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Asymptotic distribution of negative eigenvalues for two dimensional Pauli operators with nonconstant magnetic fields

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1. Results

The aim here is to study the asymptotic distribution of discrete eigenvalues near the bottom of the essential spectrum for two and three dimensional Pauli operators perturbed by electric fields falling off at infinity.

The Pauli operator describes the motion of a particle with spin in a magnetic field and it acts on the space $L^2(\boldsymbol{R}^3) \otimes \boldsymbol{C}^2$. The unperturbed Pauli operator without electric field is given by

$$H_p = (-i\nabla - A)^2 - \sigma \cdot B$$

under a suitable normalization of units, where $A : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ is a magnetic potential, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ with components

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the vector of 2×2 Pauli matrices and $B = \nabla \times A$ is a magnetic field. We write $(x, z) = (x_1, x_2, z)$ for the coordinates over the three dimensional space $\boldsymbol{R}^3 = \boldsymbol{R}_x^2 \times \boldsymbol{R}_z$. We now assume that the magnetic field B has a constant direction. For brevity, the field is assumed to be directed along the positive z axis, so that B takes the form

$$B(x) = (0, 0, b(x)).$$

Since the magnetic field B is a closed two form, it is easily seen that B is independent of the z variable. We identify $B(x)$ with the function $b(x)$. Let $A(x) = (a_1(x), a_2(x), 0)$, $a_j \in C^1(\boldsymbol{R}^2)$, be a magnetic potential associated with $b(x)$. Then

$$b(x) = \nabla \times A = \partial_1 a_2 - \partial_2 a_1, \quad \partial_j = \partial / \partial x_j,$$

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and the Pauli operator takes the simple form

$$H_p = \begin{pmatrix} H_+ - \partial_z^2 & 0 \\ 0 & H_- - \partial_z^2 \end{pmatrix}$$

where

$$H_\pm = (-i\nabla - A)^2 \mp b = \Pi_1^2 + \Pi_2^2 \mp b, \quad \Pi_j = -i\partial_j - a_j.$$

The magnetic field b is represented as the commutator $b = i[\Pi_2, \Pi_1]$ and hence $H_\pm$ can be rewritten as

$$H_\pm = (\Pi_1 \pm i\Pi_2)^*(\Pi_1 \pm i\Pi_2).$$

This implies that $H_\pm \geq 0$ is nonnegative. If, in particular, $b(x) \geq 0$ is nonnegative, then it is known ([1, 4, 14]) that H_+ has zero as an eigenvalue and its essential spectrum begins at zero for a fairly large class of magnetic fields. We states several basic spectral properties of $H(V)$ in section 2.

We first discuss the two dimensional case. We now write H for H_+ and consider the Pauli operator

$$H(V) = H - V, \quad H = (-i\nabla - A)^2 - b,$$

perturbed by electric field $V(x)$. As stated above, the essential spectrum of unperturbed operator $H = H_+$ begins at zero. If the electric field $V(x)$ falling off at infinity is added to this operator as a perturbation, then the above operator $H(V)$ has negative discrete eigenvalues. If, in addtion, $b(x) \geq c > 0$ is strictly positve, then $H = H_+$ has a spectral gap above zero, and $H(V)$ has discrete eigenvalues in the gap accumulating at zero. Our aim is to study the asymptotic distribution of these eigenvalues.

Let $\langle x \rangle = (1 + |x|^2)^{1/2}$. We first make the following assumptions on $b(x)$ and $V(x)$:

(b) $b(x) \in C^1(\boldsymbol{R}^2)$ is a positive function and

$$\langle x \rangle^{-d}/C \leq b(x) \leq C\langle x \rangle^{-d}, \quad |\nabla b(x)| \leq C\,\langle x \rangle^{-d-1}, \quad C > 1,$$

for some $d \geq 0$.

(V) $V(x) \in C^1(\boldsymbol{R}^2)$ is a real function and

$$|V(x)| \leq C\,\langle x \rangle^{-m}, \quad |\nabla V(x)| \leq C\,\langle x \rangle^{-m-1}, \quad C > 0,$$

for some $m > 0$.

Under these assumptions, the operator $H(V)$ formally defined above admits a unique self–adjoint realization in $L^2 = L^2(R^2)$ with natural domain $\{u \in L^2 : Hu \in L^2\}$, where Hu is understood in the distributional sense. We denote by the same notation $H(V)$ this self–adjoint realization. The result is the following:

Theorem 1 *Let the notations be as above. Assume that* (b) *and* (V) *are fulfilled. We further assume* $V(x)$ *to satisfy*

$$\liminf_{\lambda\to 0} \lambda^{2/m} \int_{V(x)>\lambda} dx > 0. \tag{1}$$

and

$$\limsup_{\lambda\to 0} \lambda^{(2-d)/m} \int_{(1-\delta)\lambda<|V(x)|<(1+\delta)\lambda} \langle x\rangle^{-d}\, dx = o(1), \quad \delta \to 0.$$

Then one has

(i) ([11] **for** $d=0$; [12] **for** $d>0$) *Let* $N(H(V) < -\lambda)$, $\lambda > 0$ *denote the number of negative eigenvalues less than* $-\lambda$. *Assume that* d *and* m *satisfy*

$$0 \le d < 2, \qquad d < m.$$

Then we have

$$N(H(V) < -\lambda) = (2\pi)^{-1} \int_{V(x)>\lambda} b(x)\, dx (1 + o(1)), \quad \lambda \to 0.$$

(ii) ([11]) *Assume that* $d = 0$ *and* $m > 0$. *Let* $0 < c < b_0/3$, $b_0 = \inf b(x)$, *be fixed and let* $N(\lambda < H(V) < c)$, $0 < \lambda < c$, *be the number of positive eigenvalues lying in the interval* (λ, c) *of operator* $H(V)$. *Then,*

$$N(\lambda < H(V) < c) = (2\pi)^{-1} \int_{V(x)<-\lambda} b(x)\, dx + o(\lambda^{-2/m}), \quad \lambda \to 0.$$

Remarks. (1) If $\lim_{|x|\to\infty} |x|^2 b(x) = \infty$, it is known that the bottom, zero, of essential spectrum of $H = H_+$ is an eigenvalue with infinite multiplicities $\dim \operatorname{Ker} H = \infty$ ([14, Theorem 3.4]). On the other hand, if $b(x) = O(|x|^{-d})$ as $|x| \to \infty$ for some $d > 2$, then it follows that $\dim \operatorname{Ker} H < \infty$ ([12, Remark 4.1]). We point out that no decay condition on the derivatives of $b(x)$ is assumed in these results.

(2) The assumption $d < m$ means that magnetic fields are stronger than electric fields at infinity. In the last section, we will briefly discuss the case $m < d$, $0 < m < 2$, when electric fields are stronger than magnetic fields. This case is much easier to deal with and $N(H(V) < -\lambda)$ is shown to obey the classical Weyl formula. Roughly speaking, it behaves like $N(H(V) < -\lambda) \sim \lambda^{(m-2)/m}$ as $\lambda \to 0$. If $d > 2$ and $m > 2$, then the number of negative eigenvalues is expected to be finite, but it seems that the problem has not yet been established.

(3) Under the same assumptions as in Theorem 1 (i), we can prove that

$$N(H_-(V) < -\lambda) = O(\lambda^{-\varepsilon})$$

for any $\varepsilon > 0$ small enough, where $H_-(V) = H_- - V$. This follows from Theorem 1 (i) at once, if we take account of the form inequality

$$H_-(V) = H_+ + 2b - V \geq H_+ - c_N \langle x \rangle^{-N}, \quad c_N > 0,$$

for any $N \gg 1$ large enough. Thus the number $N(H_{p,2}(V) < -\lambda)$ of negative eigenvalues less than $-\lambda$ of the two dimensional perturbed Pauli operator

$$H_{p,2}(V) = H_{p,2} - V = \begin{pmatrix} H_+ - V & 0 \\ 0 & H_- - V \end{pmatrix} \quad \text{on } L^2(\boldsymbol{R}^2) \otimes \boldsymbol{C}^2$$

obeys the same asymptotic formula as in Theorem 1 (i).

Next we proceed to the three dimensional case. Let $b(x) \in C^1(R_x^2)$ be a magnetic field satisfying the assumption (b) with $d = 0$. We consider the three dimensional perturbed Pauli operator

$$H_3(V) = \Pi_1^2 + \Pi_2^2 - \partial_z^2 - b - V,$$

which acts on the space $L^2(R^3) = L^2(R_x^2 \times R_z)$, where $V = V(x,z)$ is a real function decaying at infinity. The essential spectrum of the unperturbed three dimensional Pauli operator $H_3(0)$ without potential V begins at the origin and occupies the whole positive axis. On the other hand, the perturbed operator $H_3(V)$ has an infinite number of negative eigenvalues accumulating the origin. The second theorem is formulated as follows.

Theorem 2 ([11]) *Let $H_3(V)$ be as above. Suppose that the magnetic field $b(x)$ fulfills the assumption (b) with $d = 0$. If a real function $V(x,z) \in C^1(R^3)$ satisfies*

$$\langle x,z \rangle^{-m}/C \leq V(x,z) \leq C \langle x,z \rangle^{-m}, \quad |\nabla V(x,z)| \leq C \langle x,z \rangle^{-m-1}, \quad C > 1,$$

for some $m > 0$, where $\langle x,z \rangle = (1 + |x|^2 + |z|^2)^{1/2}$, then one has

(i) If $0 < m < 2$, then

$$N(H_3(V) < -\lambda) = 2(2\pi)^{-2} \int_{V(x,z)>\lambda} b(x)(V(x,z) - \lambda)^{1/2}\, dxdz\, (1 + o(1))$$

as $\lambda \to 0$.

(ii) Assume that $m > 2$. Let $w(x)$ be defined as

$$w(x) = \int V(x,z)\, dz,$$

where the integration without domain attached is taken over the whole space. If $w(x)$ fulfills

$$\limsup_{\lambda \to 0} \lambda^{2/(m-1)} \int_{(1+\delta)\lambda > w(x) > (1-\delta)\lambda} dx = o(1), \quad \delta \to 0,$$

then

$$N(H_3(V) < -\lambda) = (2\pi)^{-1} \int_{w(x) > 2\lambda^{1/2}} b(x)\, dx\, (1 + o(1)), \quad \lambda \to 0.$$

The proof of Theorem 2 is based on the asymptotic formula in two dimensions. The argument there seems to extend to the case $0 < d < 2$ without any essential changes, if we make use of the two dimensional formula obtained in Theorem 1 (i).

There are a lot of works on the problem of spectral asymptotics for magnetic Schrödinger operators. An extensive list of literatures can be found in the survey [13]. The problem of asymptotic distribution of discrete eigenvalues below the bottom of essential spectrum has been studied by [13, 15] when $b(x) = b$ is a uniform magnetic field. Both the works make an essential use of the uniformity of magnetic fields and the argument there does not extend directly to the case of nonconstant magnetic fields. Roughly speaking, the difficulty arises from the fact that magnetic potentials which actually appear in Pauli operators undergo nonlocal changes even under local changes of magnetic fields. This makes it difficult to control nonconstant magnetic fields by a local approximation of uniform magnetic fields. Much attention is now paid on the Lieb–Thirring estimate on the sum of negative eigenvalues of Pauli operators with nonconstant magnetic fields in relation to the magnetic Thomas–Fermi theory ([5, 6, 7, 8, 16]). The present work is motivated by these works.

2. Basic spectral properties of the unperturbed operator

In this section we state a basic fact of the spectral properties of unperturbed two dimensional Pauli operators without electric fields (see [4]).

We consider the following operators

$$\tilde{H}_\pm = (-i\nabla - \tilde{A})^2 \mp \tilde{b} = \tilde{\Pi}_1^2 + \tilde{\Pi}_2^2 \mp \tilde{b} \quad \text{on } L^2 = L^2(\boldsymbol{R}^2),$$

where $\tilde{A}(x) = (\tilde{a}_1(x), \tilde{a}_2(x))$, $\tilde{\Pi}_j = -i\partial_j - \tilde{a}_j$ and $\tilde{b}(x) = \nabla \times \tilde{A}$. As stated in the previous section, these operators can be rewritten as

$$\tilde{H}_\pm = (\tilde{\Pi}_1 \pm i\tilde{\Pi}_2)^*(\tilde{\Pi}_1 \pm i\tilde{\Pi}_2)$$

and hence they become nonnegative operators. If, in particular, $\tilde{b}$ satisfies

$$\tilde{b}(x) > \tilde{c} > 0, \tag{2}$$

then $\tilde{H}_- \geq \tilde{c}$ becomes a strictly positive operator. On the other hand, it is known ([1, 14]) that $\tilde{H}_+$ has zero as an eigenvalue with infinite multiplicities. If we choose the magnetic potential $\tilde{A}(x)$ in the divegenceless form

$$\tilde{A}(x) = (\tilde{a}_1(x), \tilde{a}_2(x)) = (-\partial_2\varphi, \partial_1\varphi)$$

for some real function $\varphi \in C^2(\boldsymbol{R}^2)$ obeying $\Delta\varphi = \tilde{b}$, then we have

$$\tilde{\Pi}_1 + i\tilde{\Pi}_2 = -ie^{-\varphi}(\partial_1 + i\partial_2)e^{\varphi}.$$

This implies that the zero eigenspace just coincides with the subspace

$$K_\varphi = \{u \in L^2 : u = he^{-\varphi} \ \text{ with } \ h \in A(\boldsymbol{C})\},$$

where $A(\boldsymbol{C})$ denotes the class of analytic functions over the complex plane $\boldsymbol{C}$. Let $P_\varphi : L^2 \to L^2$ be the orthogonal projection on the zero eigenspace K_φ of $\tilde{H}_+$. We write Q_φ for $Id - P_\varphi$, Id being the identity operator. We also know ([4]) that the non–zero spectra of $\tilde{H}_\pm$ coincide with each other. Hence it follows that

$$Q_\varphi \tilde{H}_+ Q_\varphi \geq \tilde{c} Q_\varphi \tag{3}$$

in the form sense, $\tilde{c} > 0$ being as in (2).

3. Propositions

In this section we collect several basic propositions needed for the proof of the theorems.

First we use the perturbation theory for singular numbers of compact operators as a basic tool to prove the theorems. We shall briefly explain several basic properties of singular numbers. We refer to [9] for details.

We denote by $N(S > \lambda)$ and $N(S < \lambda)$ the number of eigenvalues more and less than λ of self–adjoint operator S, respectively. Let $T : X \to X$ be a compact (not necessarily self–adjoint) operator acting on a separable Hilbert space X. We write $|T|$ for $\sqrt{TT^*}$. The singular number $\{s_n(T)\}$, $n \in \boldsymbol{N}$, of compact operator T is defined as the non–increasing sequence of eigenvalues of $|T|$ and it has the following properties : $s_n(T) = s_n(T^*)$ and

$$s_{n+m-1}(T_1 + T_2) \leq s_n(T_1) + s_m(T_2) \tag{4}$$

for two compact operators T_1 and T_2. We now write

$$N(|T| > \lambda) = \#\{n \in \boldsymbol{N} : s_n(T) > \lambda\}, \quad \lambda > 0,$$

according to the above notation. If $T : X \to X$ is a compact self–adjoint operator, then

$$N(|T| > \lambda) = N(T > \lambda) + N(T < -\lambda), \quad \lambda > 0.$$

If, in particular, $T \geq 0$, it follows that $N(|T| > \lambda) = N(T > \lambda)$. The next proposition, which is a direct consequence of (4), is repeatedly used throughout the entire discussion.

Proposition 1 *Assume that T_1 and T_2 are compact operators. Let $\lambda_1, \lambda_2 > 0$ be such that $\lambda_1 + \lambda_2 = \lambda$. Then*

$$N(|T_1 + T_2| > \lambda) \leq N(|T_1| > \lambda_1) + N(|T_2| > \lambda_2).$$

If, in particular, $T_1, T_2 \geq 0$, *then*

$$\begin{aligned} N(T_1 + T_2 > \lambda) &\leq N(T_1 > (1-\delta)\lambda) + N(T_2 > \delta\lambda), \\ N(T_1 - T_2 > \lambda) &\geq N(T_1 > (1+\delta)\lambda) - 2N(T_2 > \delta\lambda) \end{aligned}$$

for any $\delta > 0$ *small enough.*

Another fundamental tool is the localization technique based on the Min-Max principle. The following relation which is often called the IMS localization formula ([4]) plays an important role. Let a smooth partition $\{\psi_j\}$ of the unity normalized by $\sum_j \psi_j(x)^2 = 1$ associated with a locally finite open cover $\{U_j\}$ of $\boldsymbol{R}^2$. Then a simple computation yields the relation

$$H(V) = \sum_j \psi_j(H(V) - \Psi)\psi_j, \qquad \Psi = \sum_j |\nabla\psi_j|^2$$

in the form sense. We then obtain the following proposition with the use of the Min-Max principle by comparing, e.g., two forms q_1 and q_2:

$$\begin{aligned} q_1[u] &= (H(V)u, u), \qquad u \in C_0^\infty(\boldsymbol{R}^2) \\ q_2[u] &= \sum_j ((H(V) - \Psi)u_j, u_j), \qquad \bigoplus_j u_j \in \bigoplus_j C_0^\infty(U_j) \end{aligned}$$

where $C_0^\infty(U)$ denotes the space of C^∞ function with compact support contained in U.

Proposition 2 *Let* $H(V)_U^D$ *denote the operator* $H(V)$ *defined on* U *with Dirichlet boundary conditions. Then one has the following:*
(i) *Let* $\{U_j\}$, $\{\psi_j\}$ *and* Ψ *be as above. Then we have*

$$N(H(V) < -\lambda) \leq \sum_j N(H(V)_{U_j}^D - \Psi < -\lambda).$$

(ii) *Let* $\{Q_j\}$ *be a family of disjoint open sets in* $\boldsymbol{R}^2$. *Then we have*

$$N(H(V) < -\lambda) \geq \sum_j N(H(V)_{Q_j}^D < -\lambda).$$

The following result about the number of eigenvalues in a cube with constant field is due to Colin de Verdière [3].

Proposition 3 *Let* Q_R *be a cube with side* R *and let*

$$S_B = (-i\nabla - \hat{A})^2, \quad \hat{A}(x) = (-Bx_2/2, Bx_1/2),$$

be the Schrödinger operator with constant magnetic field $B = \nabla \times \hat{A} > 0$. Then there exists $c > 0$ independent of μ, R and Λ, $0 < \Lambda < R/2$, such that :

(1) $N((S_B)^D_{Q_R} < \mu) \le (2\pi)^{-1} B\, |Q_R|\, F(\mu/B)$

(2) $N((S_B)^D_{Q_R} < \mu) \ge (2\pi)^{-1}(1 - \Lambda/R)^2 B\, |Q_R|\, F((\mu - c\Lambda^{-2})/B)$,

where $|Q_R| = R^2$ is the measure of cube Q_R and

$$F(\mu) = \#\{n \in \boldsymbol{N} \cup \{0\} : 2n + 1 \le \mu\}.$$

The following proposition allows us to compare the number of eigenvalues for two different magnetic fields. This can be shown by a simple use of the Min-Max principle.

Proposition 4 *Assume that $U(x) \ge 0$ is a bounded function with compact support. Let $\varphi_j \in C^2(\boldsymbol{R}^2)$, $1 \le j \le 2$, be real functions. If $\varphi_1(x) \le \varphi_2(x)$, then*

$$N(P_{\varphi_1} U P_{\varphi_1} > \mu) \le N(P_{\varphi_2} U P_{\varphi_2} > \mu/\gamma), \quad \mu > 0,$$

where

$$\gamma = \max_{x \in \operatorname{supp} U} \exp(2\omega(x)), \quad \omega(x) = \varphi_2(x) - \varphi_1(x) \ge 0.$$

The following proposition gives the existence of a solution to the Poisson equation in two dimensions which has a control on the increase order. This can be shown by using the Fourier series.

Proposition 5 *Assume that a real function $b \in C^1(\boldsymbol{R}^2)$ satisfies $|b(x)| = O(r^{-d}), d \ge 0$ as $r = |x| \to \infty$. Then there exists a real solution $\varphi_0 \in C^2(\boldsymbol{R}^2)$ to equation $\Delta\varphi_0 = b$ with bound*

$$\varphi_0(x) = \begin{cases} O(r^{2-d}), & 0 < d < 2, \ d \ne 1, \\ O(r^{2-d} \log r), & d = 0, 1, \\ O((\log r)^2), & d = 2, \\ O(\log r) & d > 2. \end{cases}$$

Our last tool is the following proposition concerning a commutator estimate, which is useful when the magnetic field $\tilde{b}(x) > \tilde{c} > 0$ is strictly positive.

Proposition 6 *Assume that $U(x) \in C^1(\boldsymbol{R}^2)$ and $|U(x)|, |\nabla U(x)|$ are bounded. Let P_φ, Q_φ be as in Section 2. Then*

$$\|P_\varphi U Q_\varphi\| \le C(\inf \tilde{b})^{-1/2} \sup |\nabla U|$$

for some $C > 0$ independent of φ, $\tilde{b}$ and U, where $\Delta\varphi = \tilde{b}$ and $\|\ \|$ denotes the norm of bounded operators acting on $L^2(\boldsymbol{R}^2)$.

4. Sketch of proof

We give a rough sketch of the idea of proof in the case where $b(x) = \tilde{b}(x) > \tilde{c} > 0$ is strictly positive, i.e., $d = 0$.
Step 1. Let $0 < c < \inf \tilde{b}/2$ be fixed and $P = P_\varphi$, $Q = Q_\varphi$ be as in Section 2. First we use the form inequality

$$\begin{aligned} H(V) &= PH(V)P + QH(V)Q - PVQ - QVP \\ &\geq PH(V)P + QH(V)Q - cQ - PV^2P/c \end{aligned}$$

and hence it follows that

$$N(H(V) < -\lambda) \leq N(P(V + V^2/c)P > \lambda) + N(Q(H(V) - c)Q < -\lambda), \quad \lambda > 0.$$

By (3) the quantity $N(Q(H(V) - c)Q < -\lambda)$ remains uniformly bounded for $\lambda > 0$. Moreover $V(x)^2 = O(|x|^{-2m})$ falls off at infinity faster than $V(x)$ and can be treated as a negligible term by a perturbation method if we use Propositions 4 and 5. On the other hand, we have

$$N(H(V) < -\lambda) \geq N(PVP > \lambda), \quad \lambda > 0,$$

because the form $(H(V)u, u)$ coincides with $(-Vu, u)$ on the range $\operatorname{Ran} P$ of P. Thus we have

$$N(H(V) < -\lambda) \sim N(PVP > \lambda) \quad \text{as } \lambda \to +0. \tag{5}$$

Step 2. When V decays sufficiently slowly, we can use Propositions 2 and 3 to obtain the asymptotics for $N(H(V) < -\lambda)$ directly. This gives at the same time the asymptotics for $N(PVP > \lambda)$ by (5).
Step 3. Use Proposition 6 to obtain

$$N(PVP > \lambda) \sim N(PV^{1/2}P > \lambda^{1/2}) \quad \text{as } \lambda \to +0,$$

since ∇V decays faster than V by the assumption (V). This allows one to extend the asymptotics of $N(PVP > \lambda)$ obtainable for slowly decaying potential in Step 2 to the case of the potentials with faster decay. This in turn produces the asymptotics of $N(H(V) < -\lambda)$ through (5). Thus we obtain the result for any potential by induction.

Finally we note that the case $d > 0$ can be treated by approximating the decaying magnetic field by some families of magnetic fields dependent on λ, which are strictly positive for each λ. The spirit of the proof is the same as that in the case $d = 0$, exploiting the relation (5), though the arguments become subtler and more involved.

5. Concluding remarks

We conclude the talk by making some comments on the asymptotic distribution of negative eigenvalues in the case that electric fields are stronger than magnetic fields at infinity in the two dimensional space. The case is much easier to deal with. We can obtain the following theorem, which can be easily proved by the simple use of the localization technique.

Theorem 3 *Assume that* (b) *and* (V) *are fulfilled. Let* $0 < m < d < 2$. *If* $V(x)$ *satisfies* (1), *then*

$$N(H(V) < -\lambda) = (4\pi)^{-1} \int_{V>\lambda} (V(x) - \lambda)\, dx + o(\lambda^{(m-2)/m}), \quad \lambda \to 0.$$

Remarks. (1) The asymptotic formula above can be rewritten as

$$N(H(V) < -\lambda) = (2\pi)^{-2}\mathrm{vol}\Big(\{(x,\xi) \in \boldsymbol{R}^2 \times \boldsymbol{R}^2 : H(x,\xi) < -\lambda\}\Big)(1 + o(1)),$$

where $H(x,\xi) = |\xi - A(x)|^2 - b(x) - V(x)$. Thus $N(H(V) < -\lambda)$ obeys the classical Weyl formula.

(2) The theorem remains true without assuming $b(x)$ to be strictly positive, and also it is still valid for the case when $d \geq 2$ and $0 < m < 2$.

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