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# Chance Constrained Bottleneck Spanning Tree with Fuzzy Random Cost

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**Abstract**

This paper considers a generalized fuzzy random version of bottleneck spanning tree problem in which edge costs are fuzzy random variables. The problem is to find an optimal spanning tree under chance constraint with respect to possibility measure of bottleneck (maximum cost) edge of spanning tree. The problem is first transformed into a deterministic equivalent problem. Then its subproblem is introduced and a close relation between these problems is clarified. Finally, fully utilizing this relation, we propose a polynomial order algorithm that finds an optimal spanning tree under two special functions.

## 1 Introduction

A spanning tree problem is one of the investigated important problem and many types of spanning tree problems have been considered, especially it has many application to communication of computer network. Ishii,et.al [1] [2] have investigated a stochastic minimum spanning tree problem that the costs of edge are assumed to be random variables. Itoh,et.al [3] have proposed fuzzy version.

In actual systems, we are often faced with the case where there exist both fuzziness and randomness. Then we introduce a fuzzy random variable to mathematical programming problem in order to treat elements containing fuzziness and randomness simultaneously.

So, this paper proposes a generalized version of spanning tree problem, i.e., fuzzy random bottleneck spanning tree problem, whose purpose is to find an optimal spanning tree under the chance constraint with respect to possibility measure of bottleneck edge of spanning tree. In other words, the problem is a fuzzy random version of [4].

Section 2 gives the definition of fuzzy random variables. Section 3 formulates the generalized fuzzy stochastic bottleneck spanning tree problem and show that it is transformed into deterministic equivalent problem $P$ by using a result of stochastic programming. Section 4 introduces maximum spanning tree problem $P^q$ with parameter $q$ and derives the close relation between $P$ and $P^q$, and shows that an optimal solution of $P$ can be found from a certain subproblem $P^q$. Further utilizing this relation, Section 5 proposes an algorithm that finds an optimal spanning tree under two special functions in a polynomial time. Finally, Section 6 concludes this paper and discusses futher research problems.

## 2 Fuzzy random variable

The concept of fuzzy random variables was introduced by Kwakernaak [5]. Puri and Ralescu [6] have established the mathematical basis of fuzzy random variables. There are many definitions of fuzzy random variables. N.Watanabe [7] gives simple but universal definition for fuzzy random variables, which are useful in applications. In this paper, we choose it as definition of fuzzy random variables.

**Definition 1** [7]
*Let $(\Omega, B_\Omega, P)$ be a probability space and $(\Lambda, B_\Lambda)$ a measurable space, where $\Omega$ is a set, Lambda is a fuzzy set, $B_\Omega$ and $B_\Lambda$ are $\sigma-$algebras, and $P$ is a probability measure. A fuzzy random variable $X$ is a measurable mapping of $\Omega$ into $\Lambda$. This means that $\{\omega | X(\omega) \in A\} \in B_\Omega$ for arbitrary $A \in B_\Lambda$.*

The following theorem is sufficient conditions for Definition 1.

**Theorem 1** [7]
*Let $x$ be a measurable mapping of a probability space $(\Omega, B_\Omega, P)$ into a measurable space $(\Gamma, B_\Gamma)$ and $X$ a mapping of $\Omega$ into $\Lambda$. If there exists a bijection $h : \Lambda \rightarrow \Gamma$, then there exists a measurable space $(\Lambda, B_\Lambda)$, and a mapping $X$ of $(\Omega, B_\Omega, P)$ into $(\Lambda, B_\Lambda)$ is a fuzzy random variable.*

Theorem implies the next corollary immediately.

**Corollary 1** [7]
*Let $X$ be a mapping of $\Omega$ into $\Lambda$. Suppose that, for $\forall\omega \in \Omega$, the membership function $\mu_{X(\omega)}$ of a fuzzy set $X(\omega)$ can be represented as $\mu_{X(\omega)}(u) = f(u; x(\omega))$ for some function $f(u;\theta)$, where $\theta$ is a parameter vector such that $\theta_1 \neq \theta_2$ implies $f(u;\theta_1) \neq f(u;\theta_2)$. Then $X$ is a fuzzy random variable.*

If the membership function of a fuzzy set $X$ is determined by the location parameter $x$ and if $x$ is a random variable, then $X$ is a fuzzy random variable from corollary. Conditions in corollary is fairly restrictive , but useful in applications. The hybrid number given by Kaufmann and Gupta [8] is such a fuzzy random variable.

## 3 Problem Formulation

Let $G = (N, E)$ denote undirected graph consisting of vertex set $N = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{e_1, e_2, \cdots, e_m\} \subset N \times N$. Moreover cost $c_j$ is attached to edge $e_j$. Spanning tree $T = T(N, S)$ of G is a partial graph satisfying the following conditions.

1. $T$ has a same vertex set as $G$.
2. $|S| = n - 1$ where $|S|$ denotes the cardinality of set $S$.
3. $T$ is connected.

$T$ can be denoted with $0-1$ variables $x_1, x_2, \ldots, x_m$ as follows.

$$T : \begin{array}{ll} x_i = 1 & e_i \in S \\ x_i = 0 & e_i \notin S \end{array}$$

Conversely, if $\{e_i | x_i = 1\}$ becomes a spanning tree of $G$ with vertex set $N$, $X = (x_1, x_2, \cdots, x_m)$ is also called spanning tree hereafter in this paper.

The ordinary minimal spanning tree problem is to seek a spanning tree $X$ minimizing $\sum_{j=1}^{m} c_j x_j$. As is easily shown, the bottleneck spanning tree problem that seeks a spanning tree $X$ minimizing $\max\{c_j | x_j = 1\}$ is equivalent to the minimal spanning tree problem by using some greedy-type algorithm.

We consider the construction of a communication network that connects some cities directly or indirectly. If each communication quantity per unit time between one city and another is constant, the problem of minimizing maximal capacity necessary for handling these quantities becomes a bottleneck spanning tree problem. In reality, however, there is a situation where these quantities vary randomly with time and experts can estimate these quantities approximately. In such a case, these quantities can be considered as fuzzy random variables. Then optimal connecting pairs of cities and capacity are to be determined under the condition that the probability that the possibility measure with respect to these quantities exceeds a certain value is greater than or equal to a satisficing level. This problem may be formulated as the problem to find an optimal spanning tree under a certain chance constraint. In other words, the problem to be considered is a discrete fuzzy random programming problem.

Suppose that $c_i$ is a fuzzy random variable characterized by the following membership function

$$\mu_{C_i(\omega)}(c_i) = \max\left\{0,\ L\left(\frac{c_i - d_i(\omega)}{\beta_i}\right)\right\}$$

where each $d_i(\omega)$ is assumed to be distributed according to the normal distribution $N(\mu_j, \sigma_j^2)$ with mean $\mu_j$ and variance $\sigma_j^2$, and they are mutually independent.

$L(\cdot)$ is a reference function satisfying the following conditions:

($a$) $L(-t) = L(t)$ for any $t \in R$.

($b$) $L(t) = 1$ *iff* $t = 0$.

($c$) $L(\cdot)$ is nonincreasing on $[0, +\infty)$.

($d$) Let $t_0 = \inf\{t > 0 | L(t) = 0\}$, then $0 < t_0 < +\infty$ ($t_0$ is called the zero point of $L$).

The less each cost of minimal spanning tree is, the better it is. So we set the fuzzy goal "each cost of minimal spanning tree is roughly smaller than $f_1$", and we give the possibility measure of the fuzzy goal as follows,

$$\Pi_{C_i(\omega)}(G) = \sup_{c_i} \min\{\mu_{C_i(\omega)}(c_i), \mu_G(c_i)\}$$

As $\mu_{C_i(\omega)}$ is a random variable, so $\Pi_{C_i(\omega)}(G)$ is .

Hereafter, we set $L$ and $\mu_G$ to the following linear functions.

$$\begin{aligned} L(t) &= 1 - \left|\frac{t}{t_0}\right| \\ \mu_G(c_i) &= \begin{cases} 1 & (c_i \le f_1) \\ \dfrac{c_i - f_1}{f_0 - f_1} & (f_1 < c_i < f_0) \\ 0 & (c_i \ge f_0) \end{cases} \end{aligned}$$

Then we propose the following problem $P_0$, which is a chance constrained programming.

$$\begin{aligned} P_0: \quad & \mathit{Maximize} \quad h + g(\alpha) \\ & \mathit{subject\ to} \quad Pr[\min\{\Pi_{C_i}(c_i) | e_i \in S\} \ge h] \ge \alpha, \quad \alpha \ge \frac{1}{2} \end{aligned}$$

where $g(\alpha)$ is a differentiable and nondecreasing function of $\alpha$. We set level $\alpha$ of the chance constraint to $1 > \alpha \ge \frac{1}{2}$. The above chance constraint is transformed into the deterministic equivalent one as follows.

$$\begin{aligned} Pr[\min\{\Pi_{C_i(\omega)}(c_i) | e_i \in T\} \ge h] \ge \alpha \quad &\Leftrightarrow \quad Pr[\{\text{all } \Pi_{C_i(\omega)}(c_i) | e_i \in T\} \ge h] \ge \alpha \\ &\Leftrightarrow \quad \Pi_{e_i \in T} Pr(\Pi_{C_i(\omega)}(c_i) \ge h) \ge \alpha \end{aligned}$$

$\Pi_{C_i(\omega)}(c_i) \geq h$ implies

$$\begin{aligned} & \sup_{c_i} \min\{\mu_{C_i(\omega)}(c_i),\ \mu_G(c_i)\} \geq h \\ \Leftrightarrow\quad & \exists\, c_i : \text{“}\mu_{C_i(\omega)}(c_i) \geq h,\ \mu_G(c_i) \geq h\text{”} \\ \Leftrightarrow\quad & \exists\, c_i : \text{“}L\left(\frac{c_i - d_i(\omega)}{\beta_i}\right) \geq h,\ \mu_G(c_i) \geq h\text{”} \\ \Leftrightarrow\quad & \exists\, c_i : \text{“}c_i \geq d_i(\omega) - L^*(h)\beta_i,\ c_i \leq \mu_G^*(h)\text{”} \\ \Leftrightarrow\quad & d_i(\omega) - L^*(h)\beta_i \leq \mu_G^*(h) \end{aligned}$$

where $\mu_G(\cdot)$ is a nondecreasing upper-semi continuous function and, $\mu_G^*(\cdot)$ and $L^*(\cdot)$ are pseudo inverse functions.

$$L^*(h) = t_0(1-h) \quad (1)$$

$$\mu_G^*(h) = h(f_1 - f_0) + f_0 \quad (2)$$

Since $Pr(d_i(\omega) \leq L^*(h)\beta_i + \mu_G^*(h)) = Pr[(d_i(\omega) - \mu_i)/\sigma_i \leq (L^*(h)\beta_i + \mu_G^*(h) - \mu_i)/\sigma_i]$ and $(d_i(\omega) - \mu_i)/\sigma_i$ is a mutually independent random variable distributed according to a standard normal distribution $N(0,1)$,

$$\begin{aligned} \Pi_{e_i\in T} Pr(d_i(\omega) \leq L^*(h)\beta_i + \mu_G^*(h)) \geq \alpha \quad & \Leftrightarrow \quad \Pi_{e_i\in T} F\left(\frac{L^*(h)\beta_i + \mu_G^*(h) - \mu_i}{\sigma_i}\right) \geq \alpha \\ & \Leftrightarrow \quad \sum_{e_i\in T} \log F\left(\frac{L^*(h)\beta_i + \mu_G^*(h) - \mu_i}{\sigma_i}\right) \geq \log\alpha \\ & \Leftrightarrow \quad \sum_{i=1}^{m} \log F\left(\frac{L^*(h)\beta_i + \mu_G^*(h) - \mu_i}{\sigma_i}\right) x_i \geq \log\alpha \end{aligned}$$

where $F$ denotes the distribution function of $N(0,1)$. Thus $P_0$ is equivalent to the following problem $P_1$.

$$\begin{aligned} P_1 : \quad & \mathit{Maximize} \quad h + g(\alpha) \\ & \mathit{subject\ to} \quad \sum_{i=1}^{m} \log F\left(\frac{L^*(h)\beta_i + \mu_G^*(h) - \mu_i}{\sigma_i}\right) x_i \geq \log\alpha \\ & \qquad 1 \geq \alpha \geq \frac{1}{2} \\ & \qquad x_i = 0 \ \mathit{or}\ 1,\ i = 1, 2, \ldots, m,\ X = (x_i) : \mathit{spanning\ tree}. \end{aligned}$$

Putting (1) and (2) into the above constraint, $P_1$ becomes the following problem $P_2$.

$$\begin{aligned} P_2 : \quad & \mathit{Maximize} \quad h + g(\alpha) \\ & \mathit{subject\ to} \quad \sum_{i=1}^{m} \log F\left(\frac{h(f_1 - f_0 - \beta_i t_0) + \beta_i t_0 + f_0 - \mu_i}{\sigma_i}\right) x_i \geq \log\alpha \\ & \qquad 1 \geq \alpha \geq \frac{1}{2} \\ & \qquad x_i = 0 \ \mathit{or}\ 1,\ i = 1, 2, \ldots, m,\ X = (x_i) : \mathit{spanning\ tree}. \end{aligned}$$

Set $\lambda = -h$, we consider the following $P$ instead of $P_2$.

$$\begin{aligned} P : \quad & \mathit{Minimize} \quad \lambda - g(\alpha) \\ & \mathit{subject\ to} \quad \sum_{i=1}^{m} \log F\left(\frac{\lambda(f_0 - f_1 + \beta_i t_0) + \beta_i t_0 + f_0 - \mu_i}{\sigma_i}\right) x_i \geq \log\alpha \\ & \qquad 1 \geq \alpha \geq \frac{1}{2} \\ & \qquad x_i = 0 \ \mathit{or}\ 1,\ i = 1, 2, \ldots, m,\ X = (x_i) : \mathit{spanning\ tree}. \end{aligned}$$

Of course, the optimal solution $h^*$ of $P_2$ is equivalent to the optimal solution $-\lambda^*$ of $P$.

## 4 Subproblem $P^q$ and Its relation to $P$

In order to solve $P$, consider the following subproblem $P^q$ with parameter $q$.

$$P^q: \quad \textit{Maximize} \quad \sum_{i=1}^{m} \log F\left(\frac{q(f_0 - f_1 + \beta_i t_0) + \beta_i t_0 + f_0 - \mu_i}{\sigma_i}\right) x_i$$

*subject to*

$$x_i = 0 \text{ or } 1,\ i = 1, 2, \ldots, m,\ X : \textit{spanning tree}.$$

For notational convenience, we define $c_i(q) = (q(f_0 - f_1 + \beta_i t_0) + \beta_i t_0 + f_0 - \mu_i)/\sigma_i$. $c_i(q)$ is an increasing function of $q$ since $f_0 > f_1$, and $t_0$, $\beta_i$, $\sigma_i > 0$. $P^q$ is the ordinary maximal spanning tree problem with edge cost $\log F(c_i(q))$ and can be efficiently solved by algorithms that have been proposed so far. $X^q$ denotes an optimal solution of $P^q$ and $Z_q$ the optimal value of objective function.

**Property 1**
*$Z_q$ is an increasing function of $q$.*

**Proof**
For $q_1 < q_2$, from the optimality of $X^{q_2}$,

$$\begin{aligned} Z_{q_2} &= \sum_{i=1}^{m} \log F(c_i(q_2)) x_i^{q_2} \geq \sum_{i=1}^{m} \log F(c_i(q_2)) x_i^{q_1} \\ &> \sum_{i=1}^{m} \log F(c_i(q_1)) x_i^{q_1} = Z_{q_1} \end{aligned}$$

where the last inequality follows from the monotonicity of $F(c_i(q))$ with respect to $q$.
Further let $(X^*, \lambda^*, \alpha^*)$ denotes an optimal solution of $P$.

**Theorem 2**

1. $Z_q > \log \alpha^* \leftrightarrow \lambda^* < q$,
2. $Z_q = \log \alpha^* \leftrightarrow \lambda^* = q$,
3. $Z_q < \log \alpha^* \leftrightarrow \lambda^* > q$,

**Proof.**
Clearly $Z_q$ is a continuous function of $q$.
**(1)$\rightarrow$**
If $Z_q = \sum_{i=1}^{m} \log F(c_i(q)) x_i^q > \log \alpha^*$, then from the continuity and monotonicity of $\log F(\cdot)$,

$$\sum_{i=1}^{m} \log F(c_i(q)) x_i^q > \sum_{i=1}^{m} F(c_i(q_1)) x_i^q \geq \log \alpha^*.$$

holds for $q_1 < q$ sufficiently close to $q$. The above relation shows $(X^q, q_1, \alpha)$ is feasible for $P$, that is, $q > q_1 \geq \lambda^*$.

**(1)←**
From the monotonicity of $F(c_i(q))$ and feasibility of $(X^*, \lambda^*, \alpha^*)$,

$$\log \alpha^* < \sum_{i=1}^{m} \log F(c_i(\lambda^*))x_i^* \leq \sum_{i=1}^{m} \log F(c_i(q))x_i^* \leq Z_q$$

**(3)→**
First, note that

$$\sum_{i=1}^{m} \log \alpha^* > Z_q \geq \sum_{i=1}^{m} \log F(c_i(q))x_i^*$$

The above relation, monotonicity of $F(c_i(q))$ and feasibility of $(X^*, \lambda^*, \alpha^*)$ show $\lambda^* > q$.
**(3)←** This is clear from the optimality of $\lambda^*$.
**(2)** Proof is automatically done after (1) and (3) are shown.
By theorem 2, the feasible solutions $(X^q,\ q,\ \alpha)$ satisfying $\sum_{i=1}^{m} \log F\left(c_i(q)\right) x_i^q = \log \alpha$ include an optimal solution $(X^*, \lambda^*, \alpha^*)$. Now, let $t = \log \alpha$, that is, $\alpha = e^t$, then $t = \sum_{j=1}^{m} \log F((c_i(q))x_j^q$ holds from the above observation. Then

$$\begin{aligned}
\frac{dt}{dq} &= \sum_{i=1}^{m} \frac{f\left(c_i(q)\right)}{F\left(c_i(q)\right)} \cdot \frac{f_0 - f_1 + \beta_i t_0}{\sigma_i} x_i^q = \sum_{i=1}^{m} \frac{\exp[-\frac{1}{2}\left\{c_i(q)\right\}^2]}{F\left(c_i(q)\right)} \cdot \frac{f_0 - f_1 + \beta_i t_0}{\sqrt{2\pi}\sigma_i} x_i^q \geq 0 \\
\frac{d^2t}{dq^2} &= \sum_{j=1}^{m} \frac{-f\left(c_i(q)\right)\left(c_i(q)F(c_i(q)) + f\left(c_i(q)\right)\right.}{F\left(c_i(q)\right)^2} \cdot \left(\frac{f_0 - f_1 + \beta_i t_0}{\sigma_i}\right)^2 x_i^q \leq 0
\end{aligned}$$

since $c_i(q) \geq 0$ from the condition $\Pi F_i(c_i(q)) \geq \alpha \geq \frac{1}{2}$. That is, $t$ is the concave function of $q$
Now we substitute $\alpha = e^t$ into $g(\alpha)$ and let $h(t) = g(e^t)$. Further, let

$$u(q) = q - g(\alpha) = q - v(t) = q - v\left(\sum_{j=1}^{m} \log F\left(c_i(q)\right) x_i^q\right)$$

Thus we seek $q^*$ minimizing $u(q)$ and then $(X^*, q^*, \tilde{\alpha})$ becomes an optimal solution of $P$ where $\tilde{\alpha}$ is the value of $\alpha$ corresponding to $q^*$, i.e., $g(\tilde{\alpha}) = u(q^*) - q^*$. By the chain rule,

$$\begin{aligned}
\frac{du}{dq} &= 1 - \frac{dv}{dt}\frac{dt}{dq} \\
\frac{d^2u}{dq^2} &= -\frac{dv}{dt}\frac{d^2t}{dq^2} - \frac{d^2v}{dt^2}\left(\frac{dt}{dq}\right)^2
\end{aligned}$$

Cross points of $c_i(q)$ and $c_j(q)$ are defined by $q_{ij}$. Then

$$q_{ij} = \frac{\dfrac{\beta_j t_0 + f_0 - \mu_j}{\sigma_j} - \dfrac{\beta_i t_0 + f_0 - \mu_i}{\sigma_i}}{\dfrac{f_0 - f_1 + \beta_i t_0}{\sigma_i} - \dfrac{f_0 - f_1 + \beta_j t_0}{\sigma_j}}$$

satisfy $q_{ij} > \max_k \dfrac{\mu_k - \beta_k - f_0}{f_0 - f_1 + \beta_k t_0}$ and let the results be $-1 = q_0 < q_1 < \cdots < q_s < q_{s+1} = 0$ where $s$ is the different number of such $q_{ij}$. Combining the above results, if $dv/dt > 0$ and $d^2v/dt^2 < 0$, $u$ is a convex function of $q$ in each subinterval and endpoints of subintervals $[q_j,\ q_{j+1}]$, $j = 0, \ldots, s$ , while, in the endpoints of subintervals or the point $q$ such that $(dv/dt)(dt/dq) = 1$ includes the optimal value of $q, q^*$. In the next section, we investigate two special types of $q(\alpha)$ satisfying the convexity of $u(q)$.

## 5 Some typical cases of $g(\alpha)$

In this section, we investigate two special cases of $g(\alpha)$, that is, $g(\alpha) = \gamma \log \alpha$ and $g(\alpha) = -\gamma/\alpha$ in these case $\gamma$ is a positive constant. These cases are especially given as example which can be solved easily.

### 5.1 CASE $g(\alpha) = \gamma \log \alpha$

In this case, $u(q) = q - \gamma \log \alpha = q - \gamma t$. For each subinterval $[q_{k-1},\ q_k]$, $k = 1, \ldots, s$, and the subinterval $[q_s, \infty), u(q)$ is described as follows.

$$u(q) = \begin{cases} u_k(q) = q - \gamma \sum_{j=1}^{m} \log F(c_i(q))\, x_i^q, & q \in [q_{k-1}, q_k],\ k = 1, \ldots, s \\ u_{s+1} = q - \gamma \sum_{j=1}^{m} \log F(c_i(q))\, x_i^q, & q \in [q_s, \infty) \end{cases}$$

By differentiating $u(q)$ in each subinterval,

$$\frac{du}{dq} = 1 - \gamma \frac{dt}{dq},\ \frac{d^2u}{dq^2} = -\gamma \frac{d^2t}{dq^2} \geq 0$$

since $d^2t/dq^2 \leq 0$ and $\gamma > 0$. That is, $u(q)$ is a convex function of $q$ in each subinterval. Thus, possible candidate points minimizing $u(q)$ are $q_1, \ldots, q_s$ or the points such that $dt/dq = 1/\gamma$.

**Property 2**
*In each subinterval, there exists at most one point satisfying $dt/dq = 1/\gamma$.*

**Proof**
Note that it is clear

$$\frac{dt}{dq} = \sum_{j=1}^{m} \frac{f(c_j(q))}{F(c_j(q))} \cdot \frac{f_0 - f_1 + \beta_j t_0}{\sigma_j} x_j^q$$

is a decreasing function of $q$ in each subinterval. As is easily shown, $dt/dq$ has a possible discontinuity at $q_1, \ldots, q_s$. Thus we obtain the following solution procedure:

1. Find all cross points $q_1, \ldots, q_s$. Then calculate left and right derivatives of $dt/dq$, that is, $L = dt/dq|_{q_r - 0}, R = dt/dq|_{q_r + 0}$ at each $q_r$, $r = 1, \ldots, s$.
2. Find the subintervals $[q_r,\ q_{r+1}]$ such that $R_r > 0$ and $L_{r+1} < 0$ and find the points $q_r^\gamma$ satisfying $dq/dt = 1/\gamma$ in these subintervals.
3. Let $Q_\gamma = \{q_r^\gamma | dt/dq = 1/\gamma\}$. Compare $u(q_r^\gamma)$, $q_r^\gamma \in Q_\gamma$ and $u(q_k)$, $k = 1, \ldots, s$ and choose the minimizer $q^*$ of $u(\cdot)$ among $q \in Q_\gamma \cup \{q_1, \ldots, q_s\}$. Then $(X^{q^*}, q^*, t(q^*))$ is an optimal solution of $P$.

**Theorem 3**
*The above procedure finds an optimal solution of $P$ in at most $O(m^3 \log m)$ computational time if $q_r^\gamma$ can be found in at most $O(m \log m)$ computational time.*

**Proof**
The validity is clear from the above discussion.
(Complexity) Calculation of $q_1, \ldots, q_s$ takes at most $O(m^2 \log m)$ computational time because there exist at most $O(m^2)$ cross points of $c_i(q) = c_j(q)$, $i < j \leq m$, and sorting them takes $O(m^2)$ and the complexity of the spanning tree algorithm is $O(m \log m)$. $|Q_\gamma|$ is at most $O(m^2)$ and $L_r$, $R_r$ can be caluculated in $O(m)$. Thus, the total complexity is $O(m^2) \cdot O(m \log m) = O(m^3 \log m)$.

## 5.2 CASE $g(\alpha) = -\gamma/\alpha$

In this case,

$$u(q) = q + \frac{\gamma}{\alpha} = q + \gamma e^{-t}$$

For each subinterval $[q_{k-1},\ q_k]$, $k = 1, \ldots, s$, and the subinterval $[q_s, \infty)$, $u(q)$ is described as follows:

$$u(q) = \begin{cases} u_k(q) = q + \gamma \Pi_{e_j \in T^q} \frac{1}{F(c_j(q)} & q \in [q_{k-1},\ q_k],\quad k = 1, \ldots, s \\ u_{s+1}(q) = q + \gamma \Pi_{e_j \in T^q} \frac{1}{F(c_j(q))}, & q \in [q_s, \infty), \end{cases}$$

where $T^q$ is the maximum spanning tree corresponding to $q$, that is, $\Pi_{e_j \in T^q} 1/F(c_j(q))$ is the product of $F(c_j(q))$ for $x_j^q = 1$, i.e., the $j$th element of $X^q = 1$. By differentiating $u(q)$ in each subinterval,

$$\frac{du}{dq} = 1 - \gamma e^{-t} \frac{dt}{dq},\ \frac{d^2u}{dq^2} = \gamma e^{-t} \left(\frac{dt}{dq}\right)^2 - \gamma e^{-t} \frac{d^2t}{dq^2} > 0$$

since $d^2t/dq^2 < 0$ and $\gamma > 0$. In this case, $u(q)$ is also a convex function of $q$ in each subinterval. But, different from the case in subsection 4.1, possible candidate points minimizing $u(q)$ are $q_1, \ldots, q_s$ or the points such that $dt/dq = (1/\gamma)e^t$. Let $w(q) = dt/dq - (1/\gamma)e^t$. Then the following property holds:

**Property 3**
*In each subinterval, there exists at most one point satisfying $u(q) = 0$.*

**Proof**
In each subinterval,

$$\frac{dw}{dq} = \frac{d^2t}{dq^2} - \frac{1}{\gamma} e^t \frac{dt}{dq} \leq 0$$

As is already mentioned, $dt/dq$ has a possible discontinuity at $q_1, \ldots, q_s$. Thus we obtain the following solution procedure:

1. Find all cross points $q_1, \ldots, q_s$ $(0 \leq q_i \leq 1)$. Then calculate

$$L_r = \lim_{q \to q_r - 0} w(q) \text{ and } R_r = \lim_{q \to q_r + 0} w(q)$$

   at each $q_r$, $r = 1, \ldots, s$.

2. Find the subintervals $[q_r,\ q_{r+1}]$ such that $R^r > 0$ and $L^{r+1} < 0$ and find the points $\tilde{q}$ satisfying $w(q) = 0$ in these subintervals.

3. Let $Q = \{\tilde{q_r} | w(\tilde{q_r}) = 0\}$. Compare $u(\tilde{q_r})$, $\tilde{q_r} \in Q$ and $u(q_k), k = 1, \ldots, s$, and choose the minimizer $q^*$ of $u(\cdot)$ among $q \in Q \cup \{q_1, \ldots, q_s\}$. Then $(X^{q^*}, q^*, t(q^*))$ is an optimal solution of $P$.

**Theorem 4**
*The above procedure finds an optimal solution of $P$ in at most $O(m^3 \log m)$ computational time if $q_r$ can be found in at most $O(m \log m)$ compurational time.*

**Proof**
The validity is clear from the above discusstion. Calculation of $q_1, \ldots, q_s$ takes $O(m^2 \log m)$ computational time as is shown in the proof of Theorem 3. Thus the maximum spanning tree problem to be solved is $O(m^2 \log m)$. $|Q|$ is at most $O(m^2)$ and $L^r$, $R^r$ can be calculated in $O(m)$. Thus, the total complexity is $O(m^3 \log m)$.

## 6 Conclusion

We have considered a generalized fuzzy random bottleneck spanning tree problem. However, we proposed have solution procedures for only two special cases. So it is better to consider more general types of $g(\alpha)$. Furthermore fuzzy random costs are not necessarily independent. Besides, we should try to extend the idea in this paper to other fuzzy random combinatorial optimization problems.

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