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A New Parametric Method for Finding Efficient Solutions in Biobjective Shortest Route Problems

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§ 1. Preliminaries

Denote the variables representing two quantities we want to minimize in the biobjective programs by x and y. We call the coordinates plane having its orthogonal coordinates x and y the *original plane*, which is denoted by $\mathcal{P}$. The points of the plane $\mathcal{P}$ are wrtten as $a=(x, y)$ and $b=(x', y')$, etc.. An order relation among the points of the plane $\mathcal{P}$ is given by the usual manner. Namely, for two points $a=(x, y)$ and $b=(x', y')$, we write

$$a \leq b \quad \text{iff} \quad x \leq x' \text{ and } y \leq y'. \tag{1}$$

Let Ω be a non-empty finite subset of $\mathcal{P}$. We consider the optimization problem :

$$(\mathrm{P}_0) \quad \text{Minimize } a \text{ subject to } a \in \Omega.$$

A point $a \in \Omega$ is said to be an *efficient solution* to the problem (P_0), if there is no point $b \in \Omega$ such that $a \geq b$ and $a \neq b$.

For $t \in (0,1)$, define a 2×2 matrix $G(t)$ by

$$G(t) = \begin{bmatrix} t & 1 \\ t & -1 \end{bmatrix}. \tag{2}$$

We call the matrix $G(t)$ a *transformation matrix*, and t is called a *transformation parameter*. Trivially, the matrix $G(t)$ is nonsingular for every $t \in (0,1)$. Let

$$\begin{bmatrix} u \\ \alpha \end{bmatrix} = G(t) \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{P}. \tag{3}$$

The coordinates plane whose orthogonal coordinates are given by u and α of (4) is called the *transformed plane*, which is denoted by $\mathcal{H}$. The points of the plane $\mathcal{H}$ are wrtten as $A=(u, \alpha)$ and $B=(v, \beta)$, etc..

Proposition 1. Let $a=(x, y)$ and $b=(x', y')$ be two points of $\mathcal{P}$. Let $0<t \leq 1$ be arbitrary but fixed. Let $A=(u, \alpha)$ and $B=(v, \beta)$ be the points of $\mathcal{H}$ transformed from a and b by the transformation matrix $G(t)$,

respectively. Then the order relation $a \leq b$ is equivalently transformed to a relation on $\mathscr{H}$ as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} x' \\ y' \end{bmatrix} \Leftrightarrow |\alpha - \beta| \leq v - u. \tag{4}$$

§ 2. An order relation on the transformed plane

Definition 1. Let $0 \leq \lambda \leq 1$ be arbitrary. For two points $A = (u, \alpha)$ and $B = (v, \beta)$ of the transformed plane, we define an order relation $\leq^{\lambda}$ with the parameter λ by the following :

$$A \leq^{\lambda} B \underset{def}{\Leftrightarrow} \begin{cases} \text{(i)} \ |\alpha - \beta| \leq v - u, \\ \text{or} \\ \text{(ii)} \ 0 < \lambda(\beta - \alpha) \leq |v - u| < \beta - \alpha, \\ \text{or} \\ \text{(iii)} \ 0 < v - u < \lambda|\alpha - \beta|, \\ \text{or} \\ \text{(iv)} \ u = v \ \text{and} \ \alpha < \beta. \end{cases} \tag{5}$$

The four cases in the right-hand side of (5) are exclusive one another. Let a and b be the points transformed from A and B, respectively, by $G(t)^{-1}$. Proposition 1, then, implies that the condition (i) in the right-hand side of (5) is equivalent to the relation $a \leq b$. This fact holds true regardless of the value of the transformation parameter t.

Proposition 2. For each $0 \leq \lambda \leq 1$, the order relation $\leq^{\lambda}$ is reflexive and asymmetric on $\mathscr{H}$.

Proposition 3. For any pair $A = (u, \alpha)$ and $B = (v, \beta)$ of points in the plane $\mathscr{H}$ and for any $0 \leq \lambda \leq 1$, either $A \leq^{\lambda} B$ or $B \leq^{\lambda} A$ necessarily holds.

Proposition 4. Let $0 \leq \lambda \leq 1$ be arbitrary, and let A and B be two points on $\mathcal{H}$. For every positive number μ and every point C on $\mathcal{H}$, then, it holds that
$$A \leq^{\lambda} B \implies \mu A \leq^{\lambda} \mu B \text{ and } A \pm C \leq^{\lambda} B \pm C.$$

§ 3. Descent sequence on the transformed plane

Proposition 5. Let $A = (u, \alpha)$ and $B = (v, \beta)$ be two points on $\mathcal{H}$ satisfying that
$$u < v \text{ and } \alpha < \beta. \tag{6}$$
Then A is smaller than B with respect to the order $\leq^{\lambda}$ for every $0 \leq \lambda \leq 1$.

Definition 2. Let $\{A_i = (u_i, \alpha_i) \; ; \; i = 1, 2, \cdots, m\}$, where $m \geq 3$, be a finite sequence of points on $\mathcal{H}$. Then the sequence is said to be *descending to the right*, iff it holds that
$$\left. \begin{array}{l} u_i < u_{i+1}, \\ \alpha_i > \alpha_{i+1}, \end{array} \right\} \quad i = 1, 2, \cdots, m-1. \tag{7}$$

For a sequence $\{A_i = (u_i, \alpha_i) \; ; \; i = 1, 2, \cdots, m\}$ of points, if it holds that
$$\left. \begin{array}{l} u_i > u_{i+1}, \\ \alpha_i < \alpha_{i+1}, \end{array} \right\} \quad i = 1, 2, \cdots, m-1, \tag{8}$$
then the sequence is, of course, descending to the right, by renumbering the index of points. But, in order to unify the numbering of points, when we speak of a sequence descending to the right, we suppose to imply the condition (7) but not (8).

We denote the gradient of the line segment connecting two points a and b on the original plane by γ_{ab}. Similarly, we denote the gradient of the line segment connecting two points A and B on the transformed plane by γ_{AB}.

Theorem 1. Let $\{a_i = (x_i, y_i) \; ; \; i = 1, 2, \cdots, m\}$, where $m \geq 3$, be a sequence of points on $\mathcal{P}$ such that

$$\left.\begin{array}{l} x_i > x_{i+1}, \\ y_i < y_{i+1}, \end{array}\right\} \qquad i = 1, 2, \cdots, m-1. \tag{9}$$

Suppose that

$$\gamma_{a_{i-1}a_i} < \gamma_{a_i a_{i+1}}, \qquad i = 2, 3, \cdots, m-1. \tag{10}$$

Choose t such that $0 < t < \mathrm{Min}\left\{ \left|\gamma_{a_{m-1}a_m}\right|, 1 \right\}$. Define (u_i, α_i), $i = 1, 2, \cdots, m$, by

$$\begin{bmatrix} u_i \\ \alpha_i \end{bmatrix} = G(t) \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \qquad i = 1, 2, \cdots, m, \tag{11}$$

and let $\mathscr{A} = \{A_i = (u_i, \alpha_i)\ ;\ i = 1, 2, \cdots, m\}$. Then it holds that

(i) The sequence $\mathscr{A}$ is descending to the right:

(ii) $0 > \gamma_{A_{i-1}A_i} > \gamma_{A_i A_{i+1}}$, $i = 2, 3, \cdots, m-1$.

§ 4. Detection of efficient solutions

Definition 3. For two points $A = (u, \alpha)$ and $B = (v, \beta)$ on $\mathscr{H}$, if it holds that

$$|\alpha - \beta| > |u - v|, \tag{12}$$

then the points are said to be *mutually nondominant*. For a finite sequence $\{A_i = (u_i, \alpha_i)\ ;\ i = 1, 2, \cdots, m\}$ of points on $\mathscr{H}$, if every two elements of the sequence are mutually nondominant, then it is said that the *sequence is mutually nondominant.*

Proposition 6. Let $A = (u, \alpha)$ and $B = (v, \beta)$ be points on $\mathscr{H}$. Let $t \in (0, 1)$ be arbitrary, and let $a = (x, y)$ and $b = (x', y')$ be the points transformed by $G(t)^{-1}$ from A and B, respectively. Then, A and B are mutually nondominant, if and only if, the relation

$$\left\{\begin{array}{l} x < x' \text{ and } y > y', \\ \quad \text{or} \\ x > x' \text{ and } y < y', \end{array}\right.$$

holds.

As we have stated in the preceding section, the order relation $\leq^{\lambda}$ is reflexive and asymmetric but not transitive on the whole plane $\mathscr{H}$. However, it can be shown that if we restrict ourselves to the family of sequences which are mutually nondominant and descending to the right, then the relation $\leq^{\lambda}$ is transitive on each of the sequences.

Theorem 2. Let $\mathscr{A} = \{A_i = (u_i, \alpha_i)\ ;\ i = 1, 2, \cdots, m\}$ be mutually nondominant and descending to the right. Then the relation $\leq^{\lambda}$ is a total order relation on the sequence $\mathscr{A}$ for every $\lambda \in [0, 1]$.

Let $\mathbf{M}$ denote the set $\{1, 2, \cdots, m\}$. Throughout the remainder of this section, it is assumed that $m \geq 3$.

Proposition 6. Let $\mathscr{A} = \{A_i = (u_i, \alpha_i)\ ;\ i = 1, 2, \cdots, m\}$ be mutually nondominant and descending to the right. Suppose that the relations

$$\gamma_{A_{i-1}A_i} > \gamma_{A_i A_{i+1}}, \quad i = 2, 3, \cdots, m-1. \tag{13}$$

hold. For each $k \in \mathbf{M}$, put

$$\lambda_{ki} = \frac{u_k - u_i}{\alpha_i - \alpha_k} \qquad \text{for} \quad i \in \mathbf{M} \setminus \{k\}. \tag{14}$$

Then we have

(i) for each $k \in \mathbf{M}$,

$$0 < \lambda_{ki} < 1 \qquad \text{for} \quad i \in \mathbf{M} \setminus \{k\}, \tag{15}$$

(ii) for each $k \in \mathbf{N}$,

$$\lambda_{ki} = \lambda_{ik} \qquad \text{for} \quad i \in \mathbf{M} \setminus \{k\}, \tag{16}$$

(iii) for each $k \in \mathbf{N}$,

$$\lambda_{ki} > \lambda_{k,i+1} \qquad \text{for} \quad i \in \mathbf{M} \setminus \{k-1, k, m\}, \tag{17}$$

(iv) $$\lambda_{k-1,k} > \lambda_{k,k+1} \qquad \text{for} \quad k = 2, 3, \cdots, m-1. \tag{18}$$

Now, let $\{a_i = (x_i, y_i)\ ;\ i = 1, 2, \cdots, m\}$ be the whole set of efficient solutions to the problem (P_0). Without of loss of generality, we may assume that

$$\left.\begin{array}{l} x_i > x_{i+1}, \\ y_i < y_{i+1}, \end{array}\right\} \quad i = 1, 2, \cdots, m-1. \tag{19}$$

It is well known that if the efficient solutions$\{a_i = (x_i, y_i)\,;\, i = 1, 2, \cdots, m\}$ satisfy the condition :

$$\gamma_{a_{i-1}a_i} > \gamma_{a_i a_{i+1}}, \qquad i = 2, 3, \cdots, m-1, \tag{20}$$

in addition to (37), then the solutions can be all detected by the usual scalarization method. In this paper, we consider another condition :

$$\gamma_{a_{i-1}a_i} < \gamma_{a_i a_{i+1}}, \qquad i = 2, 3, \cdots, m-1. \tag{21}$$

Theorem 3. Let $\{a_i = (x_i, y_i)\,;\, i = 1, 2, \cdots, m\}$ be the whole set of efficient solutions to the problem (P_0). Suppose the conditions (19) and (21) to be satisfied. Choose t such that $0 < t < \mathrm{Min}\left\{\left|\gamma_{a_{m-1}a_m}\right|, 1\right\}$, and define $\mathscr{A} = \{A_i = (u_i, \alpha_i)\,;\, i = 1, 2, \cdots, m\}$ by (11).

Then the sequence$\{\lambda_{ki}\}$ defined by (14) generates a partition of $[0, 1]$:

$$\begin{aligned} 1 > \lambda_{12} > \lambda_{23} > \cdots > \lambda_{k-1,k} > \lambda_{k,k+1} \\ > \cdots > \lambda_{m-2,m-1} > \lambda_{m-1,m} > 0, \end{aligned} \tag{22}$$

such that, with respect to the order criterion $\leq^{\lambda}$,

(i) A_1 is the smallest one among $\mathscr{A}$ iff $1 \geq \lambda > \lambda_{12}$,

(ii) for each $k\,(2 \leq k \leq m-1)$, A_k is the smallest one among $\mathscr{A}$ iff

$$\lambda_{k-1,k} \geq \lambda > \lambda_{k,k+1},$$

(iii) A_m is the smallest among $\mathscr{A}$ iff $\lambda_{m-1,m} \geq \lambda \geq 0$.

§ 5. Applications to biobjective shortest route problems

We consider a directed network (N, A, Γ), where $N = \{1, 2, \cdots\cdots, N\}$ is a finite set of nodes, A is a set of arcs whose elements are ordered pairs (i, j)

of distinct nodes, and $\boldsymbol{\Gamma}=\{\gamma_{ij}=(\gamma^1{}_{ij},\gamma^2{}_{ij})^T \mid (i,j)\in \boldsymbol{A}\}$: $\gamma_{ij}=(\gamma^1{}_{ij},\gamma^2{}_{ij})^T$ denotes a biobjective distance attached to the directed arc (i, j). Node 1 is assigned to a starting node, and node N to a terminal node.

Choose a transformation parameter t of an appropriate value, and transform the original data by the matrix $G(t)$. Put

$$T_{ij}=G(t)\,\gamma_{ij}, \qquad \text{for } (i,j)\in \boldsymbol{A}.$$

Let $A(i)$ denote the set of terminal nodes of all arcs emanating from node i.

Definition 4. For two points $A=(u,\alpha)$ and $B=(v,\beta)$ on $\mathcal{H}$, we define the relations $\prec$ by

$$A \prec B \quad \text{iff} \quad [\,|\alpha-\beta|\le v-u \ \text{ and } \ A\neq B\,].$$

Algorithm modified Dijkstra ;

begin
  Choose $\lambda\in(0,1)$ arbitrarily;
  $S:=\boldsymbol{N}$;
  $d(i):=+\infty$ for each node $i\in \boldsymbol{N}\setminus\{1\}$;
  $d(1):=0$ and $\mathrm{pred}(1):=0$;
  while $S\neq\emptyset$ **do**
  begin
    $D:=\{\,j\in S \mid (\exists i\in S)(\,d(i)\prec d(j)\,)\}$;
    $M:=S\setminus D$;
    let $i_0\in M$ be a node satisfying that $d(i_0)\le^{\lambda} d(j)$ for $\forall j\in M$;
    $S:=S\setminus\{i_0\}$;
    for each $j\in A(i_0)$ **do**
      if $d(i_0)+T_{i_0 j}\le^{\lambda} d(j)$ and $d(i_0)+T_{i_0 j}\neq d(j)$ **then**
        $d(j):=d(i_0)+T_{i_0 j}$ and $\mathrm{pred}(j):=i_0$;
  end;
end.