A GEOMETRIC INTERPRETATION OF ISOPARAMETRIC HYPERSURFACES

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1. Isoparametric hypersurfaces in a real space form.

This part is a joint work with Sadahiro Maeda [KM]. In differential geometry it is interesting to know the shape of a Riemannian submanifold by observing the extrinsic shape of geodesics of the submanifold. For example: A hypersurface $M^n$ isometrically immersed into a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature $c$ (that is, $\widetilde{M}^{n+1}(c) = \mathbb{R}^{n+1}, S^{n+1}(c)$ or $H^{n+1}(c)$ according as the curvature $c$ is zero, positive, or negative) is totally umbilic in $\widetilde{M}^{n+1}(c)$ if and only if every geodesic of $M$, considered as a curve in the ambient space $\widetilde{M}^{n+1}(c)$, is a circle. Here we treat a geodesic as a circle of null curvature.

In this talk we are interested in a hypersurface $M^n$ of a real space form $\widetilde{M}^{n+1}(c)$ satisfying that there exists such an orthonormal basis $\{v_1, \cdots, v_n\}$ at each point $p$ of the hypersurface $M^n$ that all geodesics of $M^n$ through $p$ in the direction $v_i$ ($1 \leq i \leq n$) are circles in the ambient space $\widetilde{M}^{n+1}(c)$. The classification problem of such hypersurfaces is concerned with studies about isoparametric hypersurfaces $M^n$'s in a real space form $\widetilde{M}^{n+1}(c)$ (that is, all principal curvatures of $M^n$ in $\widetilde{M}^{n+1}(c)$ are constant).

Theory of isoparametric submanifolds is one of the most interesting objects in differential geometry. In particular, É. Cartan studied extensively isoparametric hypersurfaces in a standard sphere. The classification problem of isoparametric hypersurfaces in a sphere is still open (for details, see [CR]).

The initial purpose of this talk is to provide a characterization of all isoparametric hypersurfaces by observing the extrinsic shape of geodesics of hypersurfaces in a real space form.

**Theorem 1.** Let $M^n$ be a connected hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature $c$. Then $M^n$ is isoparametric in $\widetilde{M}^{n+1}(c)$ if and only if there exists such an orthonormal basis $\{v_1, \cdots, v_m\}$ of the orthogonal complement of $\ker A$ in $T_p(M)$ ($m = \text{rank} \, A$) that all geodesics of $M$ through $p$ in the direction $v_i$ ($1 \leq i \leq m$) are circles of nonzero curvature in the ambient space $\widetilde{M}^{n+1}(c)$.

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Theorem 2. Let $M^n$ be a connected hypersurface of a real space form $\overline{M}^{n+1}(c)$ of constant curvature $c$. Then $M^n$ is isoparametric with nonzero constant principal curvatures in $\overline{M}^{n+1}(c)$ if and only if for each point $p$ of $M$, there exists such an orthonormal basis $\{v_1, \cdots, v_n\}$ of $T_p(M)$ that all geodesics of $M$ through $p$ in the direction $v_i$ $(1 \leq i \leq n)$ are circles of nonzero curvature in the ambient space $\overline{M}^{n+1}(c)$.

Theorem 3. Let $M^n$ be a connected hypersurface of a real space form $\overline{M}^{n+1}(c)$ of constant curvature $c$. Then $M^n$ is isoparametric in $\overline{M}^{n+1}(c)$ if and only if for each point $p$ of $M$, there exists such an orthonormal basis $\{v_1, \cdots, v_n\}$ of $T_p(M)$ of principal curvature vectors that all geodesics of $M$ through $p$ in the direction $v_i$ $(1 \leq i \leq n)$ are circles in the ambient space $\overline{M}^{n+1}(c)$.

2. Homogeneous real hypersurfaces in a complex projective space.

This part is a joint work with Toshiaki Adachi and Sadahiro Maeda [AKM].

Let $P_n(\mathbb{C})$ be an $n$-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4, and let $M$ be a real hypersurface of $P_n(\mathbb{C})$. Then $M$ has an almost contact metric structure $(\phi, \xi, \eta, g)$ inherited from the Kaehler structure of $P_n(\mathbb{C})$. Many differential geometers have studied $M$ by using this structure (cf. [O]). Typical examples of real hypersurfaces in $P_n(\mathbb{C})$ are homogeneous real hypersurfaces, that is, real hypersurfaces given as orbits under subgroups of the projective unitary group $PU(n+1)$.

Takagi ([T]) classified homogeneous real hypersurfaces in $P_n(\mathbb{C})$. Due to his work, we find that a homogeneous real hypersurface in $P_n(\mathbb{C})$ is locally congruent to one of the six model spaces of type $A_1, A_2, B, C, D$ and $E$. They are realized as tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or 2. A homogeneous real hypersurface of type $A_1$ is usually called a geodesic hypersphere.

In the study of real hypersurfaces in $P_n(\mathbb{C})$, there can be the following two problems:

(A) Give a characterization of homogeneous real hypersurfaces in $P_n(\mathbb{C})$.

(B) Construct non-homogeneous nice real hypersurfaces in $P_n(\mathbb{C})$ and characterize such examples.

In this talk we are interested in Problem (A). In differential geometry it is interesting to know the shape of a Riemannian submanifold by observing the extrinsic shape of geodesics of the submanifold. From this point of view we here recall the fact that a hypersurface $M^n$ in $\mathbb{R}^{n+1}$ is locally a standard sphere if and only if all geodesics of $M$ are circles of positive curvature in $\mathbb{R}^{n+1}$. We shall provide a characterization of all homogeneous real hypersurfaces in $P_n(\mathbb{C})$ by observing the shape of geodesics on the real hypersurfaces as curves in $P_n(\mathbb{C})$.

The purpose of this part is to prove the following result which is an improvement of the previous paper [MO].

Theorem 4. Let $M$ be a connected real hypersurface of $P_n(\mathbb{C})$. Then $M$ is congruent to a homogeneous real hypersurface if and only if there exist such orthonormal vectors $v_1, v_2, \cdots, v_{2n-2}$ orthogonal to $\xi$ at each point $p$ of $M$ that all geodesics $\gamma_i = \gamma_i(s)$ on $M$ with $\gamma_i(0) = p$ and $\gamma_i(0) = v_i(1 \leq i \leq 2n-2)$ are circles in $P_n(\mathbb{C})$ with positive curvature.
In the hypothesis of our Theorem we do not need to suppose that we take the vectors \( \{v_1, \cdots, v_{2n-2}\} \) as a local field of orthonormal frames in \( M \). However, for all homogeneous real hypersurfaces \( M's \) in \( P_n(\mathbb{C}) \), we can take a local field of orthonormal frames in \( M \) satisfying the hypothesis of our Theorem.

It is well-known that there does not exist a real hypersurface all of whose geodesics are circles in \( P_n(\mathbb{C}) \). Every circle in Theorem is a simple closed curve which lies on some totally real totally geodesic \( P^2(\mathbb{R}) \) in \( P_n(\mathbb{C}) \). We note that for any homogeneous real hypersurface \( M \), at each point \( p \) of \( M \) the geodesic \( \gamma = \gamma(s) \) with \( \gamma(0) = p \) and \( \dot{\gamma}(0) = \xi \) is also a circle in \( P_n(\mathbb{C}) \) which is a simple closed curve lying on some holomorphic totally geodesic \( P_1(\mathbb{C}) \) in \( P_n(\mathbb{C}) \). All circles in \( P_n(\mathbb{C}) \) are simple curves. However, a circle in \( P_n(\mathbb{C}) \) is not necessarily closed (see, [AMU]).

REFERENCES


