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Hamiltonian stationary normal bundles

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1 Introduction

Let N be a symplectic manifold with the symplectic structure Ω, that is, Ω is a nondegenerate closed 2-form on N. For example, cotangent bundles of manifolds, tangent bundles of Riemannian manifolds and almost Kähler manifolds have canonical symplectic structures. For a function f on N, there exists a unique vector field X_f on N such that $i_{X_f}\Omega = df$. This vector field X_f is called a Hamiltonian vector field associated to f, and generates a one-parameter group of (local) symplectomorphisms. A submanifold M in N is called Lagrangian if $\dim M = (1/2)\dim N$ and $\Omega|_M = 0$.

Here we consider a variational problem for the volume of Lagrangian submanifolds in almost Kähler manifolds, under deformations along Hamiltonian vector fields on the ambient spaces.

2 Preliminaries

Let N be an almost Kähler manifold with the almost complex structure J and the Hermitian metric g. The Kähler form Ω of N is defined by $\Omega(X,Y) = g(JX,Y)$ for $X, Y \in T_xN$, which is a symplectic structure on N. For a function f on N, $X_f = -J(\mathrm{grad}(f))$ is the Hamiltonian vector field associated to f. Let M be a Lagrangian submanifold in N. Then J becomes a bijection between T_xM and $T_x^{\perp}M$ for $x \in M$.

Let $i : M \to N$ be the inclusion map. A compactly supported deformation $\phi_t : M \to N(-\epsilon < t < \epsilon, \phi_0 = i)$ of M is called a Hamiltonian deformation if its variation vector field V satisfies $V = J(\mathrm{grad}(f))$ for some compactly supported function f on M. We say that M is Hamiltonian stationary if

$$\frac{d}{dt}\mathrm{vol}(\phi_t(M))|_{t=0} = 0$$

for all Hamiltonian deformations ϕ_t (cf.[4]). The Euler-Lagrange equation is given as follows:

PROPOSITION (cf. [4]). Let N be an almost Kähler manifold with the almost complex structure J. A Lagrangian submanifold M in N is Hamiltonian stationary if and only if its mean curvature vector H satisfies $\operatorname{div}(JH) = 0$ on M.

Locally, a Lagrangian submanifold in an almost Kähler manifold is Hamiltonian stationary if and only if it is a critical point of the volume functional for all deformations along Hamiltonian vector fields associated to compactly supported functions on the ambient space.

REMARK 1. We should choose the deformations more carefully if we consider the second variation (cf. [4]).

3 Problems

Of course, any minimal Lagrangian submanifold in an almost Kähler manifold is Hamiltonian stationary. From the proposition, we can also see that a Lagrangian submanifold with parallel mean curvature in a Kähler manifold is Hamiltonian stationary. So we have the following:

PROBLEM 1. Construct Hamiltonian stationary Lagrangian submanifolds with non-parallel mean curvature in Kähler manifolds, and non-minimal Hamiltonian stationary Lagrangian submanifolds in non-Kähler almost Kähler manifolds.

On the other hand, let M be a submanifold in a Riemannian manifold N. The normal bundle $T^{\perp}M$ of M may be naturally included in the tangent bundle TN of N. We consider the almost Kähler structure on TN, which is compatible with the canonical symplectic structure and the Sasaki metric on TN (cf. [8]). Then $T^{\perp}M$ is a Lagrangian submanifold in TN. So we have the following:

PROBLEM 2. Which submanifolds in Riemannian manifolds have Hamiltonian stationary normal bundles ?

4 Results

First we solve Problem 2 in the case of surfaces in R^3.

THEOREM 1 ([6]). Let S be a surface in R^3. Then $T^{\perp}S$ is Hamiltonian stationary if and only if S is either minimal, a part of a round sphere, or a part of a cone with vertex angle $\pi/2$.

REMARK 2. (i) The normal bundles of a round sphere and a cone with vertex angle $\pi/2$ in R^3, are Hamiltonian stationary Lagrangian submanifolds with non-parallel mean curvature in C^3, which are (noncompact) solutions to Problem 1.

(ii) Harvey and Lawson determined submanifolds in R^n with minimal normal bundles (see [2, III, Th.3.11, Prop.2.17]). In particular, they showed that a surface S in R^n has minimal normal bundle if and only if S is minimal.

Next we give a result for Problem 2 in the case of curves in Riemannian manifolds.

THEOREM 2 ([7]). Let c be a regular curve in a Riemannian manifold N. Suppose that N satisfies one of the following conditions:
(1) N is 2-dimensional,
(2) N has positive Ricci curvature,
(3) N has negative Ricci curvature,
(4) N has nonnegative sectional curvature,
(5) N is a space of constant curvature.
Then $T^{\perp}c$ is Hamiltonian stationary if and only if c is a geodesic.

REMARK 3. The normal bundle of a geodesic in a Riemannian manifold is minimal.

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