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Kyoto University
SP-property for a pair of $C^*$-algebras

Hiroyuki Osaka

Abstract

Recall that $C^*$-algebra $A$ has the SP-property if every non-zero hereditary $C^*$-subalgebra of $A$ has a non-zero projection. Let $1 \in A \subset B$ be a pair of $C^*$-algebras.

In this paper we investigate a sufficient condition for $B$ to have the SP-property under $A$ holds. As an application, we will present the cancellation property for crossed products of simple $C^*$-algebras by discrete groups.

This paper basically comes from joint works with Ja A Jeong ([7][8]).

1 The SP-Property

In this section we present a sufficient condition for $B$ to have the SP-property under $A$ holds.

The argument in [11, Lemma 10] gives the following general result.

Theorem 1.1 Let $1 \in A \subset B$ be a pair of $C^*$-algebras. Suppose that $A$ has the SP-property and there is a conditional expectation $E$ from $B$ to $A$. If for any non-zero positive element $x$ in $B$ and an arbitrary positive number $\varepsilon > 0$ there is an element $y$ in $B$ such that

$$
\|y^*(x - E(x))y\| < \varepsilon,
\|y^*E(x)y\| \geq \|E(x)\| - \varepsilon
$$

then $B$ has the SP-property. Moreover, every non-zero hereditary $C^*$-subalgebra of $B$ has a projection which is equivalent to some projection in $A$ in the sense of Murray-von Neumann.

Next we consider the following stronger assumption on a conditional expectation $E$ from $B$ to $A$.

Definition 1.2 Let $1 \in A \subset B$ be a pair of $C^*$-algebras. A conditional expectation $E$ from $B$ to $A$ is called outer if for any element $x \in B$ with
$E(x) = 0$ and any non-zero hereditary $C^*$-subalgebra $C$ of $A$

\[ \inf \{ \| cxc \| ; c \in C^+ , \| c \| = 1 \} = 0. \]

The following result comes from the same argument as in [10, Lemma 3.2] and Theorem 1.1.

**Corollary 1.3** Let $1 \in A \subset B$ be a pair of $C^*$-algebras. Suppose that $A$ has the SP-property and there is a conditional expectation $E$ from $B$ to $A$. If $E$ is outer, then $B$ has the SP-property.

We present some examples of a pair of $C^*$-algebras with an outer conditional expectations.

**Example 1.4** Let $\rho$ be a corner endomorphism on a unital $C^*$-algebra $A$, and let $E$ be a canonical conditional expectation from a crossed product $A \times_{\rho} \mathbb{N}$ to $A$. Suppose that

\[ \hat{T}(\rho) = \{ \lambda \in \mathbb{T} | \hat{\rho}(I) = I \text{ for } \forall I \in \text{Prime}(A \times_{\rho} \mathbb{N}) \} = \mathbb{T}. \]

Then, $E$ is outer.

**Proof.** See Jeong-Kodaka-Osaka [6]. \qed

**Example 1.5** (Kishimoto[10]) Let $G$ be a discrete group and let $\alpha$ be a representation of $G$ by automorphisms of a simple unital $C^*$-algebra $A$. Suppose $\alpha$ is outer. Then, a canonical conditional expectation from a crossed product $A \times_{\alpha} G$ to $A$ is outer.

In the case of a crossed product of a simple unital $C^*$-algebra with the SP-property by a finite group $G$, we can deduce the SP-property for the crossed product algebra $A \times_{\alpha} G$ by any automorphism $\alpha$ on $A$.

**Theorem 1.6** ([7]) Let $A$ be a simple unital $C^*$-algebra with the SP-property, and let $\alpha$ be an action by a finite group $G$. Then, a crossed product algebra $A \times_{\alpha} G$ has the SP-property.

## 2 C*-Index Theory

In this section, we brief the $C^*$-index theory by Watatani ([16]).

Let $1 \in A \subseteq B$ be a pair of $C^*$-algebras. By a conditional expectation $E : B \to A$ we mean a positive faithful linear map of norm one satisfying

\[ E(aba') = aE(b)a', \quad a,a' \in A, b \in B. \]
A finite family \(\{(u_1, v_1), \cdots, (u_n, v_n)\}\) in \(B \times B\) is called a quasi-basis for \(E\) if
\[
\sum_{i=1}^{n} u_i E(v_i b) = \sum_{i=1}^{n} E(b u_i) v_i = b
\]
for \(b \in B\).

We say that a conditional expectation \(E\) is of index-finite type if there exists a quasi-basis for \(E\). In this case the index of \(E\) is defined by
\[
\text{Index } E = \sum_{i=1}^{n} u_i v_i.
\]

Note that \(\text{Index } E\) does not depend on the choice of a quasi-basis and every conditional expectation \(E\) of index-finite type on a \(C^*\)-algebra has a quasi-basis of the form \(\{(u_1, u_1^*), \cdots, (u_n, u_n^*)\}\) (16, Lemma 2.1.6). Moreover, \(\text{Index } E\) is always contained in the center of \(B\), so that it is a scalar whenever \(B\) has the trivial center, in particular when \(B\) is simple.

Let \(E : B \to A\) be a conditional expectation. Then \(B_A(= B)\) is a pre-Hilbert module over \(A\) with an \(A\)-valued inner product
\[
\langle x, y \rangle = E(x^* y), \ x, y \in B_A.
\]

Let \(E\) be the completion of \(B_A\) with respect to the norm on \(B_A\) defined by
\[
\|x\|_{B_A} = \|E(x^* x)\|_A^{1/2}, \ x \in B_A.
\]

Then \(E\) is a Hilbert \(C^*\)-module over \(A\). Since \(E\) is faithful, the canonical map \(B \to E\) is injective. Let \(L_A(E)\) be the set of all (right) \(A\)-module homomorphisms \(T : E \to E\) with an adjoint \(A\)-module homomorphism \(T^* : E \to E\) such that
\[
\langle T \xi, \zeta \rangle = \langle \xi, T^* \zeta \rangle \ \xi, \zeta \in E.
\]

Then \(L_A(E)\) is a \(C^*\)-algebra with the operator norm \(\|T\| = \sup\{\|T \xi\| : \|\xi\| = 1\}\). There is an injective \(*\)-homomorphism \(\lambda : B \to L_A(E)\) defined by
\[
\lambda(b)x = bx
\]
for \(x \in B_A, b \in B\), so that \(B\) can be viewed as a \(C^*\)-subalgebra of \(L_A(E)\).

Note that the map \(e_A : B_A \to B_A\) defined by
\[
e_A x = E(x), \ x \in B_A
\]
is bounded and thus it can be extended to a bounded linear operator, denoted by \(e_A\) again, on \(E\). Then \(e_A \in L_A(E)\) and \(e_A = e_A^2 = e_A^*\), that is, \(e_A\) is a projection in \(L_A(E)\).
The (reduced) $C^*$-basic construction is a $C^*$-subalgebra of $L_A(\mathcal{E})$ defined to be
\[ C^*(B, e_A) = \overline{\text{span}\{\lambda(x)e_A\lambda(y) : x, y \in B\}} \|\cdot\| \]
see [16, Definition 2.1.2].

Then,

**Lemma 2.1** ([16, Lemma 2.1.4]) (1) $e_A C^*(B, e_A) e_A = \lambda(A) e_A$.
(2) $\psi : A \to e_A C^*(B, e_A) e_A, \psi(a) = \lambda(a) e_A$, is a *-isomorphism (onto).

**Lemma 2.2** ([16, Lemma 2.1.5]) The following are equivalent:
(1) $E : B \to A$ is of index-finite type
(2) $C^*(B, e_A)$ has an identity and there exists a number $c$ with $0 < c < 1$ such that
\[ E(x^* x) \geq c(x^* x) \quad x \in B. \]

The above inequality was shown first in [13] by Pimsner and Popa for the conditional expectation $E_N : M \to N$ from a type $\text{II}_1$ factor $M$ onto its subfactor $N$ ($c$ can be taken as the inverse of the Jones index $[M : N]$).

The conditional expectation $E_B : C^*(B, e_A) \to B$ defined by
\[ E_B(\lambda(x)e_A\lambda(y)) = (\text{Index}E)^{-1} xy, x, y \in B \]
is called the dual conditional expectation of $E : B \to A$. If $E$ is of index-finite type, so is $E_B$ with a quasi-basis $\{(w_i, w_i^*)\}$, where $w_i = \sqrt{\text{Index}E} u_i e_A$, and $\{(u_i, u_i^*)\}$ are quasis-basis for $E$ ([16, Proposition 2.3.4]).

3 The Stable Rank for $C^*$-Crossed Products

Let $\alpha$ be an action of a finite group $G$ on a unital $C^*$-algebra $A$ by automorphisms, and let $A \times_\alpha G$ be its crossed product, that is, it is the universal $C^*$-algebra generated by a copy of $A$ and implementing unitaries $\{u_g | g \in G\}$ with $\alpha_g(a) = u_g au_g^*$ for every $g \in G$ and $a \in A$. Then there exists a canonical conditional expectation $E : A \times_\alpha G \to A$ defined by
\[ E(\sum_g a_g u_g) = a_e, \]
for $a_g \in A$ and $g \in G$, where $e$ denotes the identity of the group $G$.

**Lemma 3.1** Under this situation, the canonical conditional expectation $E$ is of index-finite type with a quasi-basis $\{(u_g, u_g^*) : g \in G\}$ and
\[ \text{Index}(E) = \sum_{g \in G} u_g u_g^* = |G|, \] the order of $G$. 

Let $B = A \times_{\alpha} G$ and $n = |G|$. Then, a dual conditional expectation $E_B$ is of index-finite type with a quasi-basis $\{(w_g, w_g^*) : g \in G\}$, where $w_g = \sqrt{n}u_ge_A$ (see section 2).

The following fact comes from a simple computation.

Lemma 3.2 ([8]) The expression $x = \sum_{g \in G} w_gb_g$ ($b_g \in B$) is unique for each $x \in C^*(B, e_A)$.

Let $A$ be a unital $C^*$-algebra and $Lg_n(A)$ denote the $n$-tuples $(x_1, \ldots, x_n)$ in $A^n$ which generate $A$ as a left ideal. The topological stable rank of $A$ ($sr(A)$) is defined to be the least integer for which $Lg_n(A)$ is dense in $A^n$. If there does not exist such an integer then $sr(A)$ is defined to be $\infty$. For a non unital $C^*$-algebra $A$ we define $sr(A) = sr(\tilde{A})$ where $\tilde{A}$ is the unitization of $A$. See [15] for details about stable rank. It is not hard to see that for a unital $C^*$-algebra $A$ $sr(A) = 1$ if and only if the set of invertible elements is dense in $A$.

Theorem 3.3 ([8]) Let $G$ be a finite group, and $\alpha$ be an action of $G$ on a unital $C^*$-algebra $A$ with $sr(A) = 1$. Then $sr(A \times_{\alpha} G) \leq |G|$.

Proof. Let $n = |G|$, and $(b_{g_1}, \ldots, b_{g_n}) \in B^n$, where $B = A \times_{\alpha} G$. Put $y = \sum_{g \in G} w_gb_g \in C^*(B, e_A)$. Since $C^*(B, e_A)$ is strong Morita equivalent to $A$ and $sr(A) = 1$, we have $sr(C^*(B, e_A)) = 1$ ([16, Proposition 1.3.4.]). Approximate $y$ by invertible elements $x$ in $C^*(B, e_A)$, and write $x = \sum_{g \in G} w_gc_g$, $c_g \in B$. Then by Lemma 3.2, $(c_{g_1}, \ldots, c_{g_n})$ is close to $(b_{g_1}, \ldots, b_{g_n})$. Note that

$$x^*x = n \sum_g c_g^*e_Ac_g.$$ 

By Lemma 2.2

$$E_B(x^*x) \geq \frac{1}{n}x^*x, \ x \in C^*(B, e_A).$$

Since $E_B(x^*x) = \sum_g c_g^*c_g$, it follows that

$$\sum_g c_g^*c_g \geq \frac{1}{n}x^*x$$

which is invertible in $C^*(B, e_A)$. Therefore $\sum_g c_g^*c_g$ is invertible in $B$, that is, $(c_{g_1}, \ldots, c_{g_n}) \in Lg_n(B)$.

$\Box$
**Remark 3.4** If $sr(A) = m$ then it can be shown that $sr(A \times_{\alpha} G) \leq |G|m$ whenever $A$ is a simple unital $C^*$-algebra. Indeed, it can come from the following two facts; (i) $C^*(B, e_A)$ is isomorphic to the matrix algebra $M_n(A)$ ([16]), (ii) $sr(M_n(A)) = \{sr(A) - 1 \over n\} + 1$, where $\{t\}$ denotes the least integer which is greater than or equal to $t$ ([15]).

4 The Cancellation Property

A $C^*$-algebra $A$ is said to have cancellation of projections if for any projections $p, q, r$ in $A$ with $p \perp r, q \perp r, p + r \sim q + r$, we have $p \sim q$. If $M_n(A)$ has cancellation of projections for each $n = 1, 2, \ldots$, then we simply say that $A$ has cancellation. Note that every $C^*$-algebra with cancellation is stably finite, that is, every matrix algebra $M_n(A)$ with entries from $A$ contains no infinite projections for $n = 1, 2, \ldots$. It can be shown that if $A$ is a $C^*$-algebra with $sr(A) = 1$ then it has cancellation. In the previous section we proved that the stable rank of the $C^*$-crossed product $A \times_{\alpha} G$ is bounded by the order of the group $G$ if $sr(A) = 1$, and actually it seems that the crossed product has stable rank 1, and therefore it would be natural to ask if it has cancellation.

**Theorem 4.1** ([2, Theorem 4.2.2]) Let $A$ be a simple unital $C^*$-algebra. Suppose $A$ contains a sequence $(p_k)$ of projections such that

1. for each $k$ there is a projection $r_k$ such that $2p_{k+1} \oplus r_k$ is equivalent to a subprojection of $p_k \oplus r_k$,
2. there is a constant $K$ such that $sr(p_k A p_k) \leq K$ for all $k$.

Then $A$ has cancellation.

**Theorem 4.2** ([8]) Let $A$ be a simple unital $C^*$-algebra with $sr(A) = 1$ and $SP$-property. If $G$ is a finite group and $\alpha$ is an action of $G$ on $A$ then the crossed product $A \times_{\alpha} G$ has cancellation.

**Sketch of a proof.**

We give a proof in the case that $A \times_{\alpha} G$ is simple.

Since the fixed point algebra $A^\alpha$ can be identified with a hereditary $C^*$-subalgebra of the crossed product it has the SP-property by Theorem 1.6. Thus there is a sequence of projections $\{p_k\} \in A^\alpha$ such that $2[p_{k+1}] \leq [p_k]$ by [9, Lemma 2.2], where $[p]$ denotes the equivalence class of $p$. Since
\( p_k \in A^\alpha, p_k(A \times_\alpha G)p_k \) is isomorphic to \((p_k A p_k) \times_\alpha G\) for each \( k \in N \). Note that each \( p_k A p_k \) has stable rank one. By Theorem 3.3 \( sr(p_k A p_k \times_\alpha G) \leq |G| \). Therefore, the assertion follows from Theorem 4.1 \((K = |G|, r_k = 0)\). □

Recall that a unital \( C^* \)-algebra \( A \) has real rank zero, \( RR(A) = 0 \), if the set of invertible self-adjoint elements is dense in \( A_{sa} \). It is well known that \( RR(A) = 0 \) is equivalent to say that every non-zero hereditary \( C^* \)-subalgebra contains an approximate identity consisting of projections (HP) ([3]). From [2, Section 4] where the HP-property is studied for simple \( C^* \)-algebras we can deduce the following.

**Corollary 4.3** ([8]) Under the assumptions of the above theorem, if \( RR(A \times_\alpha G) = 0 \) then its stable rank is one.

For crossed products by the integer group \( Z \) we have the following cancellation theorem:

**Theorem 4.4** ([8]) Let \( A \) be a simple unital \( C^* \)-algebra with \( sr(A) = 1 \) and SP-property. If \( \alpha \) is an outer action of the integer group \( Z \) on \( A \) such that \( \alpha_* \circ id \) on the \( K_0 \) group \( K_0(A) \) of \( A \) then the crossed product \( A \times_\alpha Z \) has cancellation.

**Example 4.5** If \( A \) is a UHF algebra or an irrational rotation algebra then the identity map is the only possible homomorphism on its \( K_0 \) group. Therefore the theorem says that any crossed product \( A \times_\alpha Z \) has cancellation.

**Corollary 4.6** ([8]) Under the same assumption of Theorem 3.5 if \( RR(A \times_\alpha Z) = 0 \), then its stable rank is one.

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**参考文献**


