } \alpha < \beta. \end{cases}$$ \hfill (5)

The four cases in the right-hand side of (5) are exclusive one another. Let $a$ and $b$ be the points transformed from $A$ and $B$, respectively, by $G(t)^{-1}$. Proposition 1, then, implies that the condition (i) in the right-hand side of (5) is equivalent to the relation $a \leq b$. This fact holds true regardless of the value of the transformation parameter $t$.

**Proposition 2.** For each $0 \leq \lambda \leq 1$, the order relation $\leq^\lambda$ is reflexive and asymmetric on $\mathfrak{H}$.

**Proposition 3.** For any pair $A=(u, \alpha)$ and $B=(v, \beta)$ of points in the plane $\mathfrak{H}$ and for any $0 \leq \lambda \leq 1$, either $A \leq^\lambda B$ or $B \leq^\lambda A$ necessarily holds.
Proposition 4. Let $0 \leq \lambda \leq 1$ be arbitrary, and let $A$ and $B$ be two points on $\mathfrak{X}$. For every positive number $\mu$ and every point $C$ on $\mathfrak{X}$, then, it holds that 

$$A \preceq^\lambda B \Rightarrow \mu A \preceq^\lambda \mu B \text{ and } A \pm C \preceq^\lambda B \pm C.$$ 

§ 3. Descent sequence on the transformed plane

Proposition 5. Let $A = (u, \alpha)$ and $B = (v, \beta)$ be two points on $\mathfrak{X}$ satisfying that 

$$u < v \text{ and } \alpha < \beta. \tag{6}$$

Then $A$ is smaller than $B$ with respect to the order $\preceq^\lambda$ for every $0 \leq \lambda \leq 1$.

Definition 2. Let $\{A_i = (u_i, \alpha_i) ; i = 1, 2, \cdots, m\}$, where $m \geq 3$, be a finite sequence of points on $\mathfrak{X}$. Then the sequence is said to be descending to the right, iff it holds that

$$\begin{align*}
&\begin{cases}
    u_i < u_{i+1}, \\
    \alpha_i > \alpha_{i+1},
\end{cases} & i = 1, 2, \cdots, m-1.
\end{align*} \tag{7}$$

For a sequence $\{A_i = (u_i, \alpha_i) ; i = 1, 2, \cdots, m\}$ of points, if it holds that

$$\begin{align*}
&\begin{cases}
    u_i > u_{i+1}, \\
    \alpha_i < \alpha_{i+1},
\end{cases} & i = 1, 2, \cdots, m-1,
\end{align*} \tag{8}$$

then the sequence is, of course, descending to the right, by renumbering the index of points. But, in order to unify the numbering of points, when we speak of a sequence descending to the right, we suppose to imply the condition (7) but not (8).

We denote the gradient of the line segment connecting two points $a$ and $b$ on the original plane by $\gamma_{ab}$. Similarly, we denote the gradient of the line segment connecting two points $A$ and $B$ on the transformed plane by $\gamma_{AB}$.

Theorem 1. Let $\{a_i = (x_i, y_i) ; i = 1, 2, \cdots, m\}$, where $m \geq 3$, be a sequence of points on $\mathcal{P}$ such that
\begin{align*}
& x_i > x_{i+1}, \quad i = 1, 2, \ldots, m-1. \\
& y_i < y_{i+1}, \quad i = 1, 2, \ldots, m-1.
\end{align*}

(9)

Suppose that
\[ \gamma_{a_{i-1}} < \gamma_{a_{i}a_{i+1}}, \quad i = 2, 3, \ldots, m-1. \]

(10)

Choose \( t \) such that \( 0 < t < \text{Min}\left\{ \left| \gamma_{a_{m-1}a_m} \right|, 1 \right\} \). Define \((u_i, \alpha_i)\), \( i = 1, 2, \ldots, m \), by
\[ \begin{bmatrix} u_i \\ \alpha_i \end{bmatrix} = G(t) \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \quad i = 1, 2, \ldots, m, \]

(11)

and let \( \mathcal{A} = \{ A_i = (u_i, \alpha_i) ; i = 1, 2, \ldots, m \} \). Then it holds that

(i) The sequence \( \mathcal{A} \) is descending to the right:

(ii) \( 0 > \gamma_{A_{i-1}A_i} > \gamma_{A_iA_{i+1}}, \quad i = 2, 3, \ldots, m-1. \)

\section*{§ 4. Detection of efficient solutions}

\textbf{Definition 3.} For two points \( A = (u, \alpha) \) and \( B = (v, \beta) \) on \( \mathcal{H} \), if it holds that
\[ |\alpha - \beta| > |u - v|, \]

(12)

then the points are said to be \textit{mutually nondominant}. For a finite sequence \( \{ A_i = (u_i, \alpha_i) ; i = 1, 2, \ldots, m \} \) of points on \( \mathcal{H} \), if every two elements of the sequence are mutually nondominant, then it is said that the sequence is \textit{mutually nondominant}.

\textbf{Proposition 6.} Let \( A = (u, \alpha) \) and \( B = (v, \beta) \) be points on \( \mathcal{H} \). Let \( t \in (0, 1) \) be arbitrary, and let \( a = (x, y) \) and \( b = (x', y') \) be the points transformed by \( G(t)^{-1} \) from \( A \) and \( B \), respectively. Then, \( A \) and \( B \) are mutually nondominant, if and only if, the relation
\[ \begin{cases} 
   x < x' \text{ and } y > y', \\
   \text{or} \\
   x > x' \text{ and } y < y', 
\end{cases} \]

holds.
As we have stated in the preceding section, the order relation $\leq^\lambda$ is reflexive and asymmetric but not transitive on the whole plane $\mathcal{H}$. However, it can be shown that if we restrict ourselves to the family of sequences which are mutually nondominant and descending to the right, then the relation $\leq^\lambda$ is transitive on each of the sequences.

**Theorem 2.** Let $\mathcal{A} = \{ A_i = (u_i, \alpha_i) ; i = 1, 2, \ldots, m \}$ be mutually nondominant and descending to the right. Then the relation $\leq^\lambda$ is a total order relation on the sequence $\mathcal{A}$ for every $\lambda \in [0, 1]$.

Let $M$ denote the set $\{1, 2, \ldots, m\}$. Throughout the remainder of this section, it is assumed that $m \geq 3$.

**Proposition 6.** Let $\mathcal{A} = \{ A_i = (u_i, \alpha_i) ; i = 1, 2, \ldots, m \}$ be mutually nondominant and descending to the right. Suppose that the relations
\[\gamma_{A_{i-1}A_i} \succ \gamma_{A_iA_{i+1}}, \quad i = 2, 3, \ldots, m-1.\] (13)
hold. For each $k \in M$, put
\[\lambda_{ki} = \frac{u_k - u_i}{\alpha_i - \alpha_k} \quad \text{for} \quad i \in M \setminus \{k\}.\] (14)
Then we have

(i) for each $k \in M$,
\[0 < \lambda_{ki} < 1 \quad \text{for} \quad i \in M \setminus \{k\},\] (15)
(ii) for each $k \in N$,
\[\lambda_{ki} = \lambda_{ik} \quad \text{for} \quad i \in M \setminus \{k\},\] (16)
(iii) for each $k \in N$,
\[\lambda_{ki} > \lambda_{k,i+1} \quad \text{for} \quad i \in M \setminus \{k-1, k, m\},\] (17)
(iv) $\lambda_{k-1,k} > \lambda_{k,k+1}$ for $k = 2, 3, \ldots, m-1$. (18)

Now, let $\{a_i = (x_i, y_i) ; i = 1, 2, \ldots, m\}$ be the whole set of efficient solutions to the problem $(P_0)$. Without of loss of generality, we may assume that
\[ x_i > x_{i+1}, \quad y_i < y_{i+1}, \quad i = 1, 2, \ldots, m - 1. \]  

(19)

It is well known that if the efficient solutions \( a_i = (x_i, y_i), \quad i = 1, 2, \ldots, m \) satisfy the condition:

\[ \gamma_{a_{i-1}a_i} > \gamma_{a_i a_{i+1}}, \quad i = 2, 3, \ldots, m - 1, \]

(20)

in addition to (37), then the solutions can be all detected by the usual scalarization method. In this paper, we consider another condition:

\[ \gamma_{a_{i-1}a_i} < \gamma_{a_i a_{i+1}}, \quad i = 2, 3, \ldots, m - 1. \]

(21)

**Theorem 3.** Let \( \{a_i = (x_i, y_i); \quad i = 1, 2, \ldots, m\} \) be the whole set of efficient solutions to the problem \( (P_0) \). Suppose the conditions (19) and (21) to be satisfied. Choose \( t \) such that \( 0 < t < \text{Min}\{\gamma_{a_{m-1}a_m}, 1\} \), and define \( \mathcal{A} = \{A_i = (u_i, \alpha_i); \quad i = 1, 2, \ldots, m\} \) by (11).

Then the sequence \( \{\lambda_{ki}\} \) defined by (14) generates a partition of \([0, 1]\):

\[ 1 > \lambda_{12} > \lambda_{23} > \ldots > \lambda_{k-1,k} > \lambda_{k,k+1} > \ldots > \lambda_{m-2,m-1} > \lambda_{m-1,m} > 0, \]

(22)

such that, with respect to the order criterion \( \leq^\lambda \),

(i) \( A_1 \) is the smallest one among \( \mathcal{A} \) iff \( 1 \geq \lambda > \lambda_{12} \).

(ii) for each \( k (2 \leq k \leq m - 1) \), \( A_k \) is the smallest one among \( \mathcal{A} \) iff \( \lambda_{k-1,k} \geq \lambda > \lambda_{k,k+1} \).

(iii) \( A_m \) is the smallest among \( \mathcal{A} \) iff \( \lambda_{m-1,m} \geq \lambda \geq 0 \).

\[ \S 5. \text{Applications to biobjective shortest route problems} \]

We consider a directed network \( (N, A, \Gamma) \), where \( N = \{1, 2, \ldots, N\} \) is a finite set of nodes, \( A \) is a set of arcs whose elements are ordered pairs \( (i, j) \)
of distinct nodes, and $\Gamma = \{ \gamma_{ij} = (\gamma_{ij}^{1}, \gamma_{ij}^{2})^{T} \mid (i, j) \in A \} : \gamma_{ij} = (\gamma_{ij}^{1}, \gamma_{ij}^{2})^{T}$
denotes a biobjective distance attached to the directed arc $(i, j)$. Node 1 is
assigned to a starting node, and node $N$ to a terminal node.

Choose a transformation parameter $t$ of an appropriate value, and transform
the original data by the matrix $G(t)$. Put

$$T_{ij} = G(t)\gamma_{ij}, \quad \text{for } (i, j) \in A.$$ 

Let $A(i)$ denote the set of terminal nodes of all arcs emanating from node $i$.

**Definition 4.** For two points $A = (u, \alpha)$ and $B = (v, \beta)$ on $\mathcal{E}$, we define
the relations $\prec$ by

$$A \prec B \iff |\alpha - \beta| \leq v - u \text{ and } A \neq B.$$ 

**Algorithm modified Dijkstra ;**

begin
Choose: $\lambda \in (0, 1)$ arbitrarily;
$S := N$;
$d(i) := +\infty$ for each node $i \in N \setminus \{1\}$;
$d(1) := 0$ and $\text{pred}(1) := 0$;
while $S \neq \emptyset$ do
begin
$D := \{ j \in S \mid (\exists i \in S) ( d(i) \prec d(j) ) \}$;
$M := S \setminus D$;
let $i_{0} \in M$ be a node satisfying that $d(i_{0}) \leq^{\lambda} d(j)$ for $\forall j \in M$;
$S := S \setminus \{i_{0}\}$;
for each $j \in A(i_{0})$ do
if $d(i_{0}) + T_{i_{0}j} \leq^{\lambda} d(j)$ and $d(i_{0}) + T_{i_{0}j} \neq d(j)$ then
$d(j) := d(i_{0}) + T_{i_{0}j}$ and $\text{pred}(j) := i_{0}$;
end;
end.